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Tikhonov Regularization and Best Approximate Inversion Formulas on Weighted Hardy Spaces

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Abstract

In this paper we introduce a weighted Hardy space $\mathscr{H}_\beta$. This space which gives a generalization of some complex Hilbert spaces like, the Dirichlet space $\mathscr{D}$ and the Bergman space $\mathscr{A}$, it plays a background to our contribution. We use the Tikhonov regularization method and determine the extremal functions associated to the difference and primitive operators T_α and L_α on $\mathscr{H}_\beta$. Moreover, we deduce approximation inversion formulas for these operators.

Keywords: *Weighted Hardy space, extremal functions, inversion formulas.*

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1. Introduction

Tikhonov regularization, named for Andrey Tikhonov, is a method of regularization of ill-posed problems. In statistics, the method is also known as ridge regression, it is particularly useful to mitigate the problem of multicollinearity in linear regression, which commonly occurs in models with large numbers of parameters [1]. In general, this method related to the Levenberg-Marquardt algorithm for solving nonlinear least squares problems. Tikhonov regularization has been invented independently in many different contexts. It became widely known from its application to integral equations from the work of Andrey Tikhonov [2, 3, 4] and David L. Phillips [5]. Some authors use the term Tikhonov-Phillips regularization.

Before the applications to the Tiknohov regularization, we shall examine the concept of the Moore-Penrose generalized inverses from the viewpoint of the theory of reproducing kernels. Here, we will be able to realize the natural and powerful method of the theory of reproducing kernels for the best approximation problems that lead to the Moore-Penrose generalized inverses.

Let E be an arbitrary set and let H_K be a reproducing kernel Hilbert space admitting the reproducing kernel K on E. For any Hilbert space H we consider a bounded linear operator T from H_K to H. Then the following problem is a classical and fundamental problem which is known as best approximate mean square norm problems

$$\inf_{f\in\mathscr{H}_K}\left\{\|Tf-h\|_H^2\right\}, \tag{1.1}$$

where $h\in H$ is given. If there exists $F_T^*(h)\in H_K$ which attains this infimum, the problem (1.1) is called solvable otherwise it is called unsolvable. If H_K is a reproducing kernel Hilbert space admitting a reproducing kernel $K(p,q)$ on a the set E then whether the problem (1.1) is solvable or not, the following problem

$$\inf_{f\in\mathscr{H}_K}\left\{\lambda\|f\|_{\mathscr{H}_K}^2+\|Tf-h\|_H^2\right\}, \tag{1.2}$$

is always solvable for all $\lambda>0$ and we obtain a method for determine the extremal function $F_{\lambda,T}^*(h)\in H_K$ which attains the infimum (1.2). The problem (1.2) is called the Tikhonov regularization for the problem (1.1) and if the problem (1.1) is solvable then we have

$$F_{\lambda,T}^*(h)\longrightarrow F_T^*(h)\quad\text{as}\quad\lambda\longrightarrow 0^+,$$

in H_K and $F_T^*(h)$ is the element which attaints the infimum (1.1).

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Let H be a Hilbert space and $T : \mathscr{H}_\beta \to H$ be a bounded linear operator. The main goal of the paper is to find the minimizers for the following extremal problem:

$$\inf_{f\in\mathscr{H}_\beta}\left\{\lambda\|f\|^2_{\mathscr{H}_\beta}+\|Tf-h\|^2_H\right\},$$

where $h \in H$ and $\lambda > 0$. Here $\mathscr{H}_\beta$ are a weighted Hardy spaces of analytic functions. These spaces and some important operators on them were studied in detail by Shields [6] and later by many followers and, for instance, we can and in [7] a standard reference for them. We expect that the results of this paper will be useful when discussing the extremal function associated to the bounded linear operator T. As applications, We come up with some results regarding the weighted Bergman space $\mathscr{A}_\nu$ and the weighted Dirichlet space $\mathscr{D}_\nu$.

Let $\mathbb{D} := \mathbb{D}(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. The Hardy space $\mathscr{H}$ (see [9, 8]) is the set of all analytic functions f in the unit disk $\mathbb{D}$ such that

$$\|f\|^2_{\mathscr{H}} := \frac{1}{2\pi}\int_0^{2\pi}|f(e^{i\theta})|^2 \mathrm{d}\theta < \infty.$$

It is a Hilbert space when equipped with the inner product

$$\langle f,g\rangle_{\mathscr{H}} := \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\mathrm{d}\theta.$$

In this paper, we introduce the weighted Hardy space $\mathscr{H}_\beta$, which is the set of all analytic functions f in the disk $\mathbb{D}$, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, such that

$$\|f\|^2_{\mathscr{H}_\beta} := \sum_{n=0}^{\infty}\beta_n|a_n|^2 < \infty,$$

where $\beta = \{\beta_n\}$ is a positive sequence so that

$$\limsup_{n\to\infty}(\beta_n)^{-1/n} = 1.$$

It is a reproducing kernel Hilbert space that gives a generalization of some complex Hilbert spaces like, the Dirichlet space $\mathscr{D}$ (see [10, 11, 12, 13, 14]) when $\beta_n = n$, and the Bergman space $\mathscr{A}$ (see [16, 15]) when $\beta_n = \frac{1}{n+1}$.

This space is the background of some applications. Especially, by using the theory of extremal functions and reproducing kernels [17, 18, 19], we examine the extremal function associated to the operator T_α defined for $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, by

$$T_\alpha f(z) := \sum_{n=0}^{\infty}\alpha_n a_{n+1} z^n,$$

where $\alpha = \{\alpha_n\}$ is a sequence satisfying, there exists $c > 0$ such that

$$|\alpha_n| \le c\sqrt{\frac{\beta_{n+1}}{\beta_n}}, \quad n \in \mathbb{N}.$$

More precisely, for any $h \in \mathscr{H}_\beta$ and for any $\lambda > 0$, the infimum

$$\inf_{f\in\mathscr{H}_\beta}\left\{\lambda\|f\|^2_{\mathscr{H}_\beta}+\|T_\alpha f-h\|^2_{\mathscr{H}_\beta}\right\},$$

is attained at one function $F^*_{\lambda,T_\alpha}(h)$ called the extremal function, and given by

$$F^*_{\lambda,T_\alpha}(h)(z) = \sum_{n=0}^{\infty}\frac{\alpha_n\beta_n h_n}{\lambda\beta_{n+1}+\beta_n(\alpha_n)^2}z^{n+1}, \quad z \in \mathbb{D}.$$

We establish also the extremal function associated to the operator L_α defined for $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, by

$$L_\alpha f(z) := \sum_{n=1}^{\infty}\alpha_n\frac{a_{n-1}}{n}z^n,$$

where $\alpha = \{\alpha_n\}$ is a sequence satisfying, there exists $c > 0$ such that

$$|\alpha_n| \le cn\sqrt{\frac{\beta_{n-1}}{\beta_n}}, \quad n \ge 1.$$

Moreover, we establish best approximate inversion formulas for the operators T_α and L_α, on $\mathscr{H}_\beta$. Finally, we come up with some results regarding the extremal functions associated to the difference operator $\mathscr{D}f(z) := \frac{1}{z}(f(z)-f(0))$, and the primitive operator $\mathscr{P}f(z) := \int_{[0,z]} f(w)\mathrm{d}w$, for the weighted Bergman space $\mathscr{A}_\nu$ and the weighted Dirichlet space $\mathscr{D}_\nu$, respectively.

The paper is organized as follows. In Section 2 we introduce the weighted Hardy space $\mathscr{H}_\beta$, and we prove that the space $\mathscr{H}_\beta$ is a reproducing kernel Hilbert space (RKHS). In Sections 3 and 4, we examine the extremal functions for operators T_α and L_α on $\mathscr{H}_\beta$; and we establish best approximate inversion formulas for them. Next, in Section 5, we give some examples of extremal functions for the difference operator and the primitive operator, on the classical Hardy space $\mathscr{H}$, on the weighted Bergman space $\mathscr{A}_\nu$, and on the weighted Dirichlet space $\mathscr{D}_\nu$, repectively. Finally, in Section 6 we give a conclusion and perspective.

2. Tikhonov regularization problem on $\mathscr{H}_\beta$

Let $\mathbb{D}$ be the disk with center 0 and radius 1. We consider a sequence $\beta = \{\beta_n\}$, with $\beta_n > 0$, such that

$$\limsup_{n\to\infty}(\beta_n)^{-1/n} = 1.$$

The weighted Hardy space $\mathscr{H}_\beta$ is the set of all analytic functions f in the disk $\mathbb{D}$, with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, such that

$$\|f\|^2_{\mathscr{H}_\beta} := \sum_{n=0}^{\infty} \beta_n |a_n|^2 < \infty. \tag{2.1}$$

It is a pre-Hilbert space when equipped with the inner product

$$\langle f, g\rangle_{\mathscr{H}_\beta} = \sum_{n=0}^{\infty} \beta_n a_n \overline{b_n},$$

where $f, g \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

We denote by K_β the kernel in $\mathscr{H}_\beta$ defined by

$$K_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\beta_n}, \quad z \in \mathbb{D}.$$

Lemma 2.1. *If $f \in \mathscr{H}_\beta$, then*

$$|f(z)| \le (K_\beta(|z|^2))^{1/2} \|f\|_{\mathscr{H}_\beta}, \quad z \in \mathbb{D}.$$

Proof. Let $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. From the Cauchy-Schwarz inequality, we have

$$|f(z)| \le \left[\sum_{n=0}^{\infty} \frac{|z|^{2n}}{\beta_n}\right]^{1/2} \left[\sum_{n=0}^{\infty} \beta_n |a_n|^2\right]^{1/2} = (K_\beta(|z|^2))^{1/2} \|f\|_{\mathscr{H}_\beta},$$

which gives the result. □

Theorem 2.2. *The weighted Hardy space $\mathscr{H}_\beta$ is a Hilbert space; and the set $\left\{\frac{z^n}{\sqrt{\beta_n}}\right\}_{n=0}^{\infty}$ forms a Hilbertian basis for this space.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{H}_\beta$. From Lemma 2.1, we have

$$|f_{n+p}(z) - f_n(z)| \le (K_\beta(|z|^2))^{1/2} \|f_{n+p} - f_n\|_{\mathscr{H}_\beta}.$$

This inequality shows that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is pointwise convergent to f. Since the function $z \to (K_\beta(|z|^2))^{1/2}$ is continuous on $\mathbb{D}$, then $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly on $\mathbb{D}$. Consequently, by Weierstrass uniform convergence Theorem [20] we deduce that f is analytic in $\mathbb{D}$. On the other hand, there exists $n_0 \in \mathbb{N}$, such that for all $n \ge n_0$ we have $\|f_n - f\|_{\mathscr{H}_\beta} \le 1$. And this we deduce that

$$\|f\|_{\mathscr{H}_\beta} \le \|f_{n_0} - f\|_{\mathscr{H}_\beta} + \|f_{n_0}\|_{\mathscr{H}_\beta} \le 1 + \|f_{n_0}\|_{\mathscr{H}_\beta} < \infty.$$

Hence $f \in \mathscr{H}_\beta$. □

From Lemma 2.1, the map $f \to f(z)$, $z \in \mathbb{D}$, is a continuous linear functional on $\mathscr{H}_\beta$. Thus from Riesz theorem [21], $\mathscr{H}_\beta$ has a reproducing kernel (see [7], Theorem 2.10).

Theorem 2.3. *The kernel k_z given for $w, z \in \mathbb{D}$, by*

$$k_z(w) = K_\beta(\bar{z}w) = \sum_{n=0}^{\infty} \frac{(\bar{z}w)^n}{\beta_n}, \tag{2.2}$$

is a reproducing kernel for the weighted Hardy space $\mathscr{H}_\beta$, meaning that $k_z \in \mathscr{H}_\beta$, and for all $f \in \mathscr{H}_\beta$, we have $\langle f, k_z\rangle_{\mathscr{H}_\beta} = f(z)$.

Proof. It is easy to see that k_z is analytic in $\mathbb{D}$. And from (2.1) and (2.2) we deduce that

$$\|k_z\|^2_{\mathscr{H}_\beta} = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\beta_n} = K_\beta(|z|^2) < \infty.$$

On the other hand, if $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\langle f, k_z\rangle_{\mathscr{H}_\beta} = \sum_{n=0}^{\infty} a_n z^n = f(z), \quad z \in \mathbb{D},$$

which completes the proof. □

If $\beta_n = 1$ the corresponding weighted Hardy space is the Hardy space $\mathscr{H}$ (see [21, 22, 23]), which is the set of all analytic functions f in the unit disk $\mathbb{D}$ such that

$$\|f\|^2_{\mathscr{H}} := \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 \mathrm{d}\theta < \infty.$$

It is a Hilbert space when equipped with the inner product

$$\langle f,g\rangle_{\mathscr{H}} := \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\mathrm{d}\theta.$$

The Szegö kernel k_z is given for $w,z \in \mathbb{D}$, by

$$k_z(w) = \sum_{n=0}^{\infty} (\bar{z}w)^n = \frac{1}{1-\bar{z}w}.$$

If $\beta_0 = 1$ and $\beta_n = n$, $n \geq 1$, the corresponding weighted Hardy space is the Dirichlet space $\mathscr{D}$ (see [11, 12, 13, 14]), which is the space of square-integrable analytic functions on $\mathbb{D}$, with inner product,

$$\langle f,g\rangle_{\mathscr{D}} := f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\frac{\mathrm{d}x\mathrm{d}y}{\pi}, \quad z = x+iy.$$

This Hilbert space [10, 21] has the reproducing kernel

$$k_z(w) = 1 + \sum_{n=1}^{\infty} \frac{(\bar{z}w)^n}{n} = 1 + \log\left(\frac{1}{1-\bar{z}w}\right), \quad z,w \in \mathbb{D}.$$

If $\beta_n = \frac{1}{n+1}$ the corresponding weighted Hardy space is the Bergman space $\mathscr{A}$ (see [16, 15]), which is the space of square-integrable analytic functions on $\mathbb{D}$, with inner product,

$$\langle f,g\rangle_{\mathscr{A}} := \int_{\mathbb{D}} f(z)\overline{g(z)}\frac{\mathrm{d}x\mathrm{d}y}{\pi}, \quad z = x+iy.$$

This Hilbert space [21] has the reproducing kernel

$$k_z(w) = \sum_{n=0}^{\infty} (n+1)(\bar{z}w)^n = \frac{1}{(1-\bar{z}w)^2}, \quad z,w \in \mathbb{D}.$$

Let H be a Hilbert space, and let $T : \mathscr{H}_\beta \to H$ be a bounded linear operator from $\mathscr{H}_\beta$ into H. For any $h \in H$ and for any $\lambda > 0$, the Tikhonov regularization problem associated to the operator T is given by

$$\inf_{f\in\mathscr{H}_\beta}\left\{\lambda\|f\|^2_{\mathscr{H}_\beta} + \|Tf-h\|^2_H\right\}. \tag{2.3}$$

Building on the ideas of Saitoh [17, 18, 19] we examine the extremal function associated to the Tikhonov regularization problem (2.3).

Theorem 2.4. *For any $h \in H$ and for any $\lambda > 0$, the Tikhonov regularization problem* (2.3) *has a unique minimizer given by*

$$F^*_{\lambda,T}(h) = (\lambda I + T^*T)^{-1}T^*h. \tag{2.4}$$

Proof. Let $\lambda > 0$. We denote by $\langle .,.\rangle_{\lambda,\mathscr{H}_\beta}$ the inner product defined on the space $\mathscr{H}_\beta$ by

$$\langle f,g\rangle_{\lambda,\mathscr{H}_\beta} := \lambda\langle f,g\rangle_{\mathscr{H}_\beta} + \langle Tf,Tg\rangle_H.$$

The two norms $\|.\|_{\mathscr{H}_\beta}$ and $\|.\|_{\lambda,\mathscr{H}_\beta}$ are equivalent. In particular, from Lemma 2.1, we have

$$|f(z)| \leq \left(\frac{K_\beta(|z|^2)}{\lambda}\right)^{1/2}\|f\|_{\lambda,\mathscr{H}_\beta}, \quad f \in \mathscr{H}_\beta, z \in \mathbb{D}.$$

Then the space $\mathscr{H}_\beta$, equipped with the norm $\|.\|_{\lambda,\mathscr{H}_\beta}$ has a reproducing kernel $k_{\lambda,z}$. Therefore, we have the functional equation

$$(\lambda I + T^*T)k_{\lambda,z} = k_z, \quad z \in \mathbb{D}, \tag{2.5}$$

where I is the unit operator and $T^* : H \longrightarrow \mathscr{H}_\beta$ is the adjoint of T. Therefore and from Saitoh ([17], Theorem 2.5, Section 2), we have

$$F^*_{\lambda,T}(h)(z) = \langle h, Tk_{\lambda,z}\rangle_H.$$

On the other hand by (2.5) we deduce that

$$F^*_{\lambda,T}(h)(z) = \langle T^*h, k_{\lambda,z}\rangle_{\mathscr{H}_\beta} = \langle T^*h, (\lambda I + T^*T)^{-1}k_z\rangle_{\mathscr{H}_\beta} = \langle(\lambda I + T^*T)^{-1}T^*h, k_z\rangle_{\mathscr{H}_\beta}.$$

Hence

$$F^*_{\lambda,T}(h)(z) = (\lambda I + T^*T)^{-1}T^*h(z),$$

which gives the desired result. □

The next Sections of this paper are devoted to the Tikhonov regularization problem (2.3) and its application to the theory of reproducing kernels, in some particular cases.

3. Tikhonov's extremal function for T_α

Let T_α be the operator defined for $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, by

$$T_\alpha f(z) := \sum_{n=0}^{\infty} \alpha_n a_{n+1} z^n, \tag{3.1}$$

where $\alpha = \{\alpha_n\}$ is a real sequence satisfying, there exists $c > 0$ such that

$$|\alpha_n| \leq c\sqrt{\frac{\beta_{n+1}}{\beta_n}}, \quad n \in \mathbb{N}. \tag{3.2}$$

In this section we solve the Tikhonov-Phillips regularization problem associated to T_α. We establish the estimate properties of its minimizer function $F^*_{\lambda,T_\alpha}(h)(z)$. And we deduce approximate inversion formulas for the operator T_α. These formula are the analogous of Calderón's reproducing formula for the Fourier-type transforms [24, 25, 26]. A pointwise approximate inversion formulas T_α are also discussed.

Lemma 3.1. *We have*
(i) The operator T_α maps continuously from $\mathscr{H}_\beta$ into $\mathscr{H}_\beta$, and $\|T_\alpha f\|_{\mathscr{H}_\beta} \leq c\|f\|_{\mathscr{H}_\beta}$.
(ii) For $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have

$$T^*_\alpha f(z) = \sum_{n=1}^{\infty} \frac{\alpha_{n-1}\beta_{n-1}}{\beta_n} a_{n-1} z^n,$$

$$T^*_\alpha T_\alpha f(z) = \sum_{n=1}^{\infty} \frac{(\alpha_{n-1})^2\beta_{n-1}}{\beta_n} a_n z^n.$$

(iii) For any $h \in \mathscr{H}_\beta$ and for any $\lambda > 0$, the problem

$$\inf_{f \in \mathscr{H}_\beta} \left\{ \lambda\|f\|^2_{\mathscr{H}_\beta} + \|T_\alpha f - h\|^2_{\mathscr{H}_\beta} \right\}$$

has a unique minimizer given by

$$F^*_{\lambda,T_\alpha}(h)(z) = \langle h, \Psi_z \rangle_{\mathscr{H}_\beta},$$

where

$$\Psi_z(w) = \sum_{n=0}^{\infty} \frac{\alpha_n (\bar{z})^{n+1}}{\lambda\beta_{n+1} + \beta_n(\alpha_n)^2} w^n, \quad w \in \mathbb{D}.$$

Proof. (i) If $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\|T_\alpha f\|^2_{\mathscr{H}_\beta} = \sum_{n=0}^{\infty} \beta_n |\alpha_n|^2 |a_{n+1}|^2 \leq c^2 \sum_{n=1}^{\infty} \beta_n |a_n|^2 \leq c^2 \|f\|^2_{\mathscr{H}_\beta}.$$

(ii) If $f, g \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle T_\alpha f, g \rangle_{\mathscr{H}_\beta} = \sum_{n=0}^{\infty} \alpha_n \beta_n a_{n+1} \bar{b}_n = \sum_{n=1}^{\infty} \alpha_{n-1}\beta_{n-1} a_n \bar{b}_{n-1} = \langle f, T^*_\alpha g \rangle_{\mathscr{H}_\beta},$$

where

$$T^*_\alpha g(z) = \sum_{n=1}^{\infty} \frac{\alpha_{n-1}\beta_{n-1}}{\beta_n} b_{n-1} z^n.$$

And therefore

$$T^*_\alpha T_\alpha f(z) = \sum_{n=1}^{\infty} \frac{(\alpha_{n-1})^2\beta_{n-1}}{\beta_n} a_n z^n.$$

(iii) We put $h(z) = \sum_{n=0}^{\infty} h_n z^n$ and $F^*_{\lambda,T_\alpha}(h)(z) = \sum_{n=0}^{\infty} c_n z^n$. From (2.4) we have $(\lambda I + T^*_\alpha T_\alpha) F^*_{\lambda,T_\alpha}(h)(z) = T^*_\alpha h(z)$. By (ii) we deduce that

$$c_0 = 0, \quad c_n = \frac{\alpha_{n-1}\beta_{n-1}h_{n-1}}{\lambda\beta_n + \beta_{n-1}(\alpha_{n-1})^2}, \quad n \geq 1.$$

Thus

$$F^*_{\lambda,T_\alpha}(h)(z) = \sum_{n=0}^{\infty} \frac{\alpha_n \beta_n h_n}{\lambda\beta_{n+1} + \beta_n(\alpha_n)^2} z^{n+1} = \langle h, \Psi_z \rangle_{\mathscr{H}_\beta}, \tag{3.3}$$

where

$$\Psi_z(w) = \sum_{n=0}^{\infty} \frac{\alpha_n (\bar{z})^{n+1}}{\lambda\beta_{n+1} + \beta_n(\alpha_n)^2} w^n.$$

The lemma is proved. □

The extremal function $F^*_{\lambda,T_\alpha}(h)$ given by (3.3) satisfies the following properties.

Lemma 3.2. *If $\lambda > 0$ and $f,h \in \mathscr{H}_\beta$, then*

(i) $|F^*_{\lambda,T_\alpha}(h)(z)| \le \frac{1}{2\sqrt{\lambda}}(K_\beta(|z|^2))^{1/2}\|h\|_{\mathscr{H}_\beta}$,

(ii) $|T_\alpha F^*_{\lambda,T_\alpha}(h)(z)| \le \frac{c}{2\sqrt{\lambda}}(K_\beta(|z|^2))^{1/2}\|h\|_{\mathscr{H}_\beta}$,

(iii) $|F^*_{\lambda,T_\alpha}(T_\alpha f)(z)| \le \frac{c}{2\sqrt{\lambda}}(K_\beta(|z|^2))^{1/2}\|f\|_{\mathscr{H}_\beta}$,

(iv) $\|F^*_{\lambda,T_\alpha}(h)\|^2_{\mathscr{H}_\beta} \le \frac{1}{4\lambda}\|h\|^2_{\mathscr{H}_\beta}$.

Proof. Let $\lambda > 0$ and $h \in \mathscr{H}_\beta$ with $h(z) = \sum_{n=0}^\infty h_n z^n$. From (3.3) we have

$|F^*_{\lambda,T_\alpha}(h)(z)| \le \|\Psi_z\|_{\mathscr{H}_\beta}\|h\|_{\mathscr{H}_\beta}$.

And by using the fact that $(x+y)^2 \ge 4xy$ we obtain

$$\|\Psi_z\|^2_{\mathscr{H}_\beta} = \sum_{n=0}^\infty \beta_n\left[\frac{\alpha_n|z|^{n+1}}{\lambda\beta_{n+1}+\beta_n(\alpha_n)^2}\right]^2 \le \frac{1}{4\lambda}\sum_{n=0}^\infty \frac{|z|^{2(n+1)}}{\beta_{n+1}} \le \frac{1}{4\lambda}K_\beta(|z|^2).$$

This gives (i). On the other hand, from (3.1) and (3.3) we have

$$T_\alpha F^*_{\lambda,T_\alpha}(h)(z) = \sum_{n=0}^\infty \frac{(\alpha_n)^2\beta_n h_n}{\lambda\beta_{n+1}+\beta_n(\alpha_n)^2}z^n = \langle h,\Phi_z\rangle_{\mathscr{H}_\beta}, \tag{3.4}$$

where

$$\Phi_z(w) = \sum_{n=0}^\infty \frac{(\alpha_n)^2(\bar{z})^n}{\lambda\beta_{n+1}+\beta_n(\alpha_n)^2}w^n.$$

And by (3.2) we deduce that

$$\|\Phi_z\|^2_{\mathscr{H}_\beta} = \sum_{n=0}^\infty \beta_n\left[\frac{(\alpha_n)^2|z|^n}{\lambda\beta_{n+1}+\beta_n(\alpha_n)^2}\right]^2 \le \frac{1}{4\lambda}\sum_{n=0}^\infty \frac{(\alpha_n)^2}{\beta_{n+1}}|z|^{2n} \le \frac{c^2}{4\lambda}K_\beta(|z|^2).$$

This gives (ii). Furthermore, if $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^\infty a_n z^n$ we have

$$F^*_{\lambda,T_\alpha}(T_\alpha f)(z) = \sum_{n=1}^\infty \frac{(\alpha_{n-1})^2\beta_{n-1}a_n}{\lambda\beta_n+\beta_{n-1}(\alpha_{n-1})^2}z^n = \langle f,\Omega_z\rangle_{\mathscr{H}_\beta}, \tag{3.5}$$

where

$$\Omega_z(w) = \sum_{n=1}^\infty \frac{(\alpha_{n-1})^2\beta_{n-1}(\bar{z})^n}{\lambda\beta_n+\beta_{n-1}(\alpha_{n-1})^2}\frac{w^n}{\beta_n}.$$

And by (3.2) we conclude that

$$\|\Omega_z\|^2_{\mathscr{H}_\beta} = \sum_{n=1}^\infty \frac{1}{\beta_n}\left[\frac{(\alpha_{n-1})^2\beta_{n-1}|z|^n}{\lambda\beta_n+\beta_{n-1}(\alpha_{n-1})^2}\right]^2 \le \frac{1}{4\lambda}\sum_{n=1}^\infty \frac{(\alpha_{n-1})^2\beta_{n-1}}{(\beta_n)^2}|z|^{2n} \le \frac{c^2}{4\lambda}K_\beta(|z|^2).$$

This gives (iii). Finally, from (3.3) we have

$$\|F^*_{\lambda,T_\alpha}(h)\|^2_{\mathscr{H}_\beta} = \sum_{n=1}^\infty \beta_n\left[\frac{\alpha_{n-1}\beta_{n-1}|h_{n-1}|}{\lambda\beta_n+\beta_{n-1}(\alpha_{n-1})^2}\right]^2.$$

Using the fact that $(x+y)^2 \ge 4xy$ we obtain

$$\|F^*_{\lambda,T_\alpha}(h)\|^2_{\mathscr{H}_\beta} \le \frac{1}{4\lambda}\sum_{n=1}^\infty \beta_{n-1}|h_{n-1}|^2 = \frac{1}{4\lambda}\|h\|^2_{\mathscr{H}_\beta},$$

which gives (iv) and completes the proof of the lemma. □

We establish approximate inversion formulas for the operator T_α.

Theorem 3.3. *If $\lambda > 0$ and $f,h \in \mathscr{H}_\beta$, then*

(i) $\lim_{\lambda\to 0^+}\|T_\alpha F^*_{\lambda,T_\alpha}(h) - h\|^2_{\mathscr{H}_\beta} = 0$,

(ii) $\lim_{\lambda\to 0^+}\|F^*_{\lambda,T_\alpha}(T_\alpha f) - f_0\|^2_{\mathscr{H}_\beta} = 0$, *where* $f_0(z) = f(z) - f(0)$.

Proof. Let $\lambda > 0$ and $h \in \mathscr{H}_\beta$ with $h(z) = \sum_{n=0}^{\infty} h_n z^n$. From (3.4) we have

$$T_\alpha F^*_{\lambda,T_\alpha}(h)(z) - h(z) = \sum_{n=0}^{\infty} \frac{-\lambda\beta_{n+1}h_n}{\lambda\beta_{n+1}+\beta_n(\alpha_n)^2} z^n. \tag{3.6}$$

Therefore

$$\|T_\alpha F^*_{\lambda,T_\alpha}(h) - h\|^2_{\mathscr{H}_\beta} = \sum_{n=0}^{\infty} \beta_n \left[\frac{\lambda\beta_{n+1}|h_n|}{\lambda\beta_{n+1}+\beta_n(\alpha_n)^2}\right]^2.$$

Again, by dominated convergence theorem and the fact that

$$\beta_n \left[\frac{\lambda\beta_{n+1}|h_n|}{\lambda\beta_{n+1}+\beta_n(\alpha_n)^2}\right]^2 \le \beta_n|h_n|^2,$$

we deduce (i). Finally, let $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then from (3.5) we have

$$F^*_{\lambda,T_\alpha}(T_\alpha f)(z) - f_0(z) = \sum_{n=1}^{\infty} \frac{-\lambda\beta_n a_n}{\lambda\beta_n+\beta_{n-1}(\alpha_{n-1})^2} z^n. \tag{3.7}$$

So, one has

$$\|F^*_{\lambda,T_\alpha}(T_\alpha f) - f_0\|^2_{\mathscr{H}_\beta} = \sum_{n=1}^{\infty} \beta_n \left[\frac{\lambda\beta_n|a_n|}{\lambda\beta_n+\beta_{n-1}(\alpha_{n-1})^2}\right]^2.$$

Using the dominated convergence theorem and the fact that

$$\beta_n \left[\frac{\lambda\beta_n|a_n|}{\lambda\beta_n+\beta_{n-1}(\alpha_{n-1})^2}\right]^2 \le \beta_n|a_n|^2,$$

we deduce (ii). □

We deduce also pointwise approximate inversion formulas for T_α.

Theorem 3.4. *If $\lambda > 0$ and $f,h \in \mathscr{H}_\beta$, then*

(i) $\lim_{\lambda\to 0^+} T_\alpha F^*_{\lambda,T_\alpha}(h)(z) = h(z)$,

(ii) $\lim_{\lambda\to 0^+} F^*_{\lambda,T_\alpha}(T_\alpha f)(z) = f_0(z)$.

Proof. Let $f,h \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $h(z) = \sum_{n=0}^{\infty} h_n z^n$. From (3.6) and (3.7), by using the dominated convergence theorem and the fact that

$$\frac{\lambda\beta_{n+1}|h_n|}{\lambda\beta_{n+1}+\beta_n(\alpha_n)^2}|z|^n \le |h_n||z|^n, \quad \frac{\lambda\beta_n|a_n|}{\lambda\beta_n+\beta_{n-1}(\alpha_{n-1})^2}|z|^n \le |a_n||z|^n,$$

we obtain (i) and (ii). □

4. Tikhonov's extremal function for L_α

Let L_α be the operator defined for $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, by

$$L_\alpha f(z) := \sum_{n=1}^{\infty} \alpha_n \frac{a_{n-1}}{n} z^n, \tag{4.1}$$

where $\alpha = \{\alpha_n\}$ is a real sequence satisfying, there exists $c > 0$ such that

$$|\alpha_n| \le cn\sqrt{\frac{\beta_{n-1}}{\beta_n}}, \quad n \ge 1. \tag{4.2}$$

For the operator L_α we establish the same applications given in Section 3.

Lemma 4.1. *We have*
(i) The operator L_α maps continuously from $\mathscr{H}_\beta$ into $\mathscr{H}_\beta$, and $\|L_\alpha f\|_{\mathscr{H}_\beta} \le c\|f\|_{\mathscr{H}_\beta}$.
(ii) For $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have

$$L^*_\alpha f(z) = \sum_{n=0}^{\infty} \frac{\alpha_{n+1}\beta_{n+1}}{(n+1)\beta_n} a_{n+1} z^n,$$

$$L^*_\alpha L_\alpha f(z) = \sum_{n=0}^{\infty} \frac{(\alpha_{n+1})^2\beta_{n+1}}{(n+1)^2\beta_n} a_n z^n.$$

(iii) For any $h \in \mathscr{H}_\beta$ and for any $\lambda > 0$, the problem

$$\inf_{f\in\mathscr{H}_\beta}\left\{\lambda\|f\|^2_{\mathscr{H}_\beta} + \|L_\alpha f - h\|^2_{\mathscr{H}_\beta}\right\}$$

has a unique minimizer given by

$$F^*_{\lambda,L_\alpha}(h)(z) = \langle h, \Psi_z\rangle_{\mathscr{H}_\beta},$$

where

$$\Psi_z(w) = \sum_{n=1}^{\infty} \frac{n\alpha_n(\bar{z})^{n-1}}{\lambda n^2\beta_{n-1} + \beta_n(\alpha_n)^2} w^n, \quad w \in \mathbb{D}.$$

Proof. (i) If $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\|L_\alpha f\|^2_{\mathscr{H}_\beta} = \sum_{n=1}^{\infty} \beta_n \frac{|\alpha_n|^2}{n^2}|a_{n-1}|^2 \le c^2 \sum_{n=0}^{\infty} \beta_n |a_n|^2 = c^2\|f\|^2_{\mathscr{H}_\beta}.$$

(ii) If $f, g \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then

$$\langle L_\alpha f, g\rangle_{\mathscr{H}_\beta} = \sum_{n=1}^{\infty} \alpha_n\beta_n \frac{a_{n-1}}{n}\bar{b}_n = \sum_{n=0}^{\infty} \alpha_{n+1}\beta_{n+1}\frac{a_n}{n+1}\bar{b}_{n+1} = \langle f, L^*_\alpha g\rangle_{\mathscr{H}_\beta},$$

where

$$L^*_\alpha g(z) = \sum_{n=0}^{\infty} \frac{\alpha_{n+1}\beta_{n+1}}{(n+1)\beta_n} b_{n+1} z^n.$$

And therefore

$$L^*_\alpha L_\alpha f(z) = \sum_{n=0}^{\infty} \frac{(\alpha_{n+1})^2\beta_{n+1}}{(n+1)^2\beta_n} a_n z^n.$$

(iii) We put $h(z) = \sum_{n=0}^{\infty} h_n z^n$ and $F^*_{\lambda,L_\alpha}(h)(z) = \sum_{n=0}^{\infty} c_n z^n$. From (2.4) we have $(\lambda I + L^*_\alpha L_\alpha)F^*_{\lambda,L_\alpha}(h)(z) = L^*_\alpha h(z)$. By (ii) we deduce that

$$c_n = \frac{(n+1)\alpha_{n+1}\beta_{n+1}h_{n+1}}{\lambda(n+1)^2\beta_n + \beta_{n+1}(\alpha_{n+1})^2}, \quad n \in \mathbb{N}.$$

Thus

$$F^*_{\lambda,L_\alpha}(h)(z) = \sum_{n=1}^{\infty} \frac{n\alpha_n\beta_n h_n}{\lambda n^2\beta_{n-1} + \beta_n(\alpha_n)^2} z^{n-1} = \langle h, \Psi_z\rangle_{\mathscr{H}_\beta}, \tag{4.3}$$

where

$$\Psi_z(w) = \sum_{n=1}^{\infty} \frac{n\alpha_n(\bar{z})^{n-1}}{\lambda n^2\beta_{n-1} + \beta_n(\alpha_n)^2} w^n.$$

The lemma is proved. □

The extremal function $F^*_{\lambda,L_\alpha}(h)$ given by (4.3) satisfies the following properties.

Lemma 4.2. *If $\lambda > 0$ and $f, h \in \mathscr{H}_\beta$, then*

(i) $|F^*_{\lambda,L_\alpha}(h)(z)| \le \frac{1}{2\sqrt{\lambda}}(K_\beta(|z|^2))^{1/2}\|h\|_{\mathscr{H}_\beta}$,

(ii) $|L_\alpha F^*_{\lambda,L_\alpha}(h)(z)| \le \frac{c}{2\sqrt{\lambda}}(K_\beta(|z|^2))^{1/2}\|h\|_{\mathscr{H}_\beta}$,

(iii) $|F^*_{\lambda,L_\alpha}(L_\alpha f)(z)| \le \frac{c}{2\sqrt{\lambda}}(K_\beta(|z|^2))^{1/2}\|f\|_{\mathscr{H}_\beta}$,

(iv) $\|F^*_{\lambda,L_\alpha}(h)\|^2_{\mathscr{H}_\beta} \le \frac{1}{4\lambda}\|h\|^2_{\mathscr{H}_\beta}$.

Proof. Let $\lambda > 0$ and $h \in \mathscr{H}_\beta$ with $h(z) = \sum_{n=0}^{\infty} h_n z^n$. From (4.3) we have

$$|F^*_{\lambda,L_\alpha}(h)(z)| \le \|\Psi_z\|_{\mathscr{H}_\beta}\|h\|_{\mathscr{H}_\beta}.$$

And by using the fact that $(x+y)^2 \ge 4xy$ we obtain

$$\|\Psi_z\|^2_{\mathscr{H}_\beta} = \sum_{n=1}^{\infty} \beta_n \left[\frac{n\alpha_n|z|^{n-1}}{\lambda n^2\beta_{n-1} + \beta_n(\alpha_n)^2}\right]^2 \le \frac{1}{4\lambda}\sum_{n=0}^{\infty}\frac{|z|^{2n}}{\beta_n} = \frac{1}{4\lambda}K_\beta(|z|^2).$$

This gives (i). On the other hand, from (4.1) and (4.3) we have

$$L_\alpha F^*_{\lambda,L_\alpha}(h)(z) = \sum_{n=1}^{\infty} \frac{(\alpha_n)^2\beta_n h_n}{\lambda n^2\beta_{n-1} + n\beta_n(\alpha_n)^2} z^n = \langle h, \Phi_z\rangle_{\mathscr{H}_\beta}, \tag{4.4}$$

where

$$\Phi_z(w) = \sum_{n=1}^{\infty} \frac{(\alpha_n)^2(\bar{z})^n}{\lambda n^2\beta_{n-1} + n\beta_n(\alpha_n)^2} w^n.$$

And by (4.2) we deduce that

$$\|\Phi_z\|^2_{\mathscr{H}_\beta} = \sum_{n=1}^{\infty} \beta_n \left[\frac{(\alpha_n)^2|z|^n}{\lambda n^2\beta_{n-1} + n\beta_n(\alpha_n)^2}\right]^2 \le \frac{1}{4\lambda}\sum_{n=1}^{\infty} \frac{(\alpha_n)^2}{n^3\beta_{n-1}}|z|^{2n} \le \frac{c^2}{4\lambda} K_\beta(|z|^2).$$

This gives (ii). Furthermore, if $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have

$$F^*_{\lambda,L_\alpha}(L_\alpha f)(z) = \sum_{n=0}^{\infty} \frac{(\alpha_{n+1})^2\beta_{n+1} a_n}{\lambda(n+1)^2\beta_n + \beta_{n+1}(\alpha_{n+1})^2} z^n = \langle f, \Omega_z\rangle_{\mathscr{H}_\beta}, \tag{4.5}$$

where

$$\Omega_z(w) = \sum_{n=0}^{\infty} \frac{(\alpha_{n+1})^2\beta_{n+1}(\bar{z})^n}{\lambda(n+1)^2\beta_n + \beta_{n+1}(\alpha_{n+1})^2} \frac{w^n}{\beta_n}.$$

And by (4.2) we conclude that

$$\|\Omega_z\|^2_{\mathscr{H}_\beta} = \sum_{n=0}^{\infty} \frac{1}{\beta_n} \left[\frac{(\alpha_{n+1})^2\beta_{n+1}|z|^n}{\lambda(n+1)^2\beta_n + \beta_{n+1}(\alpha_{n+1})^2}\right]^2 \le \frac{1}{4\lambda}\sum_{n=0}^{\infty} \frac{(\alpha_{n+1})^2\beta_{n+1}}{(n+1)^2(\beta_n)^2}|z|^{2n} \le \frac{c^2}{4\lambda} K_\beta(|z|^2).$$

This gives (iii). Finally, from (4.3) we have

$$\|F^*_{\lambda,L_\alpha}(h)\|^2_{\mathscr{H}_\beta} = \sum_{n=0}^{\infty} \beta_n \left[\frac{(n+1)\alpha_{n+1}\beta_{n+1}|h_{n+1}|}{\lambda(n+1)^2\beta_n + \beta_{n+1}(\alpha_{n+1})^2}\right]^2.$$

Using the fact that $(x+y)^2 \ge 4xy$ we obtain

$$\|F^*_{\lambda,L_\alpha}(h)\|^2_{\mathscr{H}_\beta} \le \frac{1}{4\lambda}\sum_{n=0}^{\infty} \beta_{n+1}|h_{n+1}|^2 \le \frac{1}{4\lambda}\|h\|^2_{\mathscr{H}_\beta},$$

which gives (iv) and completes the proof of the lemma. □

We establish approximate inversion formulas for the operator L_α.

Theorem 4.3. *If $\lambda > 0$ and $f, h \in \mathscr{H}_\beta$, then*

(i) $\lim_{\lambda\to 0^+} \|L_\alpha F^*_{\lambda,L_\alpha}(h) - h^*\|^2_{\mathscr{H}_\beta} = 0$, *where* $h^*(z) = L_1 T_1 h(z)$,

(ii) $\lim_{\lambda\to 0^+} \|F^*_{\lambda,L_\alpha}(L_\alpha f) - f\|^2_{\mathscr{H}_\beta} = 0$.

Proof. Let $\lambda > 0$ and $h \in \mathscr{H}_\beta$ with $h(z) = \sum_{n=0}^{\infty} h_n z^n$. From (4.4) we have

$$L_\alpha F^*_{\lambda,L_\alpha}(h)(z) - h^*(z) = \sum_{n=1}^{\infty} \frac{-\lambda\beta_{n-1}h_n}{\lambda n\beta_{n-1} + \beta_n(\alpha_n)^2} z^n, \tag{4.6}$$

where

$$h^*(z) = \sum_{n=1}^{\infty} \frac{h_n}{n} z^n.$$

Therefore

$$\|L_\alpha F^*_{\lambda,L_\alpha}(h) - h^*\|^2_{\mathscr{H}_\beta} = \sum_{n=1}^{\infty} \beta_n \left[\frac{\lambda\beta_{n-1}|h_n|}{\lambda n\beta_{n-1} + \beta_n(\alpha_n)^2}\right]^2.$$

Again, by dominated convergence theorem and the fact that

$$\beta_n \left[\frac{\lambda\beta_{n-1}|h_n|}{\lambda n\beta_{n-1} + \beta_n(\alpha_n)^2}\right]^2 \le \beta_n|h_n|^2,$$

we deduce (i). Finally, let $f \in \mathscr{H}_\beta$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then from (4.5) we have

$$F^*_{\lambda,L_\alpha}(L_\alpha f)(z) - f(z) = \sum_{n=0}^{\infty} \frac{-\lambda(n+1)^2\beta_n a_n}{\lambda(n+1)^2\beta_n + \beta_{n+1}(\alpha_{n+1})^2} z^n. \tag{4.7}$$

So, one has

$$\|F^*_{\lambda,L_\alpha}(L_\alpha f)-f\|^2_{\mathscr{H}_\beta}=\sum_{n=0}^{\infty}\beta_n\left[\frac{\lambda(n+1)^2\beta_n|a_n|}{\lambda(n+1)^2\beta_n+\beta_{n+1}(\alpha_{n+1})^2}\right]^2.$$

Using the dominated convergence theorem and the fact that

$$\beta_n\left[\frac{\lambda(n+1)^2\beta_n|a_n|}{\lambda(n+1)^2\beta_n+\beta_{n+1}(\alpha_{n+1})^2}\right]^2\leq\beta_n|a_n|^2,$$

we deduce (ii). □

We deduce also pointwise approximate inversion formulas for L_α.

Theorem 4.4. *If $\lambda>0$ and $f,h\in\mathscr{H}_\beta$, then*

(i) $\lim_{\lambda\to0^+}L_\alpha F^*_{\lambda,L_\alpha}(h)(z)=h^*(z)$,

(ii) $\lim_{\lambda\to0^+}F^*_{\lambda,L_\alpha}(L_\alpha f)(z)=f(z)$.

Proof. Let $f,h\in\mathscr{H}_\beta$ with $f(z)=\sum_{n=0}^{\infty}a_nz^n$ and $h(z)=\sum_{n=0}^{\infty}h_nz^n$. From (4.6) and (4.7), by using the dominated convergence theorem and the fact that

$$\frac{\lambda\beta_{n-1}|h_n|}{\lambda n\beta_{n-1}+\beta_n(\alpha_n)^2}|z|^n\leq|h_n||z|^n,\qquad\frac{\lambda(n+1)^2\beta_n|a_n|}{\lambda(n+1)^2\beta_n+\beta_{n+1}(\alpha_{n+1})^2}|z|^n\leq|a_n||z|^n,$$

we obtain (i) and (ii). □

5. Applications

In this section we examine the extremal functions associated to some particular cases of the weighted Hardy space $\mathscr{H}_\beta$, when T_α is the difference operator $\mathscr{D}$; and when L_α is the primitive operator $\mathscr{P}$, respectively.

5.1. The difference operator. Let $\mathscr{D}$ be the difference operator defined by

$$\mathscr{D}f(z):=\frac{1}{z}(f(z)-f(0)).$$

If $f(z)=\sum_{n=0}^{\infty}a_nz^n$, then $\mathscr{D}f(z)=\sum_{n=0}^{\infty}a_{n+1}z^n$.

In the following we determine the extremal function $F^*_{\lambda,\mathscr{D}}$, $\lambda>0$, for some particular cases of the space $\mathscr{H}_\beta$.

a) The Hardy space $\mathscr{H}$. In this case $\beta_n=1$, see [21, 22, 23]. We obtain the following results.

(i) The operator $\mathscr{D}$ maps continuously from $\mathscr{H}$ into $\mathscr{H}$, and $\|\mathscr{D}f\|_{\mathscr{H}}\leq\|f\|_{\mathscr{H}}$.

(ii) For $f\in\mathscr{H}$ we have

$$\mathscr{D}^*f(z)=zf(z),\quad\mathscr{D}^*\mathscr{D}f(z)=f(z)-f(0).$$

(iii) For any $h\in\mathscr{H}$ and for any $\lambda>0$, the problem

$$\inf_{f\in\mathscr{H}}\left\{\lambda\|f\|^2_{\mathscr{H}}+\|\mathscr{D}f-h\|^2_{\mathscr{H}}\right\}$$

has a unique minimizer given by

$$F^*_{\lambda,\mathscr{D}}(h)(z)=\frac{zh(z)}{\lambda+1}.$$

b) The weighted Bergman space $\mathscr{A}_\nu$. In this case $\beta_n=\frac{n!}{(\nu+2)_n}$, where $\nu>-1$ and $(a)_n=\frac{\Gamma(n+a)}{\Gamma(a)}$, see [15]. We deduce the following results.

(i) The operator $\mathscr{D}$ maps continuously from $\mathscr{A}_\nu$ into $\mathscr{A}_\nu$, and

$$\|\mathscr{D}f\|_{\mathscr{A}_\nu}\leq(\nu+2)^{1/2}\|f\|_{\mathscr{A}_\nu}.$$

(ii) For $f\in\mathscr{A}_\nu$ with $f(z)=\sum_{n=0}^{\infty}a_nz^n$ we have

$$\mathscr{D}^*f(z)=\sum_{n=1}^{\infty}\frac{n+\nu+1}{n}a_{n-1}z^n,$$

$$\mathscr{D}^*\mathscr{D}f(z)=\sum_{n=1}^{\infty}\frac{n+\nu+1}{n}a_nz^n.$$

(iii) For any $h\in\mathscr{A}_\nu$ and for any $\lambda>0$, the problem

$$\inf_{f\in\mathscr{A}_\nu}\left\{\lambda\|f\|^2_{\mathscr{A}_\nu}+\|\mathscr{D}f-h\|^2_{\mathscr{A}_\nu}\right\}$$

has a unique minimizer given by $F^*_{\lambda,\mathscr{D}}(h)(z) = \langle h, \Psi_z \rangle_{\mathscr{A}_\nu}$, where

$$\Psi_z(w) = \sum_{n=0}^{\infty} \frac{(\nu+2)_{n+1}(\bar{z})^{n+1}}{(\lambda+1)(n+1)+\nu+1} \cdot \frac{w^n}{n!}.$$

c) The weighted Dirichlet space $\mathscr{D}_\nu$. In this case $\beta_0 = 1$, $\beta_n = (\nu+1)\frac{nn!}{(\nu+1)_n}$, $n \geq 1$, where $\nu \geq 0$, see [10, 12]. We deduce the following results.
(i) The operator $\mathscr{D}$ maps continuously from $\mathscr{D}_\nu$ into $\mathscr{D}_\nu$, and

$$\|\mathscr{D}f\|_{\mathscr{D}_\nu} \leq (\nu+1)^{1/2}\|f\|_{\mathscr{D}_\nu}.$$

(ii) For $f \in \mathscr{D}_\nu$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have

$$\mathscr{D}^* f(z) = a_0 z + \sum_{n=2}^{\infty} \frac{(n-1)(n+\nu)}{n^2} a_{n-1} z^n,$$

$$\mathscr{D}^*\mathscr{D} f(z) = a_1 z + \sum_{n=2}^{\infty} \frac{(n-1)(n+\nu)}{n^2} a_n z^n.$$

(iii) For any $h \in \mathscr{D}_\nu$ and for any $\lambda > 0$, the problem

$$\inf_{f \in \mathscr{D}_\nu} \left\{ \lambda \|f\|^2_{\mathscr{D}_\nu} + \|\mathscr{D}f - h\|^2_{\mathscr{D}_\nu} \right\}$$

has a unique minimizer given by $F^*_{\lambda,\mathscr{D}}(h)(z) = \langle h, \Psi_z \rangle_{\mathscr{D}_\nu}$, where

$$\Psi_z(w) = \frac{\bar{z}}{\lambda+1} + \frac{1}{\nu+1} \sum_{n=1}^{\infty} \frac{(\nu+1)_{n+1}(\bar{z})^{n+1}}{\lambda(n+1)^2 + n(n+\nu+1)} \cdot \frac{w^n}{n!}.$$

5.2. The primitive operator. Let $\mathscr{P}$ be the primitive operator defined by

$$\mathscr{P} f(z) := \int_{[0,z]} f(w)\,\mathrm{d}w,$$

where $[0,z] = \{tz,\ t \in [0,1]\}$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\mathscr{P} f(z) = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} z^n$.
In the following we determine the extremal function $F^*_{\lambda,\mathscr{P}}$, $\lambda > 0$, for some particular cases of the space $\mathscr{H}_\beta$.
a) The Hardy space $\mathscr{H}$. We obtain the following results.
(i) The operator $\mathscr{P}$ maps continuously from $\mathscr{H}$ into $\mathscr{H}$, and $\|\mathscr{P}f\|_{\mathscr{H}} \leq \|f\|_{\mathscr{H}}$.
(ii) For $f \in \mathscr{H}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have

$$\mathscr{P}^* f(z) = \sum_{n=0}^{\infty} \frac{a_{n+1}}{n+1} z^n, \quad \mathscr{P}^*\mathscr{P} f(z) = \sum_{n=0}^{\infty} \frac{a_n}{(n+1)^2} z^n.$$

(iii) For any $h \in \mathscr{H}$ and for any $\lambda > 0$, the problem

$$\inf_{f \in \mathscr{H}} \left\{ \lambda \|f\|^2_{\mathscr{H}} + \|\mathscr{P}f - h\|^2_{\mathscr{H}} \right\}$$

has a unique minimizer given by $F^*_{\lambda,\mathscr{P}}(h)(z) = \langle h, \Psi_z \rangle_{\mathscr{H}}$, where

$$\Psi_z(w) = \sum_{n=1}^{\infty} \frac{n(\bar{z})^{n-1}}{\lambda n^2 + 1} w^n.$$

b) The weighted Bergman space $\mathscr{A}_\nu$. We deduce the following results.
(i) The operator $\mathscr{P}$ maps continuously from $\mathscr{A}_\nu$ into $\mathscr{A}_\nu$, and

$$\|\mathscr{P}f\|_{\mathscr{A}_\nu} \leq (\nu+2)^{-1/2}\|f\|_{\mathscr{A}_\nu}.$$

(ii) For $f \in \mathscr{A}_\nu$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have

$$\mathscr{P}^* f(z) = \sum_{n=0}^{\infty} \frac{a_{n+1}}{n+\nu+2} z^n,$$

$$\mathscr{P}^*\mathscr{P} f(z) = \sum_{n=0}^{\infty} \frac{a_n}{(n+1)(n+\nu+2)} z^n.$$

(iii) For any $h \in \mathscr{A}_\nu$ and for any $\lambda > 0$, the problem

$$\inf_{f \in \mathscr{A}_\nu} \left\{ \lambda \|f\|^2_{\mathscr{A}_\nu} + \|\mathscr{P}f - h\|^2_{\mathscr{A}_\nu} \right\}$$

has a unique minimizer given by $F^*_{\lambda,\mathscr{P}}(h)(z) = \langle h, \Psi_z \rangle_{\mathscr{A}_\nu}$, where

$$\Psi_z(w) = \sum_{n=1}^{\infty} \frac{n(\nu+2)_n(\bar{z})^{n-1}}{\lambda n(n+\nu+1)+1} \cdot \frac{w^n}{n!}.$$

c) The weighted Dirichlet space $\mathscr{D}_\nu$. We deduce the following results.
(i) The operator $\mathscr{P}$ maps continuously from $\mathscr{P}_\nu$ into $\mathscr{D}_\nu$, and

$$\|\mathscr{P}f\|_{\mathscr{D}_\nu} \leq \|f\|_{\mathscr{D}_\nu}.$$

(ii) For $f \in \mathscr{D}_\nu$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we have

$$\mathscr{P}^* f(z) = a_1 + \sum_{n=1}^{\infty} \frac{(n+1)a_{n+1}}{n(n+\nu+1)} z^n,$$

$$\mathscr{P}^* \mathscr{P} f(z) = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{n(n+\nu+1)} z^n.$$

(iii) For any $h \in \mathscr{D}_\nu$ and for any $\lambda > 0$, the problem

$$\inf_{f \in \mathscr{D}_\nu} \left\{ \lambda \|f\|^2_{\mathscr{D}_\nu} + \|\mathscr{P}f - h\|^2_{\mathscr{D}_\nu} \right\}$$

has a unique minimizer given by $F^*_{\lambda,\mathscr{P}}(h)(z) = \langle h, \Psi_z \rangle_{\mathscr{D}_\nu}$, where

$$\Psi_z(w) = \frac{w}{\lambda+1} + \frac{1}{\nu+1} \sum_{n=2}^{\infty} \frac{(\nu+1)_n(\bar{z})^{n-1}}{\lambda(n-1)(n+\nu)+1} \cdot \frac{w^n}{n!}.$$

6. Conclusion

The concrete results of Section 5 are the goal to give best approximation problems in numerical analysis. In the future work, by using computers, we shall illustrate numerical experiments approximation formulas for the difference and primitive operators in $\mathbb{D}$ in some particular cases of the weighted Hardy space $\mathscr{H}_\beta$, when $\lambda \to 0$.

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