

# Different Variants of Bernstein Kantorovich Operators and Their Applications in Sciences and Engineering Field

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**ABSTRACT** In this article, we investigate various Bernstein-Kantorovich variants together with their approximation properties. Nowadays, these variants of Bernstein-Kantorovich operators have been a source of inspiration for researchers as it helps to approximate integral functions also which is not feasible in the case of discrete operators. Chaos theory has also been referred to as complexity theory. Using chaos theory complexity is also reduced as in approximation theory. Thus in order to reduce complexity and to have better understanding of images in sciences and engineering field, sampling Kantorovich operators of approximation theory are widely used in this regard for enhancement of images. Thus, we discuss the important applications of Kantorovich operators depicting pragmatic and theoretical aspects of approximation theory.

**KEYWORDS**  
Bernstein  
Kantorovich operators  
q-calculus  
Lupas-Stancu operators  
Polya distribution

## INTRODUCTION

The objective of this paper is to highlight the different variants of Bernstein-Kantorovich operators which are widely used for approximation of functions in  $L^p$  spaces. The advantage of using Kantorovich variants over discrete operators is that discrete operators are not suitable for approximating functions which are not continuous, therefore these operators were generalized into operators of integral type and one such technique is Kantorovich which helps to approximate integral functions and thus Kantorovich variant of various linear positive operators have been a source of inspiration for many scholars.

Approximation theory is an area of mathematical analysis which is mainly concerned with approximation of complicated quantities by simpler functions. This unique feature of approximation theory forces us to study this field alongwith the study of some ideas of functional analysis. In (Rashid *et al.* 2022), discrete proportional fraction operators are used to contribute to the major effects of some innovative variants of reverse Minkowski and related H-order-type inequalities. Approximation theory gained

popularity with the emergence of Weierstrass theorem (1885). Karl Weierstrass presented the first proof of his fundamental theorem on approximation by algebraic and trigonometric polynomials but the complications in proof provoked many famous mathematicians to work on this fundamental theorem of approximation theory. Most commonly used proofs of Weierstrass theorem are of Fejer (1900) and Bernstein (1912) as proofs given by other famous mathematicians were not very productive and satisfactory (see e.g. (Bartle 1976; Cheney 1966; Lubinsky 1995; Pinkus 2000)). S.N Bernstein (Bernštem 1912) gave the most simplest and constructive proof of Weierstrass theorem using Bernstein operators  $E_{\Theta} : \check{c}[0, 1] \rightarrow \check{c}[0, 1]$  defined in (Bernštem 1912) as follows:

$$E_{\Theta}(g'; \varkappa) = \sum_{l=0}^{\Theta} \binom{\Theta}{l} \varepsilon_{\Theta, l}(\varkappa) g' \left( \frac{l}{\Theta} \right), \quad (1)$$

where  $\varepsilon_{\Theta, l}(\varkappa) = \binom{\Theta}{l} \varkappa^l (1 - \varkappa)^{\Theta-l}$ ,  $g' \in \check{c}[0, 1]$ ,  $\Theta \geq 1$  and  $0 \leq \varkappa \leq 1$ .

Bernstein operators are considered as foundational operators due to its immense contribution in the field of approximation theory. These operators occupy prominent position among all linear positive operators because of their efficient and notable approxi-

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mation properties but its favourable properties gets overshadowed because of slow rate of convergence. Keeping in view these drawbacks, various modifications has been made to these operators using bezier basis (Agrawal et al. 2022). These operators cannot be used to approximate the function in integral metrics. In order to overcome this problem, Kantorovich made small modification of Bernstein operators in 1930 (see (Kantorovich 1930)). In that paper the author L.V Kantorovich, introduces modified operators known as nth Bernstein-Kantorovich operators  $L_{\Theta} : L^1([0, 1]) \rightarrow \check{c}[0, 1]$  defined by

$$L_{\Theta}(g'; \varkappa) = \sum_{l=0}^{\Theta} (\Theta + 1) \left( \int_{\frac{l}{\Theta+1}}^{\frac{l+1}{\Theta+1}} g'(t) dt \right) \binom{\Theta}{l} \varkappa^l (1 - \varkappa)^{\Theta-l}, \quad (2)$$

where  $g' \in L^1([0, 1])$  and  $\varkappa \in [0, 1]$ .

Study of Lebesgue integrable functions in  $L^1$  space became possible due to modification of Bernstein operators by Kantorovich. Moreover, Altomare et al. (Altomare and Campiti 2011), established the approximation properties using Korovkin theorem and investigate the rate of convergence associated with these operators. The idea of Kantorovich modifications of sequence of linear positive operators inspires many other mathematicians to investigate some new operators within approximation theory. Many authors constructed and studied the Kantorovich type modification of some various operators (see e.g. (Agratini 2001; Barbosu 2004; Dogru and Ozalp 2001; Duman et al. 2006; Özarslan et al. 2008; Kac and Cheung 2001; Karaca 2022))

In this article, we try to give some important information about different variants of Bernstein-Kantorovich operators, hoping that this will act as a beneficial tool for all those reserachers that work in approximation theory and intend to apply Kantorovich technique in order to modify various linear positive operators.

## LITERATURE REVIEW

In this section, we review some variants of Bernstein-Kantorovich operators which are being very popularly used in approximation theory. We refer readers to some papers such as (Acu 2015; Agrawal et al. 2015; Altomare et al. 2013; Barbosu 2004; Bardaro et al. 2007; de la Cal and Valle 2000; Deo et al. 2016; Gonska et al. 2011; Igoz 2012) for generalization of these operators in Kantorovich form.

Let us recall some notations. Throughout this paper,  $\check{c}[a, b]$  is the space of all continuous real valued functions on  $[a, b]$ ,  $L^p_{[0,1]}$  is the class of all p power integrable functions on the interval  $[0, 1]$ . For basic definitions and results regarding Banach spaces with the proof as well as applications, one may refer (Karaca 2022).

Kantorovich's idea has been applied also to Bernstein operators involving q-calculus. For definition and notations from q-calculus, we refer readers to (Andrews et al. 1999; Kac and Cheung 2001). In (Lupas 1987) Lupas introduced modified form of Bernstein operators using q-calculus and explored its approximation and shape-preserving properties. Subsequently, philips in (Phillips 2003) has done q-generalization of Bernstein operators known as Bernstein operators in q-calculus (q-Bernstein operators) defined, for every positive integer  $\Theta$  and  $g' \in [0, 1]$ , by

$$B_{\Theta}(g'; q; \varkappa) = \sum_{l=0}^{\Theta} g' \left( \frac{[l]}{[\Theta]} \right) \left[ \frac{\Theta}{l} \right] \varkappa^l \prod_{s=0}^{\Theta-l-1} (1 - q^s \varkappa). \quad (3)$$

and also studied various results including the theorem confirming uniform convergence (korovkin theorem), order of

convergence and asymptotic expansion of these operators given by voronovskaya theorem. See (Ostrovskaya 2016) for similarities and distinctions of operators given by Lupas and Philips.

In (Dalmanoğlu 2007) modification of Bernstein operators in q-calculus using Kantorovich technique is instigated and its approximation properties are satisfied, while in (Radu 2008) Bernstein-Kantorovich operators in q-calculus are extended and their statistical convergence propertis are prepensed.

In (Dalmanoğlu 2007; Radu 2008), the another modification of the Bernstein operators using Kantorovich technique is elucidate in q-calculus, for every  $\Theta \in \mathbb{N}$ ,  $\varkappa \in [0, 1]$  and  $0 < q < 1$ , by

$$K_{\Theta}(g'; q; \varkappa) = [\Theta + 1] \sum_{l=0}^{\Theta} \binom{\Theta}{l} \left( \frac{\varkappa}{q} \right)^l \prod_{s=0}^{\Theta-l-1} (1 - q^s \varkappa) \times \int_{\frac{[l]}{[\Theta+1]}}^{\frac{[l+1]}{[\Theta+1]}} g'(t) d_q t. \quad (4)$$

Subsequently, the study of operators has been intensified by Dalmanog et al. in (Dalmanog et al. 2010), where they reconceive the Kantorovich variant of Bernstein operators in q-calculus using the definition of q-integral of Riemann type (see (Marinković et al. 2008)) into the operator instead of general q-integral as:

$$B_{\Theta}^*(g'; q; \varkappa) = [\Theta + 1] \sum_{l=0}^{\Theta} q^{-l} \binom{\Theta}{l}^{\Theta-l-1} \prod_{s=0}^{\Theta-l-1} (1 - q^s \varkappa) \int_{\frac{[l]}{[\Theta+1]}}^{\frac{[l+1]}{[\Theta+1]}} g'(t) d_q^{\tau} t. \quad (5)$$

The need of redefining the operators arises because the study of statistical convergence of operators  $K_{\Theta}(g'; q; \varkappa)$  to the function  $g'$  is problematical with the usage of classical q-integral.

Stancu operators are instead object of a modification for various researchers. In (Gadjiev and Ghorbanalizadeh 2010) Gadjiev et al. introduced Bernstein-Stancu type polynomials with shifted knots:

$$S_{\Theta, \alpha, \beta}(g'; \varkappa) = \left( \frac{\Theta + \beta_2}{\Theta} \right)^{\Theta} \sum_{\tau=0}^{\Theta} g' \left( \frac{\tau + \alpha_1}{\Theta + \beta_1} \right) \binom{\Theta}{\tau} \left( \varkappa - \frac{\alpha_2}{\Theta + \beta_2} \right)^{\tau} \left( \frac{\Theta + \alpha_2}{\Theta + \beta_2} - \varkappa \right)^{\Theta-\tau}, \quad (6)$$

where  $\frac{\alpha_2}{\Theta + \beta_2} \leq \varkappa \leq \frac{\Theta + \alpha_2}{\Theta + \beta_2}$  and  $\alpha_l, \beta_l$  ( $l = 1, 2$ ) are such that  $0 \leq \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2$ .

Motivated by above operators, İcöz in (İcöz 2012) designate a generalized form of Bernstein-Stancu operators using Kantorovich technique under same assumptions as:

$$S_{\Theta, \alpha, \beta}^*(g'; \varkappa) = (\Theta + \beta_1 + 1) \left( \frac{\Theta + \beta_2}{\Theta} \right)^{\Theta} \sum_{\tau=0}^{\Theta} \binom{\Theta}{\tau} \left( \varkappa - \frac{\alpha_2}{\Theta + \beta_2} \right)^{\tau} \left( \frac{\Theta + \alpha_2}{\Theta + \beta_2} - \varkappa \right)^{\Theta-\tau} \int_{\frac{\tau + \alpha_1}{\Theta + \beta_1 + 1}}^{\frac{\tau + \alpha_1 + 1}{\Theta + \beta_1 + 1}} g'(s) ds. \quad (7)$$

To show the extend of research in direction of q-calculus, we mention the work done by Muraru in (Muraru 2011). She bring

forward the  $q$  analogue of Bernstein-Schurer operators that are given by

$$E_{\Theta,p}(g'; q; \varkappa) = \sum_{l=0}^{\Theta+p} p_{\Theta,l}(q, \varkappa) g' \left( \frac{[l]_q}{[\Theta]_q} \right), \quad (8)$$

where  $p_{\Theta,l}(q, \varkappa) = \begin{bmatrix} \Theta+p \\ l \end{bmatrix} \varkappa^l (1-\varkappa)_q^{\Theta+p-l}$ ,  $\varkappa \in [0,1]$  and

premise various results involving necessary and sufficient condition for convergence of operators given by the Korovkin and Voronovskaja theorem concerning asymptotic convergence of linear positive operators.

Kantorovich modification of the operators (8) have been introduced by Ozarslan et al. in (Özarslan and Vedi 2013) defined, for every  $g' \in \mathcal{C}[0, p+1]$ ,  $0 < q < 1$  and  $p \in \Theta_0 = \{0, 1, 2, \dots\}$ , by

$$K_{\Theta}^p(g'; q; \varkappa) = \sum_{r=0}^{\Theta+p} \begin{bmatrix} \Theta+p \\ r \end{bmatrix} \varkappa^r \prod_{s=0}^{\Theta+p-r-1} (1-q^s \varkappa) \times \int_0^1 g' \left( \frac{[t]}{[\Theta+1]} + \frac{1+(q-1)[t]}{[\Theta+1]} t \right) d_q t. \quad (9)$$

Ren et al. modified the operators (8) in (Ren and Zeng 2013). A new variant of Bernstein-Schurer operators in  $q$ -calculus are given by

$$\tilde{\varepsilon}_{\Theta,p}(g'; q; \varkappa) = \sum_{l=0}^{\Theta+p} \tilde{p}_{\Theta,l}^*(q, \varkappa) g' \left( \frac{[l]_q}{[\Theta]_q} \right), \quad (10)$$

where  $\tilde{p}_{\Theta,l}^*(q, \varkappa) = \frac{[\Theta]_q^{\Theta+p}}{[\Theta+p]_q^{\Theta+p}} \begin{bmatrix} \Theta+p \\ l \end{bmatrix} \varkappa^l \left( \frac{[\Theta+p]_q}{[\Theta]_q} - \varkappa \right)_q^{\Theta+p-l}$

and  $\varkappa \in [0,1]$ .

Authors investigated compulsory Korovkin type statistical convergence theorem for uniform convergence, Voronovskaja theorem concerning asymptotic convergence, the rate of statistical convergence using various tools such as modulus of continuity and a Lipschitz function for the operators in (Ren and Zeng 2013).

Agrawal et al. in (Agrawal et al. 2015) present a stancu variant of operators (8) using Kantorovich technique defined, for every  $g' \in C[0, 1+p]$  endowed with the norm  $\|g'\| = \sup_{\varkappa \in [0,1]} |g'(\varkappa)|$ ,  $\alpha, \beta \in \mathbb{R}$  such that  $0 \leq \alpha \leq \beta$  and  $0 < q < 1$ , by

$$\kappa_{\Theta,p}^{(\alpha,\beta)}(g'; q; \varkappa) = \sum_{l=0}^{\Theta+p} \tilde{p}_{\Theta,l}^*(q, \varkappa) \int_0^1 g' \left( \frac{[l]_q + q^l t + \alpha}{[\Theta+1]_q + \beta} \right) d_q t. \quad (11)$$

Kantorovich (Kantorovich 1930) gave the integral modification of Bernstein operators so as to approximate integrable functions defined on  $[0, 1]$ . In (Ozarslan and Duman 2016) Özarslan et al. introduced modified Kantorovich operators based on non-negative parameter  $\rho$  as:

$$K_{\Theta,\rho}(g'; \varkappa) = \sum_{l=0}^{\Theta} \varepsilon_{\Theta,l}(\varkappa) \int_0^1 g' \left( \frac{l+t^\rho}{\Theta+1} \right) dt. \quad (12)$$

and investigated the order of convergence and various approximation properties of these operators using various mathematical tools concerning smoothness of operators. In that paper, authors also showed that modified Kantorovich operators based on non-negative parameter  $\rho$  depicts faster rate of convergence to a function than that of Kantorovich operators in classical form.

Another new general approach is considered by Mursaleen et al. (Mursaleen et al. 2015). In that paper authors prepensed another modified form of Bernstein operators in  $((p,q)$ -calculus) known as  $(p,q)$ -Bernstein operators defined, for every  $\varkappa \in [0,1]$ ,  $0 < p < q \leq 1$ , by

$$B_{\Theta,p,q}(g'; \varkappa) = \sum_{l=0}^{\Theta} \begin{bmatrix} \Theta \\ l \end{bmatrix}_{p,q} \varkappa^l \prod_{s=0}^{\Theta-l-1} (p^s - q^s \varkappa) g' \left( \frac{[l]_{p,q}}{[\Theta]_{p,q}} \right). \quad (13)$$

These operators turned into classical Bernstein operators in  $q$ -calculus for  $p=1$ .

For notations used in operators (13), we refer reader to (Mursaleen et al. 2015). Also details on  $(p,q)$ -calculus can be found in (Hounkonnou et al. 2013; Katriel and Kibler 1992; Sahai and Yadav 2007).

Afterward, Mursaleen et al. (Mursaleen et al. 2016) constructed of Bernstein-Kantorovich operators in  $((p,q)$ -calculus) as:

$$K_{\Theta}^{(p,q)}(g'; \varkappa) = \frac{[\Theta]_{p,q}}{p^{\frac{\Theta(\Theta-1)}{2}}} \sum_{l=0}^{\Theta} \frac{b_{\Theta,l}^{(p,q)}(\varkappa)}{p^{\Theta-l} q^l} \int \frac{[t+1]_{p,q}}{p^{l-\Theta} [\Theta]_{p,q}} g'(t) d_p t, \quad (14)$$

where  $\varkappa \in [0,1]$ ,  $b_{\Theta,l}^{(p,q)}(\varkappa) = \begin{bmatrix} \Theta \\ l \end{bmatrix}_{p,q} \varkappa^l (1-\varkappa)_{p,q}^{\Theta-l}$  and  $(\varkappa)_{p,q}^l = \varkappa(p\varkappa)(p^2\varkappa) \dots (p^{l-1}\varkappa) = p^{\frac{l(l-1)}{2}} \varkappa^l$ .

Moreover, authors study the local approximation property of  $K_{\Theta}^{(p,q)}(g'; \varkappa)$  and obtain faster rate of convergence and better error estimates of operators as compared to Bernstein-Kantorovich operators in  $q$ -calculus.

Realising the essentials of Bernstein-Stancu, Mursaleen et al. in (Mursaleen et al. 2017) introduce another variant of Bernstein-Stancu in  $q$ -calculus using Kantorovich technique as follows:

$$\begin{aligned} K_{\Theta,q}^{(\alpha,\beta)} &= \left( \frac{[\Theta] + \beta_2}{[\Theta]} \right)^{\Theta} \sum_{l=0}^{\Theta} \begin{bmatrix} \Theta \\ l \end{bmatrix} \left( \varkappa - \frac{\alpha_2}{[\Theta] + \beta_2} \right)_q^l \\ &\times \left( \frac{[\Theta] + \alpha_2}{[\Theta] + \beta_2} - \varkappa \right)_q^{\Theta-l} \int_0^1 g' \left( \frac{[l]_q t + \alpha_1}{[\Theta+1] + \beta_1} \right) d_q t. \end{aligned} \quad (15)$$

New Kantorovich-type operators based on Pólya-Eggenberger distribution (Eggenberger and Pólya 1923) are introduced by Kajla et al. in (Kajla and Araci 2017) as:

$$K_{\Theta,\rho}^{[\alpha]}(g'; \varkappa) = \sum_{l=0}^{\Theta} p_{\Theta,l}^{[\alpha]}(\varkappa) \int_0^1 g' \left( \frac{l+t^\rho}{\Theta+1} \right) dt, \quad (16)$$

where  $\rho > 0$  and  $p_{\Theta,l}^{[\alpha]}(\varkappa) = \binom{\Theta}{l} \frac{1}{1^{[\Theta-l]}} \varkappa^{l-\alpha} (1-\varkappa)^{[\Theta-l-\alpha]}$ .

In (Acu et al. 2018) Acu et al. study a new type of Bernstein operators depending on the parameter  $\lambda \in [-1, 1]$  proposed by Cai et al. (Cai et al. 2018) as follows:

$$B_{\Theta,\lambda}(g'; \varkappa) = \sum_{l=0}^{\Theta} \tilde{\varepsilon}_{\Theta,l}(\lambda; \varkappa) g' \left( \frac{l}{\Theta} \right), \quad (17)$$

where  $\tilde{\varepsilon}_{\Theta,l}(\lambda; \varkappa)$ ,  $l = 0, 1, 2, 3, \dots$ , are defined as:

$$\tilde{\varepsilon}_{\Theta,0}(\lambda; \varkappa) = \varepsilon_{\Theta,0}(\varkappa) - \frac{\lambda}{\Theta+1} \varepsilon_{\Theta+1,1}(\varkappa),$$

$$\begin{aligned} \tilde{e}_{\Theta, l}(\lambda; \varkappa) &= e_{\Theta, l}(\varkappa) \\ &+ \lambda \left( \frac{\Theta - 2k + 1}{\Theta^2 - 1} e_{\Theta+1, l}(\varkappa) - \frac{\Theta - 2k - 1}{\Theta^2 - 1} e_{\Theta+1, l+1}(\varkappa) \right) \\ \tilde{e}_{\Theta, \Theta}(\lambda; \varkappa) &= e_{\Theta, \Theta}(\varkappa) - \frac{\lambda}{\Theta + 1} e_{\Theta+1, \Theta}(\varkappa). \end{aligned}$$

Moreover, Ana et al. considered a Kantorovich modification of  $\lambda$ -Bernstein operators, namely

$$K_{\Theta, \lambda}(g'; \varkappa) = (\Theta + 1) \sum_{l=0}^{\Theta} \tilde{e}_{\Theta, l}(\varkappa) \int_{\frac{l}{\Theta+1}}^{\frac{l+1}{\Theta+1}} g'(t) dt. \quad (18)$$

In particular, in (Acu et al. 2018) (see Example 4) authors obtained better error estimate for  $\lambda$ -Bernstein operators as compared to classical Kantorovich operators.

In (Chen et al. 2017) Chen et al. studied another modification of Bernstein operators, depending on a non-negative real parameter  $\alpha$  ( $\alpha$ -Bernstein operators) defined, for every  $g' \in \mathfrak{C}[0, 1], \Theta \geq 2$  and  $0 \leq \varkappa \leq 1$ , by

$$T_{\Theta}^{\alpha}(g'; \varkappa) = \sum_{l=0}^{\Theta} p_{\Theta, l}^{(\alpha)}(\varkappa) g' \left( \frac{l}{\Theta} \right), \quad (19)$$

where

$$\begin{aligned} p_{\Theta, l}^{(\alpha)}(\varkappa) &= \left[ \binom{\Theta - 2}{l} (1 - \alpha) \varkappa + \binom{\Theta - 2}{l-2} (1 - \alpha) (1 - \varkappa) \right. \\ &\left. + \binom{\Theta}{l} \alpha \varkappa (1 - \varkappa) \right] \varkappa^{l-1} (1 - \varkappa)^{\Theta-l-1} \end{aligned}$$

The authors also proved the degree of convergence, Voronovskaja theorem concerning asymptotic formula and shape preserving properties for operator (19).

In (Mohiuddine et al. 2017) Mohiuddine et al. proposed and investigated the Kantorovich type modification of operators (19) defined, for every  $g' \in \mathfrak{C}[0, 1], \Theta \geq 2$  and  $0 \leq \varkappa \leq 1$ , by

$$\widehat{I}_{\Theta, \alpha}(g'; \varkappa) = (\Theta + 1) \sum_{s=0}^{\Theta} p_{\Theta, s}^{(\alpha)} \int_{\frac{s}{\Theta+1}}^{\frac{s+1}{\Theta+1}} g'(t) dt. \quad (20)$$

Araci et al. also proved various approximation properties with the help of Bohman-Korovkin's principle which is necessary and sufficient criteria for uniform convergence and used various tools such as the modulus of smoothness and Lipschitz type function to study the approximation rate of operators. They also derived Voronovskaja type asymptotic convergence theorem and Korovkin type A-statistical approximation theorem of these operators.

A short time ago, the Kantorovich modification of the operators (19) by Araci et al. is defined in (Araci et al. 2019), for every  $g' \in \mathfrak{C}[0, 1], \alpha > 0$  and  $\rho > 0$ , by

$$\kappa_{m, \rho}^{\alpha, \alpha}(g'; \varkappa) = \sum_{l=0}^{\Theta} p_{\Theta, l}^{(\alpha)}(\varkappa) \int_0^1 f \left( \frac{l + at^{\rho}}{\Theta + a} \right) dt, \quad (21)$$

where  $\varkappa \in [0, 1]$  and  $p_{\Theta, l}^{(\alpha)}$  is defined above.

Afterward,  $\alpha, q$ -Bernstein operators by Cai et al. in (Cai and Xu 2018) are presented as:

$$\mathfrak{T}_{\Theta, q, \alpha}(g'; \varkappa) = \sum_{\ell=0}^{\Theta} p_{\Theta, \ell}^{(\alpha)} f \left( \frac{[\ell]_q}{[\Theta]_q} \right), \quad (22)$$

where  $\alpha \in [0, 1], q \in (0, 1], \varkappa \in [0, 1], g' \in \mathfrak{C}[0, 1]$  and

$$\begin{aligned} p_{1, q, 0}^{(\alpha)}(\varkappa) &= 1 - \varkappa, p_{1, q, 1}^{(\alpha)}(\varkappa) = \varkappa, \\ p_{\Theta, q, \ell}^{(\alpha)}(\varkappa) &= \left( \begin{bmatrix} \Theta - 2 \\ \ell \end{bmatrix}_q (1 - \alpha) \varkappa + \begin{bmatrix} \Theta - 2 \\ \ell - 2 \end{bmatrix}_q \right. \\ &\left. (1 - \alpha) q^{\Theta - \ell - 2} (1 - q^{\Theta - \ell - 1} \varkappa) + \begin{bmatrix} \Theta \\ \ell \end{bmatrix}_q \times \right. \\ &\left. \alpha \varkappa (1 - q^{\Theta - \ell - 1} \varkappa) \right) \varkappa^{\ell-1} (1 - \varkappa)_q^{\Theta - \ell - 1}, \Theta \geq 2 \end{aligned}$$

Subsequently, Cai et al. establish a more general approach to Kantorovich operators known as bivariate  $\alpha, q$ -Bernstein-Kantorovich operators in (Cai et al. 2019) as:

$$\begin{aligned} \kappa_{m_1, m_2, q_1, q_2}^{(\alpha_1, \alpha_2)}(g'; \varkappa; \mathfrak{s}) &= [\Theta_1 + 1]_{q_1} [\Theta_2 + 1]_{q_2} \\ &\times \sum_{\ell_1=0}^{\Theta_1} \sum_{\ell_2=0}^{\Theta_2} p_{\Theta_1, q_1, \ell_1}^{(\alpha_1)}(\varkappa) p_{\Theta_2, q_2, \ell_2}^{(\alpha_2)}(\mathfrak{s}) q_1^{-\ell_1} q_2^{-\ell_2} \\ &\times \int_{\frac{[\ell_1]_{q_1}}{[\Theta_1+1]_{q_1}}}^{\frac{[\ell_1+1]_{q_1}}{[\Theta_1+1]_{q_1}}} \int_{\frac{[\ell_2]_{q_2}}{[\Theta_2+1]_{q_2}}}^{\frac{[\ell_2+1]_{q_2}}{[\Theta_2+1]_{q_2}}} g'(t; u) dq_1 t dq_2 u, \end{aligned}$$

where  $\varkappa, \mathfrak{s} \in [0, 1], g' \in \mathfrak{C}([0, 1] \times [0, 1]), 0 < q_1 < q_2 < 1$  and  $\alpha_1, \alpha_2 \in [0, 1]$ .

## APPLICATIONS

Approximation Theory is rigorous branch of study, developed in different directions by mathematicians. It's centrality in the development of many area of mathematics and its diverse application in Sciences and Engineering field makes it attractive field of study and research. Approximation Theory has two aspects:

- Pragmatic: Which is concerned largely with computational practicalities.
- Theoretical: Which is more concerned with applications to theoretical issues.

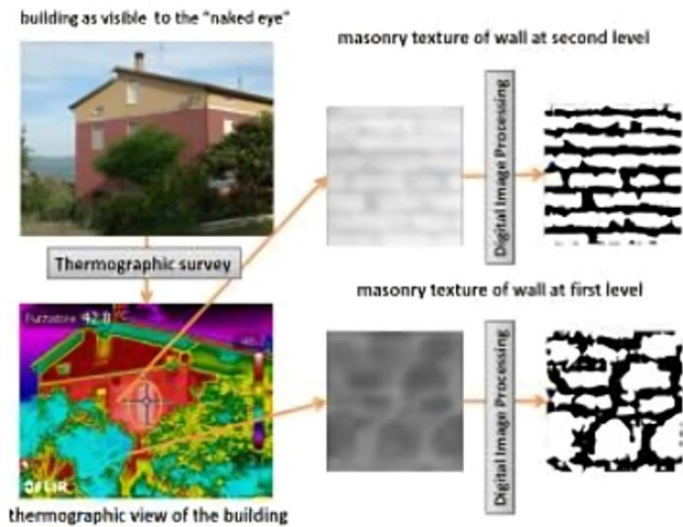
Sampling Kantorovich operators are concerned more with pragmatic aspects. In (Bardaro et al. 2007) the authors introduced the sampling Kantorovich operators and studied their convergence in the general setting of orlicz spaces in the one-dimensional space. Afterward, the results to multivariate setting have been extended in (Costarelli and Vinti 2011), to the nonlinear case in (Costarelli and Vinti 2013; Vinti and Zampogni 2009; Bardaro and Mantellini 2012) in a more general context. Results regarding the order of approximation of these operators are shown in (Costarelli et al. 2014b). In (Cluni et al. 2013) the authors obtain application to civil engineering by using multivariate sampling Kantorovich operators  $(S_{\omega})_{\omega > 0}$ , defined by:

$$(S_{\omega} g')(\varkappa) = \sum_{\ell \in \mathbb{Z}^{\Theta}} \chi(\omega \varkappa - t_{\ell}) \left[ \frac{\omega^{\Theta}}{A_{\ell}} \int_{R_{\ell}^{\omega}} g'(u) du \right], \quad (23)$$

where  $(\varkappa \in \mathbb{R}^{\Theta})$  and  $g' : \mathbb{R}^{\Theta} \rightarrow \mathbb{R}$  is locally integrable function.

For notations and more details about operators, we refer reader to (Cluni et al. 2013). In that paper, authors realized the importance of sampling Kantorovich operators to seismic engineering by demonstrating that structural analysis using sampling Kantorovich

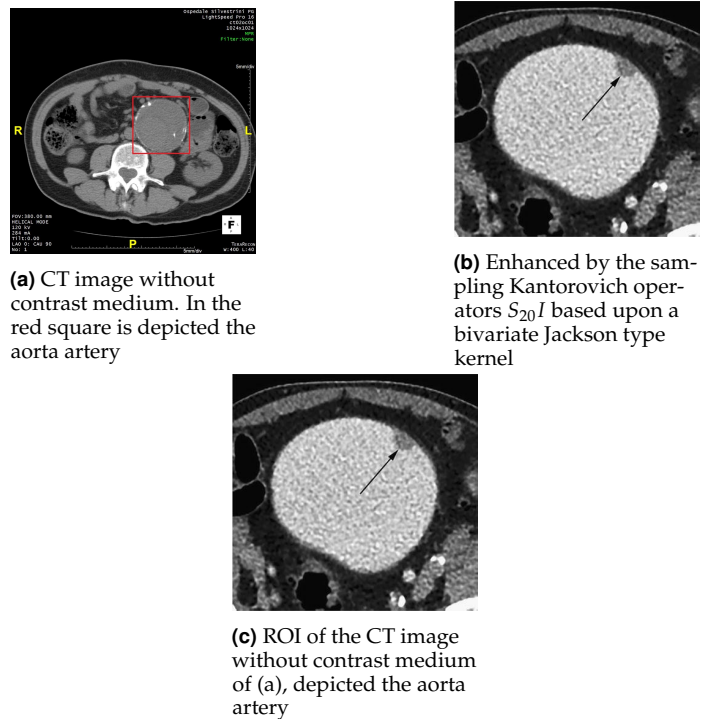
operators produce clear understanding of masonry texture, the geometry of buildings and possible structural damage, which further helps to estimate seismic risk of structure. The models for the reproduction of the behaviour of structures under seismic action and comparison of behaviour of the building using various models are obtained in (Costarelli *et al.* 2014a). In (Cheney 1966) a real world case study in terms of structural analysis is analyzed. For better understanding readers can refer Figure in (Cluni *et al.* 2013)



**Figure 1** Image depicting importance of Sampling Kantorovich operators in structural analysis of building.

Sampling Kantorovich operators are very useful in sampling and signal theories. Cluni *et al.* in (Costarelli *et al.* 2014a) also recognized that the use of sampling Kantorovich operators in approximating discontinuous signals is predominant as it reduces "time-jitter" errors. Angeloni *et al.* in (Angeloni *et al.* 2005) developed that sampling Kantorovich operators represents an approximate version of classical sampling series, based on Whittaker-Kotelnikov-Shannon sampling theorem.

In (Costarelli and Vinti 2014), Costarelli *et al.* presented an application of sampling Kantorovich operators to digital image processing (D.I.P) and also studied the various results regarding the convergence of sampling Kantorovich operators. In that paper, the various usage of the D.I.P technique are discussed from mathematical and medical point of view. Moreover, some new applications are obtained by considering biomedical images. A concrete example is showed in (Costarelli and Vinti 2014) and deduce that enhancement of images using sampling Kantorovich operators are very useful from medical point of view as it allows doctor to perform a better diagnosis. For better understanding one can refer images below (Costarelli *et al.* 2014a)



**Figure 2** Processing a portion of a CT (computer tomography) image depicting the aorta artery using Sampling Kantorovich operators.

In (Karaca *et al.* 2019) Karaca *et al.* main contribution was to provide a unique method for assessing key stroke subtypes' features using a mobile phone connected to a cloud system depicting huge progress in innovative healthcare technologies that rendered healthcare data bigger.

Angeloni *et al.* (Angeloni *et al.* 2020) examined the convergence properties of a family of multidimensional sampling Kantorovich type operators. Besides that, quantitative estimates, order of approximation and Voronovskaja type asymptotic convergence theorem have been established. Very recently, Bawa *et al.* (Bawa *et al.* 2022) elucidate the approximation properties of a Kantorovich-Lupaş-Stancu operators based on Pòlya distribution.

## CONCLUSION

We conclude that approximation theory is an intensive research area that is widely used in sciences and engineering field. In this paper, we have discussed different variants of Bernstein Kantorovich operators showing their pragmatic and theoretical applications in different areas of research. The importance of approximation theory in numerous scientific fields makes it one of the most active research areas. The theory is relevant to both engineering and mathematical fields, such as constructive approximation of functions, partial and integral equation solutions, machine learning and image processing.

### Availability of data and material

Not applicable.

### Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

## Ethical standard

The authors have no relevant financial or non-financial interests to disclose.

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