Approximation of A Class of Non-linear Integral Operators

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Abstract

In this study,we investigate the problem of pointwise convergence at lebesgue points of f funtions for the family of non-linear integral operators

$$L_\lambda(f,x)=\sum_{m=1}^{\infty}\int_a^b f^m(t)K_{\lambda,m}(x,t)dt$$

where λ is a real parameter, $K_{\lambda,m}(x,t)$ is non-negative kernels and f is the function in $L_1(a,b)$.

We consider two cases where (a,b) is a finite interval and when is the whole real axis.

Keywords — *Approximation, nonlinear integral operators, lebesque point*

1 Introduction

In [1], the concept of singularity was extended to cover the case of nonlinear integral operators,

$$T_\lambda f(s)=\int_G K_\lambda(t-x,\ f(t))\,dt,\quad x\in G,$$

the assumption of linearity of the operators being replaced by an assumption of a Lipschitz condition for K_λ with respect to the second variable. Later, Swiderski and Wachnicki [2] investigated the problem of convergence of above the same operators to f as $(w,s)\to(w_0,s_0)$ where s_0 is an accumulation point of the locally compact abelian group G in $L_p(-\pi,\pi)$ and $L_p(R)$.

In [3] Karsli examined both the pointwise convergence and the rate of pointwise convergence of above operators on a μ – generalized Lebesque point to $f\in L_1(a,b)$ as $(x,\lambda)\to(x_0,\lambda_0)$. And in [4] it is studied the rate of convergence of nonlinear integral operators for functions of bounded variation at a point x, which has a discontinuity of the first kinds as $\lambda\to\lambda_0$. In [5] they obtained estimates, convergence results and rate of approximation for functions belonging to BV-spaces for a family of nonlinear integral operators of the convolution type

$$(T_w f)(s)=\int_{-\pi}^{\pi}K_w(t,\ f(s-t))\,dt,\qquad w>0,\ s\in R$$

in the periodic case. In paper [6], they obtained pointwise convergence and rate of pointwise convergence results at Lebesgue points for a family of nonlinear integral operators of the form

$$T_\lambda(f,x)=\int_0^\infty K_\lambda(x,z,\ f(z))\frac{dz}{z},\quad x>0,\ \lambda>0,\ \lambda\in\Lambda$$

with (K_λ) is a family of kernel satisfying a Lipschitz condition.

Karsli [7] stated some approximation theorems about pointwise convergence and its rate for a class of non-convolution type nonlinear integral operators.

Soon, in [8], Almali and Gadjiev proved convergence of exponentially nonlinear integrals in Lebesgue points of generated function, having many applications in approximation theory [9,10].

The aim of the article is to obtain pointwise convergence results for a family of non-linear operators of the form

$$L_\lambda(f,x)=\sum_{m=1}^{\infty}\int_a^b f^m(t)K_{\lambda,m}(x,t)dt \qquad (1)$$

where $K_{\lambda,m}(x,t)$ is a family of kernels depending on λ. We study convergence of the family (1) at every Lebesque point of the function f in the spaces of $L_1(a,b)$ and $L_1(-\infty,\infty)$ with $\sum_{m=1}^{\infty}\int_a^b f^m(t)K_{\lambda,m}(x,t)dt$ is convergence.

Now we give the following definition

Definition 1 *(Class A): We take a family* $(K_\lambda)_{\lambda\in\Lambda}$ *of functions* $K_{\lambda,m}(x,t): RXR \to R$. *We will say that the function* $K_\lambda(x,t)$ *belongs the class A, if the following conditions are satisfied:*

a) $K_{\lambda,m}(x,t)$ is a non-negative function defined for all t and x on (a,b) and $\lambda\in\Lambda$.

b) As function of t, $K_{\lambda,m}(x,t)$ is non-decreasing on $[a,x]$ and non-increasing on $[x,b]$ for any fixed x and $\lambda\in\Lambda$.

c)For any fixed x, $\int_a^b K_{\lambda,m}(x,t)dt = C_m$.

d) $\sum_{m=1}^{\infty} C_m$ is convergence.

e) For $y\neq x$, $\lim_{\lambda\to\infty}\sum_{m=1}^{\infty} K_{\lambda,m}(x,y)=0$.

2. Main Result

We are going to prove the family of non-linear integral operators (1) with the positive kernel convergence to the functions $f\in L_1(a,b)$

Theorem 1. *Suppose that* $f\in L_1(a,b)$ *and* f *is bounded on* (a,b). *If non-negative the kernel* $K_{\lambda,m}$ *belongs to Class A, then, for the operator* $L_\lambda(f,x)$ *which is defined in (1)*

$$\lim_{\lambda\to\infty} L_\lambda(f,x_0)=\sum_{m=1}^{\infty} C_m f^m(x_0)$$

holds at every x_0 – Lebesque point of f function with $\sum_{m=1}^{\infty} C_m f^m(x_0)$ is convergence.

Proof. For integral (1), from c), we can write

$$L_\lambda(f,x_0)-\sum_{m=1}^{\infty} C_m f^m(x_0) = \sum_{m=1}^{\infty}\int_a^b [f^m(t)-f^m(x_0)]K_{\lambda,m}(x_0,t)dt$$

and in view of a)

$$\left|L_\lambda(f,x_0)-\sum_{m=1}^{\infty} C_m f^m(x_0)\right| \le \sum_{m=1}^{\infty}\int_a^b \left|f^m(t)-f^m(x_0)\right|K_{\lambda,m}(x_0,t)dt = I(x_0,\lambda)$$

Now we consider $I(x_0,\lambda)$. For any fixed $\delta>0$, we can write $I(x_0,\lambda)$ as follow.

$$I(x_0,\lambda)=\sum_{m=1}^{\infty}\left[\int_a^{x_0-\delta}+\int_{x_0-\delta}^{x_0}+\int_{x_0}^{x_0+\delta}+\int_{x_0+\delta}^{b}\right]\left|f^m(t)-f^m(x_0)\right|K_{\lambda,m}(x_0,t)dt$$
$$=I_1(x_0,\lambda,m)+I_2(x_0,\lambda,m)+I_3(x_0,\lambda,m)+I_4(x_0,\lambda,m) \qquad (2)$$

Firstly we shall calculate $I_1(x_0,\lambda,m)$, that's

$$I_1(x_0,\lambda,m)=\sum_{m=1}^{\infty}\int_a^{x_0-\delta}\left|f^m(t)-f^m(x_0)\right|K_{\lambda,m}(x_0,t)dt.$$

By the condition b), we have

$$I_1(x_0,\lambda,m)\le \sum_{m=1}^{\infty} K_{\lambda,m}(x_0,x_0-\delta)\left\{\int_a^{x_0-\delta}\left|f^m(t)\right|dt + \int_a^{x_0-\delta}\left|f^m(x_0)\right|dt\right\}$$

and

$$\le \sum_{m=1}^{\infty} K_{\lambda,m}(x_0,x_0-\delta)\left\{\left\|f^m\right\|_{L_1(a,b)}+\left|f^m(x_0)\right|(b-a)\right\}. \quad (3)$$

In the same way, we can estimate $I_4(x_0,\lambda,m)$ .From property b)

$$I_4(x_0,\lambda,m)\le \sum_{m=1}^{\infty} K_{\lambda,m}(x_0,x_0+\delta)\left\{\int_{x_0+\delta}^{b}\left|f^m(t)\right|dt + \int_{x_0+\delta}^{b}\left|f^m(x_0)\right|dt\right\}$$

$$\le \sum_{m=1}^{\infty} K_{\lambda,m}(x_0,x_0+\delta)\left\{\left\|f^m\right\|_{L_1(a,b)}+\left|f^m(x_0)\right|(b-a)\right\}. \quad (4)$$

On the other hand, since x_0 is a Lebesque point of f, for every $\varepsilon>0$, there exists a $\delta>0$ such that

$$\int_{x_0}^{x_0+h}\left|f(t)-f(x_0)\right|dt<\varepsilon h \quad (5)$$

and

$$\int_{x_0-h}^{x_0}\left|f(t)-f(x_0)\right|dt<\varepsilon h \quad (6)$$

for all $0<h\le\delta$. Now let's define a new function as follows,

$$F(t)=\int_{x_0}^{t}\left|f(u)-f(x_0)\right|du.$$

Then from (5), for $t-x_0\le\delta$ we have

$$F(t)\le\varepsilon(t-x_0).$$

Also, since f is bounded, there exists $M>0$ such that

$$\left|f^m(t)-f^m(x_0)\right|\le\left|f(t)-f(x_0)\right|M$$

is satisfied. Therefore, we can estimate $I_3(x_0,\lambda,m)$ as follows.

$$I_3(x_0,\lambda,m)\le M\sum_{m=1}^{\infty}\int_{x_0}^{x_0+\delta}\left|f(t)-f(x_0)\right|K_{\lambda,m}(x_0,t)dt$$

$$\le M\sum_{m=1}^{\infty}\int_{x_0}^{x_0+\delta}K_{\lambda,m}(x_0,t)dF(t).$$

We apply integration by part, then we obtain the following result.

$$\left|I_3(x_0,\lambda,m)\right|\le M\sum_{m=1}^{\infty}\left\{F(x_0+\delta,x_0)K_{\lambda,m}(x_0+\delta,x_0) + \int_{x_0}^{x_0+\delta}F(t)d\left(-K_{\lambda,m}(x_0,t)\right)\right\}$$

Since $K_{\lambda,m}$ is decreasing on $[x_0,b]$, it is clear that $-K_{\lambda,m}$ is increasing. Hence its differential is positive. Therefore, we can wirte

$$\left|I_3(x_0,\lambda,m)\right|\le M\sum_{m=1}^{\infty}\left\{\varepsilon\delta K_{\lambda,m}(x_0+\delta,x_0) + \varepsilon\int_{x}^{x+\delta}(t-x_0)d\left(-K_{\lambda,m}(x_0,t)\right)\right\}$$

Integration by parts again, we have the following inequality

$$\left|I_3(x_0,\lambda,m)\right|\le\varepsilon M\sum_{m=1}^{\infty}\int_{x_0}^{x_0+\delta}K_{\lambda,m}(x_0,t)dt$$

$$\le\varepsilon M\sum_{m=1}^{\infty}\int_a^b K_{\lambda,m}(x_0,t)dt. \quad (7)$$

Now, we can use similar method for evaluation $I_2(x_0,\lambda,m)$. Let

$$G(t)=\int_t^x\left|f(y)-f(x)\right|dy.$$

Then, the statement

$$dG(t)=-\left|f(t)-f(x_0)\right|dt$$

is satisfied. For $x_0-t\le\delta$, by using (6), it can be written as follows

$$G(t)\le\varepsilon\left|x_0-t\right|$$

Hence, we get

$$I_2(x_0,\lambda,m) \le M \sum_{m=1}^{\infty} \int_{x_0-\delta}^{x_0} \left| f(t) - f(x_0) \right| K_{\lambda,m}(x_0,t)dt.$$

Then, we shall write

$$\left| I_2(x_0,\lambda,m) \right| \le M \sum_{m=1}^{\infty} \left[-\int_{x_0-\delta}^{x_0} K_{\lambda,m}(x_0,t)dG(t) \right].$$

By integration of parts, we have

$$\left| I_2(x,\lambda,m) \right| \le M \sum_{m=1}^{\infty} \Big\{ G(x-\delta K_{\lambda,m}(x-\delta,x) + \int_{x-\delta}^{x} G(t)d_t(K_{\lambda,m}(x,t)) \Big\}$$

From (6), we obtain

$$\left| I_2(x_0,\lambda,m) \right| \le M \sum_{m=1}^{\infty} \Big\{ \varepsilon\delta K_{\lambda,m}(x_0,x_0-\delta) + \varepsilon \int_{x_0-\delta}^{x_0} (x_0-t)d_t(K_{\lambda,m}(x_0,t)) \Big\}$$

By using integration of parts again, we find

$$\left| I_2(x_0,\lambda,m) \right| \le \varepsilon M \sum_{m=1}^{\infty} \int_a^b K_{\lambda,m}(x_0,t)dt. \qquad (8)$$

Combined (7) and (8), we get

$$\left| I_2(x_0,\lambda,m) \right| + \left| I_3(x_0,\lambda,m) \right| \le 2\varepsilon M \sum_{m=1}^{\infty} \int_a^b K_{\lambda,m}(x_0,t)dt. \qquad (9)$$

From condition d), (9) tends to 0 as $\lambda \to \infty$. Finally,from (3), (4) and (9),the terms on right hand side of these inequalitys tend to 0 as $\lambda \to \infty$. That's

$$\lim_{\lambda\to\infty} L_\lambda(f,x_0) = \sum_{m=1}^{\infty} C_m f^m(x_0).$$

Thus, the proof is completed.

In this theorem, specially interval (a,b) may be expanded interval $(-\infty,\infty)$. In this case, we can give the following theorem.

Theorem 2 *Let $f \in L_1(-\infty,\infty)$ and f is bounded. If non-negative the kernel $K_{\lambda,m}$ belongs to Class A and satisfies also the following properties ,*

$$\lim_{\lambda\to\infty} \sum_{m=1}^{\infty} \int_{-\infty}^{x-\delta} K_{\lambda,m}(t,x)dt = 0 \qquad (10)$$

and

$$\lim_{\lambda\to\infty} \sum_{m=1}^{\infty} \int_{x+\delta}^{\infty} K_{\lambda,m}(t,x)dt = 0, \qquad (11)$$

then the statement

$$\lim_{\lambda\to\infty} L_\lambda(f,x) = \sum_{m=1}^{\infty} C_m f^m(x)$$

is satisfied at almost every $x \in R$ with $\sum_{m=1}^{\infty} C_m f^m(x)$ is convergence.

Proof. We can write, for a fixed $\delta > 0$

$$\left| L_\lambda(f,x) - \sum_{m=1}^{\infty} C_m f^m(x) \right| \le \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \left| f^m(t) - f^m(x) \right| K_{\lambda,m}(x,t)dt$$

$$= \sum_{m=1}^{\infty} \left[\int_{-\infty}^{x-\delta} + \int_{x-\delta}^{x} + \int_{x}^{x+\delta} + \int_{x+\delta}^{\infty} \right] \left| f^m(t) - f^m(x) \right| K_{\lambda,m}(x,t)dt$$

$$= A_1(x,\lambda,m) + A_2(x,\lambda,m) + A_3(x,\lambda,m) + A_4(x,\lambda,m)$$

$A_2(x,\lambda,m)$ and $A_3(x,\lambda,m)$ integrals are calculated as the proof in Theorem1. For proof, it is sufficent to show that $A_1(x,\lambda,m)$ and $A_4(x,\lambda,m)$ tend to zero as $\lambda \to \infty$.

Firstly, we consider $A_1(x,\lambda,m)$. Since f is bounded and by the property b), this integration is written in the form

$$A_1(x,\lambda,m) \le M \sum_{m=1}^{\infty} \int_{-\infty}^{x-\delta} \left| f(t) - f(x) \right| K_{\lambda,m}(x,t)dt$$

$$\le M \sum_{m=1}^{\infty} K_{\lambda,m}(x,x-\delta) \left\{ \int_{-\infty}^{x-\delta} \left| f(t) \right| dt \right\} + M \left| f(x) \right| \sum_{m=1}^{\infty} \int_{-\infty}^{x-\delta} K_{\lambda,m}(x,t)dt.$$

$$\leq \|f\|_{L_1(-\infty,\infty)} M \sum_{m=1}^{\infty} K_{\lambda,m}(x, x-\delta)$$

$$+ M\left|f(x)\right|\sum_{m=1}^{\infty}\int_{-\infty}^{x-\delta} K_{\lambda,m}(x,t)dt$$

In addition to, we obtain the inequality

$$A_4(x,\lambda,m) \leq M \sum_{m=1}^{\infty}\int_{x+\delta}^{\infty}\left|f(t)-f(x)\right|K_{\lambda,m}(x,t)dt$$

$$\leq \|f\|_{L_1(-\infty,\infty)} M \sum_{m=1}^{\infty} K_{\lambda,m}(x, x+\delta)$$

$$+ M\left|f(x)\right|\sum_{m=1}^{\infty}\int_{x+\delta}^{\infty} K_{\lambda,m}(x,t)dt.$$

According to the conditions d), (10) and (11),we find that $A_1(x,\lambda,m) + A_4(x,\lambda,m) \to 0$ as $\lambda \to \infty$. This completes the proof.

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