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THE ECHELON CONCEPT AND A PROOF IN ARBORESCENT STRUCTURES

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Abstract:

In this paper it is shown that, in an arborescent structure at any instant, total echelon holding cost is being accumulated at the same rate as total physical holding cost, and a policy which is optimal under the echelon stock charging scheme is also optimal under a charging scheme based on stock physically at each stage.

Özet:

Hiyerarşik Düzeyler Kavramı ve Ters Ağaç Yapısı için Bir İspat

Bu çalışmada, ters ağaç yapısında olan sistemlerde, herhangibir anda hiyerarşik düzeylerde biriken toplam stokta tutma maliyetinin, gerçekleşen toplam stokta tutma maliyeti ile aynı olduğu ve hiyerarşik düzeylere bağlı olarak stok maliyetleme yaklaşımı çerçevesinde optimal olan politikanın, gerçek stok maliyetlerini esas alan yaklaşım için de optimal olacağı gösterilmiştir.

1. INTRODUCTION

The general field of inventory theory is concerned with providing methods for managing and controlling inventories under different policy constraints and environmental situations. In many such inventory environments, rational management decisions for inventories cannot be made without explicit consideration of the interrelations among activities in an overall supply system

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and with transportation, manufacturing, or other processes. In particular recognition of such relations, multi-echelon inventory theory has evolved as an important specialist area.

In this paper, a proposition in the area of multi-echelon inventory theory will be proved. Schwarz and Schrage (1978) show that in a serial system, at any instant, total echelon holding cost is being accumulated at the same rate as total physical holding cost. Here, we extend their proof to an arborescent system and show that a policy which is optimal under the echelon stock charging scheme is also optimal under a charging scheme based on stock physically at each stage.

2. MULTI-ECHELON SYSTEMS AND ECHELON STOCK CONCEPT

In a representative multi-echelon inventory system, customer demands occur only at the stocking points in echelon 1, the lowest echelon of the system considered. Echelon n has its stocks replenished by shipments from echelon $n+1$. A multi-echelon inventory system can also be portrayed as a directed network, wherein the nodes represent the various stocking points in the system and the linkages represent flows of goods. If the network has at most one incoming link for each node and flows are acyclic, it is called an arborescence or inverted tree structure. More complex interconnected systems of facilities can exist; however, most of the work in multi-echelon inventory theory has been confined to arborescent structures.

A significant contribution to the multi-echelon inventory theory is made by noting the following distinction between an installation stock and an echelon stock. Considering a serial system, the stock at installation i refers only to the stock physically at that location; however, echelon stock at echelon i refers to the sum of all the stocks at installations i, i-1, ..., 2, 1 plus all the stock in transit between installations i, i-1, ..., 2, 1. This definition may permit some very convenient mathematical simplifications.

The essential innovation of an alternative formulation is the interpretation of the inventory system as a nested set of echelons (i.e., in terms of echelon stocks and echelon holding costs) rather than as individual stocking points. The alternative formulation associates, with each stocking point, an echelon consisting of all stock in the system at that activity and below. With this interpretation, a certain multi-stage problem for the system as a whole can be decomposed into a set of interconnected one-stage problems, one for each echelon in the system. The echelon stock and echelon holding cost concepts were first introduced by Clark and Scarf (1960) and used by many others see for example Blackburn and Millen (1978), Crowston et al. (1973a, 1973b), Schwarz and Schrage (1975).

3. NOTATION

To illustrate the notation, a particular echelon $m \in \{1,....M\}$ of the multi-echelon structure is considered. Let the number of stocking points at this echelon be N_m. For each of these stocking points, we define $W(i,m,n)$ to be the set of descending ($n<m$) or immediate ascending ($n=m+1$) stocking points in echelon n, that are connected to the ith, $i \in \{1,....N_m\}$, stocking point of the mth echelon. $G(i,m)$ is the set of all successors of the stocking point i in the mth echelon (i.e., $G(i,m) \equiv W(i,m,\{n|n<m\})$). $V(i,m)$ is the set of all installations that are in the first echelon and originate from stocking point i of the mth echelon (i.e., $V(i,m) \equiv W(i,m,1)$). Each stocking point is defined by a pair of numbers (i,m), where i and m denote the stocking point and echelon numbers respectively. The echelon stock at stocking point i in echelon j at the end of period t is denoted by E_{ijt}, and e_{ijt} is the echelon holding cost. The definitions of e_{ijt} and E_{ijt} are as follows:

$$\left.\begin{aligned} e_{ij} &= C_{ij} - \{C_m \mid m \in W(i,j,j+1)\} \\ E_{ijt} &= X_{ijt} + \sum_{m \in G(i,j)} X_{mt} \end{aligned}\right\} \quad t = 1,\dots,n \quad j = 1,\dots,M \quad i = 1,\dots,N_j \,. \quad (1)$$

where C_{ij} is the inventory holding cost at stocking point (i,j) and X_{ijt} is the closing inventory level at the end of period t.

4. PROBLEM DEFINITION

We want to show that at any instant echelon holding costs are being accumulated at the same rate as physical holding costs in an arborescent structure. The physical holding cost function is given below as equation (2).

$$\sum_{j=1}^{M}\sum_{i=1}^{N_j} C_{ij} X_{ijt} \quad t = 1,\dots,n \qquad (2)$$

The standart inventory transition equations are

$$X_{ijt} = X_{ij(t-1)} + U_{ijt} - \sum_{m \in W(i,j,j-1)} U_{mt} \quad i = 1,\dots,N_j \quad t = 1,\dots,n \quad j = 1,\dots,M \quad (3)$$

and

$$U_{ijt}, X_{ijt} \geq 0, \tag{4}$$

where U_{ijt} is the order received at stocking point (i,j) in period t. Equations. (2)-(4) form the model I.

It is going to be shown that, using the above linear transformations, without loss of generality, model I of multi-echelon systems being represented by, Eqs. (2)-(4), can be rewritten as Eqs. (5)-(7) given below. This alternative formulation is called model II. The concept behind this transformation is known in the MRP literature (for assembly systems) as "explosion" (Afentakis and Gavish 1983).

$$\sum_{j=1}^{M}\sum_{i=1}^{N_j} e_{ij} E_{ijt} \quad t = 1,\ldots,n \tag{5}$$

$$E_{ijt} = E_{ij(t-1)} + U_{ijt} - \sum_{m\in I'(i,j)} d_{mt} \quad t = 1,\ldots,n \quad j = 1,\ldots,M \quad i = 1,\ldots,N_j \tag{6}$$

$$0 \leq \sum_{m\in W'(i,j,j-1)} E_{mt} \leq E_{ijt} \quad t = 1,\ldots,n \quad j = 1,\ldots,M \quad i = 1,\ldots,N_j \tag{7}$$

where customer demands, d_{mt}, occur only at the stocking points in echelon 1.

The validity of the echelon stock concept in arborescent structures is discussed and the equivalence of models I and II is proved in §5.

5. A PROOF OF THE PROPOSITION

Proposition: Given $E_{ijt} = X_{ijt} + \sum_{m\in G(i,j)} X_{mt}$, models I and II are equivalent.

Proof:

We start the proof by processing echelon 2 of model I. From equation (3), we can write

$$X_{i2t} = X_{i2(t-1)} + U_{i2t} - \sum_{m \in W'(i,2,1)} U_{mt} \quad i = 1,\dots,N_2 \tag{8}$$

The summation of equations (3) for all stocking points that are in the first echelon and connected to $(i,2)$ is

$$\sum_{m \in W'(i,2,1)} X_{mt} = \sum_{m \in W'(i,2,1)} X_{m(t-1)} + \sum_{m \in W'(i,2,1)} U_{mt} - \sum_{m \in W'(i,2,0)} U_{mt} \quad . \tag{9}$$

Thereby, using the definition $U_{i0t} \equiv d_{it}$, considering the equivalence of $G(i,2)$ and $W(i,2,1)$, and adding equations (8) and (9) give

$$\left(X_{i2t} + \sum_{m \in G(i,2)} X_{mt} \right) = \left(X_{i2(t-1)} + \sum_{m \in G(i,2)} X_{m(t-1)} \right) + U_{i2t} - \sum_{m \in V'(i,2)} d_{mt} \ .$$

Applying the above process to the higher echelons $(3,\dots,M)$, we can write the general expression as

$$\left(X_{ijt} + \sum_{m \in G(i,j)} X_{mt} \right) = \left(X_{ij(t-1)} + \sum_{m \in G(i,j)} X_{m(t-1)} \right) + U_{ijt} - \sum_{m \in V'(i,j)} d_{mt} \ . \tag{10}$$

By making use of equations (1) (i.e., definition of E_{ijt}) and (10), equation (6) is found. Hence, the equivalence of (3) and (6) is shown.

Also, equation (1) can be solved for X_{ijt} giving:

$$\begin{aligned} X_{ijt} &= E_{ijt} - \sum_{m \in G(i,j)} X_{mt} \\ X_{ijt} &= E_{ijt} - \sum_{m \in W'(i,j,j-1)} E_{mt} \ ; \end{aligned} \tag{11}$$

from which the nonnegativity constraint of X_{ijt} leads directly to equation (7).

Finally, by substituting equation (11) in the cost function of model I, rearranging by E_{ijt}, and using the definition of echelon holding cost e_{ij}, (i.e., equation (1)), the cost function of model II is obtained.

Also. the equivalence of the total echelon holding cost accumulation and the total physical cost accumulation must be proved (Szendrovitz 1978). The echelon holding cost, e_{ij}. is calculated from the physical holding cost, C_{ij}, by the rule

$$e_{ij} = C_{ij} - \{C_{ij} | m \in W(i, j, j+1)\} \quad j = 1, \ldots, M \quad i = 1, \ldots, N_j . \tag{12}$$

Schwarz and Scharge (1978) show that in a serial system. at any instant

$$\sum_{j=1}^{M} \sum_{i=1}^{N_j} C_{ij} X_{ij} = \sum_{j=1}^{M} \sum_{i=1}^{N_j} e_{ij} E_{ij} , \tag{13}$$

that is, total echelon holding cost is being accumulated at the same rate as total physical holding cost.

Here, we extend their proof to an arborescent system and show that the two expressions are equal by induction on the number of stages in an arborescent system. First suppose that the system consists only of stocking points of stage 1; then $E_{i1} = X_{i1}$, $e_{i1} = C_{i1}$ and $\sum e_{i1} E_{i1} = \sum C_{i1} X_{i1}$ for $i = 1, \ldots, N_1$. A new stocking point is introduced as stage 2, and all the stocking points of stage 1 are connected to this new stocking point as successors. Then it follows

$$\begin{aligned} & e_2 = C_2 , \\ & e_{i1} \leftarrow e_{i1} - C_2 \quad i = 1, \ldots, N_1, \\ & E_2 = \sum_{k=1}^{2} \sum_{i=1}^{N_k} X_{ik} = X_2 + \sum_{i=1}^{N_1} E_{i1} \end{aligned} \tag{14}$$

from which the physical inventory cost increase may be calculated as

$$\begin{aligned} e_2 E_2 - \sum_{i=1}^{N_1} C_2 E_{i1} &= C_2 \left(E_2 - \sum_{i=1}^{N_1} E_{i1} \right) \\ &= C_2 X_2 \end{aligned} \tag{15}$$

Since this is the actual amount of increase in the physical inventory cost, one can conclude that echelon stock charging scheme gives the correct physical inventory cost for a 2-echelon system.

Now assume an inventory system of N_{j-1} independent j-1 stage systems. A new stage, stage j, is introduced and all the j-1 stage systems are connected to it as successors. The upstream echelon inventories are unaffected. However, the following modifications take place:

$$e_j = C_j ,$$
$$e_{i(j-1)} \leftarrow e_{i(j-1)} - C_j \quad i = 1, \ldots, N_{j-1} . \qquad (16)$$
$$E_j = \sum_{k=1}^{j} \sum_{i=1}^{N_k} X_{ik} = X_j + \sum_{i=1}^{N_{j-1}} E_{i(j-1)} .$$

Thus, the echelon holding cost expression increases by

$$e_j E_j - \sum_{i=1}^{N_{j-1}} C_{h_i} E_{i(j-1)} = C_{h_j} \left(E_j - \sum_{i=1}^{N_{j-1}} E_{j-1} \right) . \qquad (17)$$
$$= C_{h_j} X_j .$$

This is exactly the amount by which the physical inventory cost expression increases, which completes the induction.

Thus a policy which is optimal under the echelon stock charging scheme is also optimal under a charging scheme based on stock physically at each stage in an arborescent structure.

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