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METHOD OF VARIATION OF PARAMETERS FOR THE THIRD-ORDER LINEAR PROPORTIONAL DYNAMIC EQUATIONS

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Abstract

The differentiation and integration of an integer order are known as fractional calculus. It is possible to think of the proportional derivative as a generalization of a congruent fractional derivative, which is one of the types of fractional calculus. In this article, utilizing the proportional derivative and its characteristics on the time scale, the method of variation of parameters for the third-order linear nonhomogeneous differential equations is given. Then, an example is provided to illustrate how to apply the provided approach.

Key Words: Time scales, variation of parameters, proportional derivative

1. Introduction

The conformable derivative is a unique mathematical operator that expands the idea of differentiation to non-integer orders (Katugampola, 2014; Khalil et al., 2014; Abdeljewad, 2015). The conformable derivative offers a more understandable and natural framework for dealing with non-integer differentiation than conventional fractional calculus operators like the Riemann-Liouville or Caputo derivatives. Due to its capacity for modeling complex events showing fractal or anomalous behavior, the idea of non-integer differentiation has received a great deal of interest (Ortigueira & Machado, 2015). For the study of dynamic equations that incorporate both continuous and discrete time periods, the conformable derivative on time scales offers a coherent framework. Time scales expand the idea of real numbers to encompass discrete and continuous time domains, making it possible to represent a wider range of equations in greater detail. The conformable derivative depends on forward and backward difference quotients instead of integrals, which are necessary for fractional derivatives, making it easier to obtain and compute.

The conformable derivative on time scales has a variety of advantageous characteristics, such as linearity, the chain rule, and compatibility with conventional differentiation operators on real numbers. On time scales, physics, engineering, biology, and finance are just a few of the disciplines that the conformable derivative is used in. The conformable derivative on time scales has received a lot of interest recently from the scientific community (Benkhettou et al., 2015; Benkhettou et al., 2016; Gulsen et al., 2017; Gülşen et al., 2018; Yilmaz et al., 2022).

Even if the more inclusive definition of the proportional derivative given in Definition 1 below meets some of the characteristics of the fractional derivative, it is still best to consider the proportional derivative independently, separate from the theory of fractional derivatives. As a result, the proportional derivative reported in Anderson & Ulness (2015) was renamed a conformable derivative, and a prospective definition for the proportional derivative on a time scale was found in Segi Rahmat (2019). A specific case of the proportional derivative is the conformable derivative. When the order is equal to 1, the proportional derivative of a function defined on the time scale becomes the Hilger derivative.

Non-homogeneous linear differential equations can be solved using the variation of parameters approach. When used with third-order differential equations on time scales, it entails locating a specific solution under the presumption that the coefficients of the solution are functions of the independent variable. This approach offers a potent tool for resolving intricate differential equations that appear in a variety of scientific and engineering disciplines. For the second-order nonhomogeneous dynamic equation via the proportional derivative, the variation of parameters is given in Anderson & Georgiev (2020), but for the third-order nonhomogeneous dynamic equation via the proportional derivative, it has not been studied before, according to our research.

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In this paper, we aim to explore the theory and applications of the variation of parameters for the third-order nonhomogeneous dynamic equation via the proportional derivative. Basic ideas and notations relating to time scales and proportional derivatives on time scales are presented in Section 2. The technique of variation of parameters produced for solving the third-order proportional derivative nonhomogeneous linear dynamic equations and an example are given in Section 3. The conclusion is provided in the final Section.

2. Materials and Methods

We examine the terms and ideas related to the time scale of proportional calculations that are necessary as they are used in the following section.

Definition 1 (Anderson & Ulness, 2015) Let $\gamma \in [0,1]$. If $\mathfrak{D}^0$ is the unit operator and $\mathfrak{D}^1$ is the standard differential operator, then the differential operator $\mathfrak{D}^\gamma$ is referred to as a proportional derivative. It is expressly stated that the derivative function $h{=}h(t)$ has a proportional operator $\mathfrak{D}^\gamma$ and that only

$$\mathfrak{D}^0 h(t){=}h(t) \quad \text{and} \quad \mathfrak{D}^1 h(t){=}h'(t), \tag{1}$$

exists for it.

Remark 2 (Anderson & Ulness, 2015) The essential principle of proportional derivative is created based on the use of a proportional-derivative controller with a controller output v at time t. This controller, $v(t)$, uses the

$$v(t){=}\kappa_p E(t) + \kappa_d \frac{d}{dt} E(t),$$

algorithm (Li et al., 2006) .

In this instance, E stands for the error between the state and process variables, while κ_p and κ_d are the proportional and derivative gains, respectively.

Definition 3 (Anderson & Ulness, 2015) Consider $\gamma \in [0,1]$, $\kappa_0, \kappa_1 \colon [0,1] \times \mathbb{R} \to \mathbb{R}_0^+$ to be continuous functions and that

$$\begin{cases} \lim\limits_{\gamma \to 0^+} \kappa_0(\gamma, t) = 0, \qquad \lim\limits_{\gamma \to 0^+} \kappa_1(\gamma, t) = 1, \\ \lim\limits_{\gamma \to 1^-} \kappa_0(\gamma, t) = 1, \qquad \lim\limits_{\gamma \to 1^-} \kappa_1(\gamma, t) = 0, \\ \kappa_0(\gamma, t) \neq 0, \gamma \in (0,1], \quad \kappa_1(\gamma, t) \neq 0, \ \gamma \in [0,1), \end{cases} \tag{2}$$

to be accurate. The differential operator $\mathfrak{D}^\gamma$ defined by

$$\mathfrak{D}^\gamma h(t){=}\kappa_1(\gamma, t) h(t) + \kappa_0(\gamma, t) h^\Delta(t), \tag{3}$$

in here, h is the error, κ_1 is a kind of proportional gain κ_p, κ_0 is a type of derivative gain κ_d, and $v{=}\mathfrak{D}^\gamma h$ is the controller output.

To obtain the fundamental conclusions for the next section, we need to keep in mind a few fundamental time scale ideas. The time scale $\mathbb{T}$ belongs to $\mathbb{R}$'s standard topology and is a closed, non-empty subset of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho \colon \mathbb{T} \to \mathbb{T}$ for $t \in \mathbb{T}$ have the following definitions:

$$\sigma(t){=}\inf\{l \in \mathbb{T} : l > t\}, \qquad \rho(t){=}\sup\{l \in \mathbb{T} : l < t\}.$$

According to this definition, $\inf\emptyset = \sup\mathbb{T}$ and $\sup\emptyset{=}\inf\mathbb{T}$. If $\sigma(t) > t, \rho(t) < t$, and $\rho(t) < t < \sigma(t)$ t are, respectively, right-scattered, left-scattered, and isolated (discrete) points. In contrast, t is said to be right-dense if $t < \sup\mathbb{T}$ and $\sigma(t){=}t$, left-dense if $t > \inf\mathbb{T}$ and $\rho(t){=}t$, and t is the dense point if $\rho(t){=}t{=}\sigma(t)$. The graininess function $\mu \colon \mathbb{T} \to [0, \infty)$ is defined as $\mu(t){=}\sigma(t) - t$. If $\mathbb{T}$ has a maximum point m, then $\mathbb{T}^k{=}\mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^k{=}\mathbb{T}$. If $\mathbb{T}$ has a left-sided limit at both its right-dense and left-scattered points, then the function $h \colon \mathbb{T} \to \mathbb{R}$ is referred to as being rd-continuous, and $C_{rd}(\mathbb{T})$ is used to represent the collection of rd-continuous functions h. Let $t \in \mathbb{T}^k$ and $h : \mathbb{T} \to \mathbb{R}$ be a function. If $\forall \varepsilon > 0$ and $h^\Delta(t)$ is a real number such that

$$|[h(\sigma(t)) - h(s)] - h^\Delta(t)[\sigma(t) - s]| \leq \varepsilon\, |\sigma(t) - s|, \ \ \forall s \in U,$$

for all s in a neighborhood U of point t, then $h^{\Delta}(t)$ is referred to as the delta derivative of h at point t. There is an inverse derivative H, H^{Δ}=h(t), for any continuous rd-function h. For $s \in \mathbb{T}$,

$$\int_s^t h(\tau)\,\Delta\tau = H(t), \qquad \forall t \in \mathbb{T}.$$

On the time scale, (Aulbach & Hilger, 1990; Agarwal et al., 2002; Bohner & Peterson, 2001, 2004; Bohner & Svetlin, 2016; Hilger, 1990) offer thorough details.

We'll now give the proportional delta derivative of the function $h: \mathbb{T} \to \mathbb{R}$ of order $\gamma \in [0,1]$ at point $t \in \mathbb{T}^k$. Suppose that in the following expressions, $\kappa_0, \kappa_1 : [0,1] \times \mathbb{T} \to \mathbb{R}_0^+$ are continuous functions and satisfy conditions (2).

Definition 4 (Segi Rahmat, 2019) Let $h: \mathbb{T} \to \mathbb{R}$ be a function, and $t \in \mathbb{T}^k$. $\forall \varepsilon > 0$, and for every s in a neighborhood U of point t, if there is a real number $\mathfrak{D}^{\gamma} h(t)$, $\gamma \in [0,1]$, such that

$$\left|\kappa_1(\gamma,t)h(t)[\sigma(t)-s] + \kappa_0(\gamma,t)\big[h\big(\sigma(t)\big) - h(s)\big] - \mathfrak{D}^{\gamma} h(t)[\sigma(t)-s]\right| \le \varepsilon\,[\sigma(t)-s], \tag{4}$$

that number is known as the γ −th order proportional delta derivative of f at point t.

The set of all proportional delta differentiable functions will be displayed with (Segi Rahmat, 2019)

$$\Omega(\mathbb{T}) = \{\, h: \mathbb{T} \to \mathbb{R} \;:\; \text{For any } t \in \mathbb{T}^k, \mathfrak{D}^{\gamma} h(t) \text{ exists and is finite}\}.$$

Theorem 5 (Segi Rahmat, 2019) Assuming that $h: \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$.
(i) If $h \in \Omega(\mathbb{T})$, then h is continuous at t.

(ii) If h is continuous at t, t is right-scattered, and

$$h^{\Delta}(t) = \frac{h(\sigma(t)) - h(t)}{\sigma(t) - t},$$

exists, then $h \in \Omega(\mathbb{T})$. In this instance,

$$\mathfrak{D}^{\gamma} h(t) = \kappa_0(\gamma,t)\,h^{\Delta}(t) + \kappa_1(\gamma,t)\,h(t). \tag{5}$$

(iii) If t is right-dense, and

$$\lim_{t\to s} \frac{h(t)-h(s)}{t-s} = h'(t),$$

exists as a finite number, then $f \in \Omega(\mathbb{T})$, and so

$$\mathfrak{D}^{\gamma} h(t) = \kappa_0(\gamma,t)\,h'(t) + \kappa_1(\gamma,t)\,h(t). \tag{6}$$

Lemma 6 (Segi Rahmat, 2019) The following characteristics are given if $h, g: \mathbb{T} \to \mathbb{R}$ are proportional delta differentiable at the $t \in \mathbb{T}^k$ point:
(i) $\mathfrak{D}^{\gamma}[\gamma h + \theta g] = \gamma \mathfrak{D}^{\gamma} h + \theta \mathfrak{D}^{\gamma} g$, all $\gamma, \theta \in \mathbb{R}$;

(ii) $\mathfrak{D}^{\gamma}[hg] = h^{\sigma} \mathfrak{D}^{\gamma} g + g \mathfrak{D}^{\gamma} \mathrm{h} - h^{\sigma} g \kappa_1(\gamma,.)$;

(iii) $\mathfrak{D}^{\gamma}\left[\frac{1}{g}\right] = -\frac{\mathfrak{D}^{\gamma}[g]}{g.g^{\sigma}} + \left(\frac{1}{g} + \frac{1}{g^{\sigma}}\right)\kappa_1, \quad gg^{\sigma} \neq 0$;

(iv) $\mathfrak{D}^{\gamma}\left[\frac{h}{g}\right] = \frac{g^{\sigma}\mathfrak{D}^{\gamma}\mathrm{h} - h\mathfrak{D}^{\gamma} g}{g.g^{\sigma}} + \frac{h^{\sigma}}{g^{\sigma}}\kappa_1(\gamma,.), \quad gg^{\sigma} \neq 0.$

Definition 7 (Segi Rahmat, 2019) Let $\gamma \in [0,1]$. If the condition

$$1 + \frac{p(\zeta) - \kappa_1(\gamma,\zeta)}{\kappa_0(\gamma,\zeta)}\mu(\zeta) \neq 0, \quad \text{all } \zeta \in \mathbb{T}^k,$$

holds, $p: \mathbb{T} \to \mathbb{R}$ is considered to be γ-regressive.

The whole collection of rd-continuous and γ-regressive functions on $\mathbb{T}$ is represented by $\mathcal{R}_{\gamma} = \mathcal{R}_{\gamma}(\mathbb{T})$.

Definition 8 (Segi Rahmat, 2019) Let $\gamma \in (0,1]$ and $p \in \mathcal{R}_{\gamma}$. Suppose that p/κ_0, κ_1/κ_0 delta integrable functions on $\mathbb{T}$. The proportional exponential function on $\mathbb{T}$ for operator $\mathfrak{D}^{\gamma}$ is defined by

$$\tilde{e}_p(t,s)=\exp\left[\int_s^t \frac{1}{\mu(\zeta)} Log\left(1+\frac{p(\zeta)-\kappa_1(\gamma,\zeta)}{\kappa_0(\gamma,\zeta)}\mu(\zeta)\right)\Delta\zeta\right], \tag{7}$$
$$\tilde{e}_0(t,s)=\exp\left[\int_s^t \frac{1}{\mu(\zeta)} Log\left(1-\frac{\kappa_1(\gamma,\zeta)}{\kappa_0(\gamma,\zeta)}\mu(\zeta)\right)\Delta\zeta\right],\ s,t\in\mathbb{T},$$

where Log is the basic logarithm function. For $\mu(t)=0$,

$$\tilde{e}_p(t,s)=\exp\left[\int_s^t\left(\frac{p(\zeta)-\kappa_1(\gamma,\zeta)}{\kappa_0(\gamma,\zeta)}\right)\Delta\zeta\right],\ \tilde{e}_0(t,s)=\exp\left[-\int_s^t \frac{\kappa_1(\gamma,\zeta)}{\kappa_0(\gamma,\zeta)}\Delta\zeta\right]. \tag{8}$$

Lemma 9 (Segi Rahmat, 2019) Let $\gamma\in(0,1]$ and $p\in\mathcal{R}_\gamma$. For fixed $s\in\mathbb{T}$,

$$\mathfrak{D}^\gamma[\tilde{e}_p(.,s)]=p(t)\tilde{e}_p(.,s),$$

and for the proportional exponential function $\tilde{e}_0$,

$$\mathfrak{D}^\gamma\left[\int_a^t \frac{h(\zeta)\tilde{e}_0(t,\sigma(\zeta))}{\kappa_0(\gamma,\zeta)}\Delta\zeta\right]=h(t). \tag{9}$$

Definition 10 (Segi Rahmat, 2019) Assume that $h\in C_{rd}(\mathbb{R})$, $\gamma\in(0,1]$, and $t_0\in\mathbb{T}$. According to (7),

$$\int \mathfrak{D}^\gamma h(\zeta)\Delta_\gamma\zeta=h(t)+c\tilde{e}_0(t,t_0),\ \forall t\in\mathbb{T}, c\in\mathbb{R},$$

defines the indefinite proportional integral (anti derivative), and according to Lemma 9

$$\int_a^t h(\zeta)\tilde{e}_0(t,\sigma(\zeta))\,\Delta_\gamma\zeta=\int_a^t \frac{h(\zeta)\tilde{e}_0(t,\sigma(\zeta))}{\kappa_0(\boldsymbol{\gamma},\zeta)}\Delta\zeta, \qquad \Delta_\gamma\zeta=\frac{1}{\kappa_0(\gamma,\zeta)}\Delta\zeta, \tag{10}$$

denotes the definite proportional integral of h on $[a,b]_\mathbb{T}$.

Lemma 11 (Segi Rahmat, 2019) Let $\gamma\in(0,1]$, $h\in C_{rd}(\mathbb{R})$. Then,

$$\mathfrak{D}^\gamma\left[\int_a^t h(\zeta)\tilde{e}_0(t,\sigma(\zeta))\,\Delta_\gamma\zeta\right]=h(t). \tag{11}$$

Lemma 12 (Segi Rahmat, 2019) If $h,g\in\Omega(\mathbb{T})$,

(i) $\int_a^t \mathfrak{D}^\gamma[h(\zeta)]\ \tilde{e}_0(t,\sigma(\zeta))\,\Delta_\gamma\zeta=[h(\zeta)\ \tilde{e}_0(t,\sigma(\zeta))]_{\zeta=a}^t$.

(ii) $\int_a^b h(\zeta)\,\mathfrak{D}^\gamma[g(\zeta)]\,\tilde{e}_0(b,\sigma(\zeta))\Delta_\gamma\zeta=\left[h(\zeta)g(\zeta)\ \tilde{e}_0(b,\sigma(\zeta))\right]_{\zeta=a}^b$

$$-\int_a^b g^\sigma(\zeta)\{\mathfrak{D}^\gamma[h(\zeta)]-\kappa_1(\gamma,\zeta)h(\zeta)\}\tilde{e}_0(b,\sigma(\zeta))\,\Delta_\gamma\zeta.$$

Theorem 13 (Anderson & Ulness, 2015) Let $p\in C_{rd}(\mathbb{T})\cap\mathcal{R}_\gamma$, $q\in C_{rd}(\mathbb{T})$, $t_0\in\mathbb{T}$, and $y_0\in\mathbb{R}$. The solution of the initial value problem

$$\mathfrak{D}^\gamma y=p(t)y+q(t),\ \ y(t_0)=y_0,$$

is given by

$$y(t)=y_0\ \tilde{e}_p(t,t_0)+\int_{t_0}^t q(\tau)\tilde{e}_g(\sigma(\tau),t)\Delta_\gamma\tau,\ \ t\in\mathbb{T}^k, \tag{12}$$

where $g=\frac{(p-k_1)(\mu k_1-k_0)}{k_0+\mu(p-k_1)}$.

Think about the linear proportional dynamic equation

$$(\mathfrak{D}^\gamma)^3y+a(t)(\mathfrak{D}^\gamma)^2y+b(t)(\mathfrak{D}^\gamma y)+c(t)y=F(t),\ \ t\in\mathbb{T}^{k^2}, \tag{13}$$

where $a, b, c \in C_{rd}(\mathbb{T})$.

Definition 14 (Anderson & Georgiev, 2020) The function $y \in C_{rd}^2(\mathbb{T})$ providing the Eq. (13) is referred to as the solution of the equation.

Theorem 15 (Anderson & Georgiev, 2020) Suppose that the solutions to Eq. (13) are y_1, y_2, and y_3. The solution to Eq. (13) for $p, q, r \in \mathbb{R}$ is thus $py_1 + qy_2 + ry_3$.

Definition 16 (Anderson & Georgiev, 2020) Any functions $y_1, y_2, y_3 \in C_{rd}^1(\mathbb{T})$ have a proportional Wronskian with definition

$$W(y_1, y_2, y_3)=\begin{vmatrix} y_1 & y_2 & y_3 \\ \mathfrak{D}^\gamma y_1 & \mathfrak{D}^\gamma y_2 & \mathfrak{D}^\gamma y_2 \\ (\mathfrak{D}^\gamma)^2 y_1 & (\mathfrak{D}^\gamma)^2 y_2 & (\mathfrak{D}^\gamma)^2 y_3 \end{vmatrix}. \tag{14}$$

Definition 17 (Anderson & Georgiev, 2020) The solutions y_1, y_2, and y_3 of (13) are referred to as the fundamental solution set for (13) if the condition

$$W(y_1, y_2, y_3)(t) \neq 0,$$

is true for any $t \in \mathbb{T}^k$.

3. Results and Discussions

In this section, the formula of variation of parameters for the third-order linear nonhomogeneous dynamic equation is found, an example is given related to this method, and then a different expression of Wronskian is demonstrated for the two solutions.

Theorem 18 Think about the linear proportional dynamic equation

$$(\mathfrak{D}^\gamma)^3 y + a(t)(\mathfrak{D}^\gamma)^2 y + b(t)(\mathfrak{D}^\gamma y) + c(t)y = F(t), \tag{15}$$

where $a, b, c, f \in C_{rd}(\mathbb{T})$. Assume that the basic solutions to the associated homogeneous equation

$$(\mathfrak{D}^\gamma)^3 y + a(t)(\mathfrak{D}^\gamma)^2 y + b(t)(\mathfrak{D}^\gamma y) + c(t)y = 0, \tag{16}$$

are y_1, y_2 and y_3. Eq. (15) in this situation has a solution of

$$\begin{aligned} y(t) &= c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) \\ &+ \left(\tilde{e}_{\kappa_1}(t, t_0) + \int_{t_0}^{t} F(s) \frac{y_2^\sigma(s)(\mathfrak{D}^\gamma y_3)^\sigma(s) - y_3^\sigma(s)(\mathfrak{D}^\gamma y_2)^\sigma(s)}{(W(y_1, y_2, y_3))^\sigma(s)} \tilde{e}_0(\sigma(s), t) \Delta_{\gamma,t} s\right) y_1(t) \\ &+ \left(\tilde{e}_{\kappa_1}(t, t_0) - \int_{t_0}^{t} F(s) \frac{y_1^\sigma(s)(\mathfrak{D}^\gamma y_3)^\sigma(s) - y_3^\sigma(s)(\mathfrak{D}^\gamma y_1)^\sigma(s)}{(W(y_1, y_2, y_3))^\sigma(s)} \tilde{e}_0(\sigma(s), t) \Delta_{\gamma,t} s\right) y_2(t) \\ &+ \left(\tilde{e}_{\kappa_1}(t, t_0) + \int_{t_0}^{t} F(s) \frac{y_1^\sigma(s)(\mathfrak{D}^\gamma y_2)^\sigma(s) - y_2^\sigma(s)(\mathfrak{D}^\gamma y_1)^\sigma(s)}{(W(y_1, y_2, y_3))^\sigma(s)} \tilde{e}_0(\sigma(s), t) \Delta_{\gamma,t} s\right) y_3(t), \qquad t \in \mathbb{T}^{k^2}, \end{aligned} \tag{17}$$

where c_1, c_2, and c_3 are constants.

Proof Let y_1, y_2, y_3 be the fundamental set of solutions to the homogeneous Eq. (16). Investigate the version

$$y(t) = p(t)y_1(t) + q(t)y_2(t) + r(t)y_3(t), \tag{18}$$

of the solution to Eq. (15).

$$\begin{aligned} \mathfrak{D}^\gamma y(t) &= (\mathfrak{D}^\gamma p)y_1^\sigma + p(\mathfrak{D}^\gamma y_1) - \kappa_1 p y_1^\sigma + (\mathfrak{D}^\gamma q)y_2^\sigma + q(\mathfrak{D}^\gamma y_2) \\ &\quad - \kappa_1 q y_2^\sigma + (\mathfrak{D}^\gamma r)y_3^\sigma + r(\mathfrak{D}^\gamma y_3) - \kappa_1 r y_3^\sigma \end{aligned}$$

$$=p(t)(\mathfrak{D}^\gamma y_1)+q(t)(\mathfrak{D}^\gamma y_2)+r(t)(\mathfrak{D}^\gamma y_3), \tag{19}$$

is deduced from the proportional derivative's product rule. Here, it is assumed that

$$(\mathfrak{D}^\gamma p)y_1^\sigma+(\mathfrak{D}^\gamma q)y_2^\sigma+(\mathfrak{D}^\gamma r)y_3^\sigma=\kappa_1 p y_1^\sigma+\kappa_1 q y_2^\sigma+\kappa_1 r y_3^\sigma. \tag{20}$$

Similarly, it is obtained

$$\begin{aligned}(\mathfrak{D}^\gamma)^2 y&=p((\mathfrak{D}^\gamma)^2 y_1)+(\mathfrak{D}^\gamma p)(\mathfrak{D}^\gamma y_1)^\sigma-\kappa_1 p(\mathfrak{D}^\gamma y_1)^\sigma+q((\mathfrak{D}^\gamma)^2 y_2)+\ (\mathfrak{D}^\gamma q)(\mathfrak{D}^\gamma y_2)^\sigma\\ &-\kappa_1 q(\mathfrak{D}^\gamma y_2)^\sigma+r((\mathfrak{D}^\gamma)^2 y_3)+(\mathfrak{D}^\gamma r)(\mathfrak{D}^\gamma y_3)^\sigma-\kappa_1 r(\mathfrak{D}^\gamma y_3)^\sigma\\ &=p((\mathfrak{D}^\gamma)^2 y_1)+q((\mathfrak{D}^\gamma)^2 y_2)+r((\mathfrak{D}^\gamma)^2 y_3),\end{aligned} \tag{21}$$

by the assumption

$$(\mathfrak{D}^\gamma p)(\mathfrak{D}^\gamma y_1)^\sigma+(\mathfrak{D}^\gamma q)(\mathfrak{D}^\gamma y_2)^\sigma+(\mathfrak{D}^\gamma r)(\mathfrak{D}^\gamma y_3)^\sigma=\kappa_1 p(\mathfrak{D}^\gamma y_1)^\sigma+\kappa_1 q(\mathfrak{D}^\gamma y_2)^\sigma+\kappa_1 r(\mathfrak{D}^\gamma y_3)^\sigma. \tag{22}$$

The third order proportional derivative of is

$$\begin{aligned}(\mathfrak{D}^\gamma)^3 y &= \mathfrak{D}^\gamma(p(\mathfrak{D}^\gamma)^2 y_1)+\mathfrak{D}^\gamma(q(\mathfrak{D}^\gamma)^2 y_2)+\mathfrak{D}^\gamma(r(\mathfrak{D}^\gamma)^2 y_3)\\ &= p((\mathfrak{D}^\gamma)^3 y_1)+(\mathfrak{D}^\gamma p)((\mathfrak{D}^\gamma)^2 y_1)^\sigma-\kappa_1 p((\mathfrak{D}^\gamma)^2 y_1)^\sigma+q((\mathfrak{D}^\gamma)^3 y_2)+(\mathfrak{D}^\gamma q)((\mathfrak{D}^\gamma)^2 y_2)^\sigma\\ &-\kappa_1 q((\mathfrak{D}^\gamma)^2 y_2)^\sigma+r((\mathfrak{D}^\gamma)^3 y_3)+(\mathfrak{D}^\gamma r)((\mathfrak{D}^\gamma)^2 y_3)^\sigma-\kappa_1 r((\mathfrak{D}^\gamma)^2 y_3)^\sigma.\end{aligned} \tag{23}$$

If the formulae (18), (19), (21), and (23) are inserted in the Eq. (15), accounting for the assumptions of (20), and (22), the equation

$$\begin{aligned}(\mathfrak{D}^\gamma)^3 y+a(t)(\mathfrak{D}^\gamma)^2 y+b(t)(\mathfrak{D}^\gamma y)+c(t)y &= p(t)[(\mathfrak{D}^\gamma)^3 y_1+a(t)(\mathfrak{D}^\gamma)^2 y_1+b(t)(\mathfrak{D}^\gamma y_1)+c(t)y_1]\\ &+q(t)[(\mathfrak{D}^\gamma)^3 y_2+a(t)(\mathfrak{D}^\gamma)^2 y_2+b(t)(\mathfrak{D}^\gamma y_2)+c(t)y_2]\\ &+r(t)[(\mathfrak{D}^\gamma)^3 y_3+a(t)(\mathfrak{D}^\gamma)^2 y_3+b(t)(\mathfrak{D}^\gamma y_3)+c(t)y_3]\\ &+(\mathfrak{D}^\gamma p)((\mathfrak{D}^\gamma)^2 y_1)^\sigma-\kappa_1 p((\mathfrak{D}^\gamma)^2 y_1)^\sigma+(\mathfrak{D}^\gamma q)((\mathfrak{D}^\gamma)^2 y_2)^\sigma\\ &-\kappa_1 q((\mathfrak{D}^\gamma)^2 y_2)^\sigma+(\mathfrak{D}^\gamma r)((\mathfrak{D}^\gamma)^2 y_3)^\sigma-\kappa_1 r((\mathfrak{D}^\gamma)^2 y_3)^\sigma\\ &= F(t),\end{aligned}$$

is discovered. We have recently found the system

$$\begin{aligned}&(\mathfrak{D}^\gamma p)y_1^\sigma+(\mathfrak{D}^\gamma q)y_2^\sigma+(\mathfrak{D}^\gamma r)y_3^\sigma = \kappa_1 p y_1^\sigma+\kappa_1 q y_2^\sigma+\kappa_1 r y_3^\sigma,\\ &(\mathfrak{D}^\gamma p)(\mathfrak{D}^\gamma y_1)^\sigma+(\mathfrak{D}^\gamma q)(\mathfrak{D}^\gamma y_2)^\sigma+(\mathfrak{D}^\gamma r)(\mathfrak{D}^\gamma y_3)^\sigma = \kappa_1 p(\mathfrak{D}^\gamma y_1)^\sigma+\kappa_1 q(\mathfrak{D}^\gamma y_2)^\sigma+\kappa_1 r(\mathfrak{D}^\gamma y_3)^\sigma,\\ &(\mathfrak{D}^\gamma p)((\mathfrak{D}^\gamma)^2 y_1)^\sigma+(\mathfrak{D}^\gamma q)((\mathfrak{D}^\gamma)^2 y_2)^\sigma+(\mathfrak{D}^\gamma r)((\mathfrak{D}^\gamma)^2 y_3)^\sigma =\\ &\qquad = \kappa_1 p((\mathfrak{D}^\gamma)^2 y_1)^\sigma+\kappa_1 q((\mathfrak{D}^\gamma)^2 y_2)^\sigma+\kappa_1 r((\mathfrak{D}^\gamma)^2 y_3)^\sigma+F(t).\end{aligned}$$

We derive

$$\begin{cases}\mathfrak{D}^\gamma p(t) = \kappa_1(\alpha,t)p(t)+F(t)\dfrac{y_2^\sigma(t)(\mathfrak{D}^\gamma y_3)^\sigma-y_3^\sigma(t)(\mathfrak{D}^\gamma y_2)^\sigma}{(W(y_1,y_2,y_3))^\sigma(t)},\\ \mathfrak{D}^\gamma q(t) = \kappa_1(\alpha,t)q(t)-F(t)\dfrac{y_1^\sigma(t)(\mathfrak{D}^\gamma y_3)^\sigma-y_3^\sigma(t)(\mathfrak{D}^\gamma y_1)^\sigma}{(W(y_1,y_2,y_3))^\sigma(t)},\\ \mathfrak{D}^\gamma r(t) = \kappa_1(\alpha,t)r(t)+F(t)\dfrac{y_1^\sigma(t)(\mathfrak{D}^\gamma y_2)^\sigma-y_2^\sigma(t)(\mathfrak{D}^\gamma y_1)^\sigma}{(W(y_1,y_2,y_3))^\sigma(t)},\quad t\in\mathbb{T}^{k^2},\end{cases}$$

from the previous system.

$$\begin{cases} p(t) = \tilde{e}_{\kappa_1}(t,t_0) + \int_{t_0}^{t} F(s) \frac{y_2^\sigma(s)(\mathfrak{D}^\gamma y_3)^\sigma(s) - y_3^\sigma(s)(\mathfrak{D}^\gamma y_2)^\sigma(s)}{(W(y_1,y_2,y_3))^\sigma(s)} \tilde{e}_0(\sigma(s),t)\Delta_{\gamma,t}s, \\ q(t) = \tilde{e}_{\kappa_1}(t,t_0) - \int_{t_0}^{t} F(s) \frac{y_1^\sigma(s)(\mathfrak{D}^\gamma y_3)^\sigma(s) - y_3^\sigma(s)(\mathfrak{D}^\gamma y_1)^\sigma(s)}{(W(y_1,y_2,y_3))^\sigma(s)} \tilde{e}_0(\sigma(s),t)\Delta_{\gamma,t}s, \\ r(t) = \tilde{e}_{\kappa_1}(t,t_0) + \int_{t_0}^{t} F(s) \frac{y_1^\sigma(s)(\mathfrak{D}^\gamma y_2)^\sigma(s) - y_2^\sigma(s)(\mathfrak{D}^\gamma y_1)^\sigma(s)}{(W(y_1,y_2,y_3))^\sigma(s)} \tilde{e}_0(\sigma(s),t)\Delta_{\gamma,t}s, \quad t \in \mathbb{T}^{k^2}, \end{cases}$$

functions are identified using the formula (12). As a result,

$$\begin{aligned} y(t) &= c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) \\ &+ \left(\tilde{e}_{\kappa_1}(t,t_0) + \int_{t_0}^{t} F(s) \frac{y_2^\sigma(s)(\mathfrak{D}^\gamma y_3)^\sigma(s) - y_3^\sigma(s)(\mathfrak{D}^\gamma y_2)^\sigma(s)}{(W(y_1,y_2,y_3))^\sigma(s)} \tilde{e}_0(\sigma(s),t)\Delta_{\gamma,t}s \right) y_1(t) \\ &+ \left(\tilde{e}_{\kappa_1}(t,t_0) - \int_{t_0}^{t} F(s) \frac{y_1^\sigma(s)(\mathfrak{D}^\gamma y_3)^\sigma(s) - y_3^\sigma(s)(\mathfrak{D}^\gamma y_1)^\sigma(s)}{(W(y_1,y_2,y_3))^\sigma(s)} \tilde{e}_0(\sigma(s),t)\Delta_{\gamma,t}s \right) y_2(t) \\ &+ \left(\tilde{e}_{\kappa_1}(t,t_0) + \int_{t_0}^{t} F(s) \frac{y_1^\sigma(s)(\mathfrak{D}^\gamma y_2)^\sigma(s) - y_2^\sigma(s)(\mathfrak{D}^\gamma y_1)^\sigma(s)}{(W(y_1,y_2,y_3))^\sigma(s)} \tilde{e}_0(\sigma(s),t)\Delta_{\gamma,t}s \right) y_3(t), t \in \mathbb{T}^{k^2}, \end{aligned}$$

is the Eq. (15)'s general solution.

Theorem 19 While y_1 and y_2 are the solutions of

$$(\mathfrak{D}^\gamma)^2 y + a(t)\mathfrak{D}^\gamma y + b(t)y = 0, \quad t \in \mathbb{T}^{k^2},$$

where $a, b \in C_{rd}(\mathbb{T})$,

$$W^\sigma(y_1,y_2) = \left[\left(1 - \frac{\mu\kappa_1}{\kappa_0}\right)\left(1 - \frac{\mu\kappa_1}{\kappa_0} - \frac{a\mu}{\kappa_0}\right) + \frac{b\mu^2}{\kappa_0^2}\right] W(y_1,y_2), \tag{24}$$

is provided.

Proof Given that

$$W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ \mathfrak{D}^\gamma y_1 & \mathfrak{D}^\gamma y_2 \end{vmatrix} = y_1 \mathfrak{D}^\gamma y_2 - y_2 \mathfrak{D}^\gamma y_1,$$

according to the Wronskian definition for two solutions,

$$W^\sigma(y_1,y_2) = \begin{vmatrix} y_1^\sigma & y_2^\sigma \\ (\mathfrak{D}^\gamma y_1)^\sigma & (\mathfrak{D}^\gamma y_2)^\sigma \end{vmatrix} = {y_1}^\sigma (\mathfrak{D}^\gamma y_2)^\sigma - {y_2}^\sigma (\mathfrak{D}^\gamma y_1)^\sigma,$$

and

$$h^\sigma = \frac{\mu\mathfrak{D}^\gamma h + \kappa_0 h - \kappa_1 \mu h}{\kappa_0} = h + \frac{\mu}{\kappa_0}(\mathfrak{D}^\gamma h - \kappa_1 h),$$

it is derived that

$$\begin{aligned} W^\sigma(y_1,y_2) &= {y_1}^\sigma \left(\mathfrak{D}^\gamma y_2 + \frac{\mu}{\kappa_0}((\mathfrak{D}^\gamma)^2 y_2 - \kappa_1 \mathfrak{D}^\gamma y_2) \right) - {y_2}^\sigma \left(\mathfrak{D}^\gamma y_1 + \frac{\mu}{\kappa_0}((\mathfrak{D}^\gamma)^2 y_1 - \kappa_1 \mathfrak{D}^\gamma y_1) \right) \\ &= \left(1 - \frac{\mu\kappa_1}{\kappa_0}\right)({y_1}^\sigma \mathfrak{D}^\gamma y_2 - {y_2}^\sigma \mathfrak{D}^\gamma y_1) + \frac{\mu}{\kappa_0}({y_1}^\sigma (\mathfrak{D}^\gamma)^2 y_2 - {y_2}^\sigma (\mathfrak{D}^\gamma)^2 y_1). \end{aligned}$$

Given that

$$\begin{vmatrix} y_1^\sigma & y_2^\sigma \\ \mathfrak{D}^\gamma y_1 & \mathfrak{D}^\gamma y_2 \end{vmatrix} = \left(\frac{\kappa_0 - \kappa_1\mu}{\kappa_0}\right) W(y_1, y_2), \qquad \mathfrak{D}^\gamma W(y_1, y_2) = \begin{vmatrix} y_1^\sigma & y_2^\sigma \\ (\mathfrak{D}^\gamma)^2 y_1 & (\mathfrak{D}^\gamma)^2 y_2 \end{vmatrix} - \frac{\kappa_1(\kappa_0-\mu\kappa_1)}{\kappa_0} W(y_1, y_2)\,,$$

([33], theorem 7.1.8, 7.1.10) in this situation,

$$W^\sigma(y_1, y_2) = \left(1 - \frac{\mu\kappa_1}{\kappa_0}\right)^2 W(y_1, y_2) + \frac{\mu}{\kappa_0}\left(\mathfrak{D}^\gamma W(y_1, y_2) + \frac{\kappa_1(\kappa_0-\mu\kappa_1)}{\kappa_0} W(y_1, y_2)\right).$$

is the outcome. On the other hand, according to ([33], theorem 7.1.12) since

$$\mathfrak{D}^\gamma W(y_1, y_2) = -\left(\frac{\mathrm{a}(\kappa_0 - \mu\kappa_1)}{\kappa_0} + \frac{\kappa_1(\kappa_0 - \mu\kappa_1)}{\kappa_0} - \frac{b\mu}{\kappa_0}\right) W(y_1, y_2),$$

the eventual result is

$$W^\sigma(y_1, y_2) = \left[\frac{(\kappa_0 - \mu\kappa_1)(\kappa_0 - \mu\kappa_1 - a\mu) + \mathrm{b}\mu^2}{{\kappa_0}^2}\right] W(y_1, y_2)$$

$$= \left[\left(1 - \frac{\mu\kappa_1}{\kappa_0}\right)\left(1 - \frac{\mu\kappa_1}{\kappa_0} - \frac{a\mu}{\kappa_0}\right) + \frac{b\mu^2}{\kappa_0^2}\right] W(y_1, y_2).$$

Example 20 Assume that $\mathbb{T} = \mathbb{Z}$, $\kappa_1(\gamma, t) = (1-\gamma)t^{2\gamma}, \kappa_0(\gamma, t) = \gamma t^{2(1-\gamma)}$, $\gamma \in (0, 1]$, $t \in \mathbb{T}$. Take into account the following differential equation

$$\left(\mathfrak{D}^{\frac{1}{3}}\right)^3 y - \mathfrak{D}^{\frac{1}{3}}\mathrm{y}=2t. \tag{25}$$

Firstly, we will find the solution of the corresponding homogeneous equation of Eq. (25)

$$\left(\mathfrak{D}^{\frac{1}{3}}\right)^3 y - \mathfrak{D}^{\frac{1}{3}}\mathrm{y}=0, \tag{26}$$

by using the method in Anderson & Georgiev (2020). The auxiliary formula for (26) is

$$\lambda^3 - \lambda = 0, \tag{27}$$

and from here the roots are easily found as $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = 1$.

We may reformat Eq. (26) as follows:

$$\mathfrak{D}^{\frac{1}{3}}\left(\mathfrak{D}^{\frac{1}{3}} + 1\right)\left(\mathfrak{D}^{\frac{1}{3}} - 1\right) = 0. \tag{28}$$

Taking

$$\left(\mathfrak{D}^{\frac{1}{3}} + 1\right)\left(\mathfrak{D}^{\frac{1}{3}} - 1\right) y = y_1, \tag{29}$$

we form

$$\mathfrak{D}^{\frac{1}{3}} y_1 = 0,$$

and from theorem 13, the solution of this equation is

$$y_1(t) = c_1\, \tilde{e}_0(t, t_0), \tag{30}$$

where c_1 is a constant. Now we get

$$\left(\mathfrak{D}^{\frac{1}{3}} - 1\right) y = y_2, \tag{31}$$

and from Eq. (28) we form

$$\left(\mathfrak{D}^{\frac{1}{3}} + 1\right) y_2 = y_1(t), \tag{32}$$

or

$$\mathfrak{D}^{\frac{1}{3}} y_2 = -y_2 + y_1(t),$$

and using theorem (13), we find its solution as

$$y_2(t)=c_2\tilde{e}_{-1}(t,t_0)+\int_{t_0}^{t} y_1(\tau)\tilde{e}_{g_1}(\tau+1,t)\varDelta_{\frac{1}{3}}\tau\,,\qquad t\in\mathbb{T}^k, \tag{33}$$

where $g_1=\frac{(-1-\frac{2}{3}t^{\frac{2}{3}})(\frac{2}{3}t^{\frac{2}{3}}-\frac{1}{3}t^{\frac{4}{3}})}{\frac{1}{3}t^{\frac{4}{3}}-1-\frac{2}{3}t^{\frac{2}{3}}}$ and c_2 is a constant. From Eq. (31) considering the formula (33) and theorem (13) it can be easily obtained that

$$y(t)=c_3\tilde{e}_1(t,t_0)+\int_{t_0}^{t} y_2(\tau)\tilde{e}_{g_2}(\tau+1,t)\varDelta_{\frac{1}{3}}\tau\,,\qquad t\in\mathbb{T}^k, \tag{34}$$

where $g_2=\frac{(1-\frac{2}{3}t^{\frac{2}{3}})(\frac{2}{3}t^{\frac{2}{3}}-\frac{1}{3}t^{\frac{4}{3}})}{\frac{1}{3}t^{\frac{4}{3}}+1-\frac{2}{3}t^{\frac{2}{3}}}$ and c_3 is a constant. If we substitute the solutions (30) and (33) in the solution (34), we obtain the solution of the homogeneous Eq. (26) as

$$y(t)=c_1\int_{t_0}^{t}\int_{t_0}^{s}\tilde{e}_0(\tau,t_0)\tilde{e}_{g_1}(\tau+1,t)\tilde{e}_{g_2}(s+1,t)\varDelta_{\frac{1}{3}}\tau\varDelta_{\frac{1}{3}}s+c_2\int_{t_0}^{t}\tilde{e}_{-1}(\tau,t_0)\tilde{e}_{g_2}(s+1,t)\varDelta_{\frac{1}{3}}\tau\;+c_3\tilde{e}_1(t,t_0).$$

Thus if we compare the formula (17), we discovered that

$y_1(t)=\int_{t_0}^{t}\int_{t_0}^{s}\tilde{e}_0(\tau,t_0)\tilde{e}_{g_1}(\tau+1,t)\tilde{e}_{g_2}(s+1,t)\varDelta_{\frac{1}{3}}\tau\varDelta_{\frac{1}{3}}s,$

$y_2(t)=\int_{t_0}^{t}\tilde{e}_{-1}(\tau,t_0)\tilde{e}_{g_2}(s+1,t)\varDelta_{\frac{1}{3}}\tau,$

$y_3(t)=\tilde{e}_1(t,t_0).$

So, it is possible to find the general solution of the given problem by while keeping in mind that these formulae and $\mathrm{F(t)}=2\mathrm{t}$ in formula (17):

$$\begin{aligned}
y(t)=&c_1y_1(t)+c_2y_2(t)+c_3y_3(t)\\
&+\left(\tilde{e}_{\frac{2}{3}t^{\frac{2}{3}}}(t,t_0)+\int_{t_0}^{t}2s\,\frac{y_2(s+1)\mathfrak{D}^{\frac{1}{3}}y_3(s+1)-y_3(s+1)\mathfrak{D}^{\frac{1}{3}}y_2(s+1)}{W(y_1,y_2,y_3)(s+1)}\tilde{e}_0(s+1,t)\varDelta_{\frac{1}{3},t}s\right)y_1(t)\\
&+\left(\tilde{e}_{\frac{2}{3}t^{\frac{2}{3}}}(t,t_0)-\int_{t_0}^{t}2s\,\frac{y_1(s+1)\mathfrak{D}^{\frac{1}{3}}y_3(s+1)-y_3(s+1)\mathfrak{D}^{\frac{1}{3}}y_1(s+1)}{W(y_1,y_2,y_3)(s+1)}\tilde{e}_0(s+1,t)\varDelta_{\frac{1}{3},t}s\right)y_2(t)\\
&+\left(\tilde{e}_{\frac{2}{3}t^{\frac{2}{3}}}(t,t_0)+\int_{t_0}^{t}2s\,\frac{y_1(s+1)\mathfrak{D}^{\frac{1}{3}}y_2(s+1)-y_2(s+1)\mathfrak{D}^{\frac{1}{3}}y_1(s+1)}{W(y_1,y_2,y_3)(s+1)}\tilde{e}_0(s+1,t)\varDelta_{\frac{1}{3},t}s\right)y_3(t).
\end{aligned}$$

4. Conclusion

The variation of parameters was examined using the proportional derivative as a general example of a conformable derivative for the third-order linear nonhomogeneous dynamic equation, and an example was given on time scales with the special choice of the functions $\kappa_0(\gamma,t)$ and $\kappa_1(\gamma,t)$.

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