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New Sequence Spaces Derived From the Composition of Integrated and Differentiated Spaces and Binomial Matrix

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Abstract. In this article, we construct new sequence spaces by combining the integrated and differentiated sequence spaces with the binomial matrix. We first construct the properties of these new sequence spaces and we examine some inclusion relations. Furthermore, we determine $\alpha-$, $\beta-$ and $\gamma-$ duals of the integrated and differentiated sequence spaces separately and provide proofs for some of them. Additionally, we characterize some matrix classes associated with these new sequence spaces, along with the obtained results. Finally, we investigate some geometric properties of new integrated sequence spaces.

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1. Introduction

Let w be the set of all real (or complex) valued sequences. w is a vector space under scalar multiplication and pointwise addition. Each vector subspace of w is called a sequence space. ℓ_∞, c_0 and c are symbolic of all bounded, null and convergent sequence spaces, respectively.

If each of the transformations $q_k : X \to \mathbb{C}$ defined by $q_k(x) = x_k$ is continuous for $\forall k \in \mathbb{N}$, then a Banach sequence space is called a BK-space [7]. Hence, we can say that the sequece spaces ℓ_∞, c_0 and c are BK-spaces with the sup-norm defined by

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} |x_n| .$$

Let T and U be two sequence spaces, $x = (x_k) \in w$ and $B = (b_{nk})$ be an infinite matrix. In this case, the B-transform of x is assumed to be convergent for all $n \in \mathbb{N}$ and is defined by

$$(Bx)_n = \sum_k b_{nk} x_k,$$

where the entries of matrix B are complex numbers. Then, using the notation $(T : U)$, we denote the class of all infinite matrices from T to U and defined as

$$(T : U) = \{B = (b_{nk}) : Bt \in U \ for\ all\ t \in T\} .$$

The matrix domain of $B = (b_{nk})$ in T is determined as follows;

$$T_B = \{t = (t_k) \in w : Bt \in T\}$$

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and which is also a sequence space [22].

Now, we define the the matrix domain of the summation matrix $S = (s_{nk})$ such that;

$$s_{nk} = \begin{cases} 1 & , \quad 0 \leq k \leq n \\ 0 & , \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}$.

Thus, the sets of all bounded and convergent series are defined as $bs = (\ell_\infty)_s$ and $cs = c_s$, respectively.

An infinite matrix $B = (b_{nk})$ is named triangle matrix if $b_{nk} = 0$ for $k > n$ and $b_{nn} \neq 0$ for all $n, k \in \mathbb{N}$. The inverse of a triangle matrix is always exist. Also, this inverse is unique and triangle.

The integrated and differentiated sequence spaces was first used by Goes and Goes [12]. Recently, these sequence spaces and some of their properties have been investigated by Kirişçi [14–16].

Also, the domain of the binomial matrix and the binomial sequence spaces was first defined by Bişgin [2, 3]. Then; Bişgin [4–6], Meng [17], Sönmez [19] and Topal [21] conducted various studies using the binomial matrix.

2. New Sequence Spaces

The definition of the Binomial matrix $B^{u,v} = \left(b^{u,v}_{nk}\right)$ is as follows;

$$b^{u,v}_{nk} = \begin{cases} \frac{1}{(v+u)^n}\binom{n}{k} v^{n-k} u^k & , \quad 0 \leq k \leq n \\ 0 & , \quad k > n \end{cases}$$

for all $u, v \in \mathbb{R}$, $v \cdot u > 0$ and $n, k \in \mathbb{N}$. We assume that $v \cdot u > 0$ from now on unless otherwise stated.

The binomial sequence spaces $b^{u,v}_0$, $b^{u,v}_c$, $b^{u,v}_p$ and $b^{u,v}_\infty$ were first defined by Bişgin in [2] and [3] as follows;

$$b^{u,v}_0 = \left\{ y = (y_k) \in w : \lim_{n\to\infty} \frac{1}{(v+u)^n} \sum_{k=0}^{n} \binom{n}{k} v^{n-k} u^k y_k = 0 \right\},$$

$$b^{u,v}_c = \left\{ y = (y_k) \in w : \lim_{n\to\infty} \frac{1}{(v+u)^n} \sum_{k=0}^{n} \binom{n}{k} v^{n-k} u^k y_k \ \ exists \right\},$$

$$b^{u,v}_p = \left\{ y = (y_k) \in w : \sum_{n} \left| \frac{1}{(v+u)^n} \sum_{k=0}^{n} \binom{n}{k} v^{n-k} u^k y_k \right|^p < \infty \right\} \quad (1 \leq p < \infty)$$

and

$$b^{u,v}_\infty = \left\{ y = (y_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{1}{(v+u)^n} \sum_{k=0}^{n} \binom{n}{k} v^{n-k} u^k y_k \right| < \infty \right\}.$$

Then, the sequence spaces $b^{u,v}_0(\nabla)$, $b^{u,v}_c(\nabla)$ and $b^{u,v}_\infty(\nabla)$ derived by composition the binomial matrix and difference operator defined by Meng and Song in [17] as follows;

$$b^{u,v}_0(\nabla) = \left\{ y = (y_k) \in w : \lim_{n\to\infty} \frac{1}{(v+u)^n} \sum_{k=0}^{n} \binom{n}{k} v^{n-k} u^k (y_k - y_{k-1}) = 0 \right\},$$

$$b^{u,v}_c(\nabla) = \left\{ y = (y_k) \in w : \lim_{n\to\infty} \frac{1}{(v+u)^n} \sum_{k=0}^{n} \binom{n}{k} v^{n-k} u^k (y_k - y_{k-1}) \ \ exists \right\}$$

and

$$b^{u,v}_\infty(\nabla) = \left\{ y = (y_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{1}{(v+u)^n} \sum_{k=0}^{n} \binom{n}{k} v^{n-k} u^k (y_k - y_{k-1}) \right| < \infty \right\}.$$

Afterward, the sequence spaces $b^{u,v}_0(G)$, $b^{u,v}_c(G)$ and $b^{u,v}_\infty(G)$ derived by combining the binomial matrix and double band matrix defined by Sönmez in [19] as follows;

$$b_0^{u,v}(G) = \left\{ y = (y_k) \in w : \lim_{n\to\infty} \frac{1}{(v+u)^n} \sum_{k=0}^{n} \binom{n}{k} v^{n-k} u^k (ay_k + by_{k-1}) = 0 \right\},$$

$$b_c^{u,v}(G) = \left\{ y = (y_k) \in w : \lim_{n\to\infty} \frac{1}{(v+u)^n} \sum_{k=0}^{n} \binom{n}{k} v^{n-k} u^k (ay_k + by_{k-1}) \ exists \right\}$$

and

$$b_\infty^{u,v}(G) = \left\{ y = (y_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{1}{(v+u)^n} \sum_{k=0}^{n} \binom{n}{k} v^{n-k} u^k (ay_k + by_{k-1}) \right| < \infty \right\},$$

where double band matrix $G = (g_{nk})$ is defined as follows;

$$g_{nk} = \begin{cases} a & , \quad k = n \\ b & , \quad k = n-1 \\ 0 & , \quad o.w \end{cases}$$

for every $n, k \in \mathbb{N}$ and $a, b \in \mathbb{R} \setminus \{0\}$.

Afterward, the sequence spaces $b_0^{u,v}(D)$, $b_c^{u,v}(D)$ and $b_\infty^{u,v}(D)$ defined by Bişgin in [6] by combining the binomial matrix and triple band matrix as follows;

$$b_0^{u,v}(D) = \left\{ y = (y_k) \in w : \lim_{n\to\infty} \frac{1}{(v+u)^n} \sum_{k=0}^{n} \binom{n}{k} v^{n-k} u^k (ay_k + by_{k-1} + cy_{k-2}) = 0 \right\},$$

$$b_c^{u,v}(D) = \left\{ y = (y_k) \in w : \lim_{n\to\infty} \frac{1}{(v+u)^n} \sum_{k=0}^{n} \binom{n}{k} v^{n-k} u^k (ay_k + by_{k-1} + cy_{k-2}) \ exists \right\}$$

and

$$b_\infty^{u,v}(D) = \left\{ y = (y_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{1}{(v+u)^n} \sum_{k=0}^{n} \binom{n}{k} v^{n-k} u^k (ay_k + by_{k-1} + cy_{k-2}) \right| < \infty \right\},$$

where triple band matrix $D = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases} a & , \quad k = n \\ b & , \quad k = n-1 \\ c & , \quad k = n-2 \\ 0 & , \quad o.w \end{cases}$$

for all $n, k \in \mathbb{N}$ and $a, b, c \in \mathbb{R} \setminus \{0\}$.

Lastly, the sequence spaces $b_0^{u,v}(Q)$, $b_c^{u,v}(Q)$ and $b_\infty^{u,v}(Q)$ defined by Topal in [21] by using composition the binomial matrix and quadruple band matrix such that;

$$b_0^{u,v}(Q) = \left\{ y = (y_k) \in w : \lim_{n\to\infty} \frac{1}{(v+u)^n} \sum_{k=0}^{n} \binom{n}{k} v^{n-k} u^k (ay_k + by_{k-1} + cy_{k-2} + dy_{k-3}) = 0 \right\},$$

$$b_c^{u,v}(Q) = \left\{ y = (y_k) \in w : \lim_{n\to\infty} \frac{1}{(v+u)^n} \sum_{k=0}^{n} \binom{n}{k} v^{n-k} u^k (ay_k + by_{k-1} + cy_{k-2} + dy_{k-3}) \ exists \right\}$$

and

$$b_\infty^{u,v}(Q) = \left\{ y = (y_k) \in w : \sup_{n\in\mathbb{N}} \left| \frac{1}{(v+u)^n} \sum_{k=0}^{n} \binom{n}{k} v^{n-k} u^k (ay_k + by_{k-1} + cy_{k-2} + dy_{k-3}) \right| < \infty \right\},$$

where quadruple band matrix $Q = (q_{nk}(a, b, c, d))$ as follows:

$$q_{nk}(a,b,c,d)=\begin{cases} a & , \quad k=n \\ b & , \quad k=n-1 \\ c & , \quad k=n-2 \\ d & , \quad k=n-3 \\ 0 & , \quad o.w. \end{cases}$$

for all $n,k \in \mathbb{N}$ and $a,b,c,d \in \mathbb{R}\setminus\{0\}$.

Now, we define the matrix ((k + 1)I) such that;

$$\text{(k+1)I} = \begin{bmatrix} 1 & 0 & 0 \dots \\ 0 & 2 & 0 \dots \\ 0 & 0 & 3 \dots \\ \vdots & \vdots & \vdots \ddots \end{bmatrix}$$

where $k \in \mathbb{N}_0 = \{0,1,2,\dots\}$. Accordingly, Goes and Goes define the integrated and differentiated sequence spaces [12]

$$\int X = \{y=(y_k)\in w : ((k+1)y_k)\in X\} = X_{(k+1)\mathrm{I}}$$

and

$$dX = \left\{y=(y_k)\in w : \left(\left(\frac{1}{k+1}\right)y_k\right)\in X\right\} = X_{\left(\frac{1}{k+1}\right)\mathrm{I}},$$

where X is a sequence space and if we take $k=0$, we obtain $\int X = X$ and $dX = X$.

Now, we define the new sequence spaces by composition of the binomial matrix and the integrated and differentiated sequence spaces. Let $X \in \left\{b_c^{u,v}, b_0^{u,v}, b_\infty^{u,v}\right\}$.

If $X = b_c^{u,v}$, new sequence spaces written as follows;

$$\int b_c^{u,v} = (b_c^{u,v})_{(k+1)\mathrm{I}} = \left\{y=(y_k)\in w : \lim_{n\to\infty}\frac{1}{(v+u)^n}\sum_{k=0}^{n}\binom{n}{k}v^{n-k}u^k(k+1)y_k \ \ exists\right\} = [(c)_{B^{u,v}}]_{(k+1)\mathrm{I}}$$

and

$$db_c^{u,v} = (b_c^{u,v})_{\left(\frac{1}{k+1}\right)\mathrm{I}} = \left\{y=(y_k)\in w : \lim_{n\to\infty}\frac{1}{(v+u)^n}\sum_{k=0}^{n}\binom{n}{k}v^{n-k}u^k\left(\frac{1}{k+1}\right)y_k \ \ exists\right\} = [(c)_{B^{u,v}}]_{\left(\frac{1}{k+1}\right)\mathrm{I}}.$$

Also, by constructing a matrix $G^{u,v} = (g_{nk}^{u,v}) = B^{u,v}(k+1)\mathrm{I}$ so that;

$$g_{nk}^{u,v} = \begin{cases} \frac{1}{(v+u)^n}\binom{n}{k}v^{n-k}u^k(k+1) & , \quad 0\le k\le n \\ 0 & , \quad k>n \end{cases}$$

for all $n,k\in\mathbb{N}$. New integrated sequence spaces can be redefined by matrix $G^{u,v} = (g_{nk}^{u,v}) = B^{u,v}(k+1)\mathrm{I}$ as follows;

$$\int b_c^{u,v} = (c)_{G^{u,v}}, \int b_0^{u,v} = (c_0)_{G^{u,v}}, \int b_\infty^{u,v} = (\ell_\infty)_{G^{u,v}}. \tag{2.1}$$

So, for given $x=(x_k)\in w$, the $G^{u,v}$-transform of x is defined as follows:

$$y_k = (G^{u,v}x)_k = \frac{1}{(v+u)^k}\sum_{i=0}^{k}\binom{k}{i}v^{k-i}u^i(i+1)x_i,$$

where $\forall k\in\mathbb{N}$.

Similarly, by constructing a matrix $F^{u,v} = (f_{nk}^{u,v}) = B^{u,v}\left(\frac{1}{k+1}\right)\mathrm{I}$ so that;

$$f_{nk}^{u,v} = \begin{cases} \frac{1}{(v+u)^n}\binom{n}{k}v^{n-k}u^k\left(\frac{1}{k+1}\right) & ; \quad 0\le k\le n \\ 0 & , \quad k>n \end{cases}$$

for all $k\in\mathbb{N}$. The new differentiated sequence spaces can be redefined by the matrix $F^{u,v} = (f_{nk}^{u,v})$ as follows;

$$db_c^{u,v} = (c)_{F^{u,v}}, db_0^{u,v} = (c_0)_{F^{u,v}}, db_\infty^{u,v} = (\ell_\infty)_{F^{u,v}}. \tag{2.2}$$

So, for given $x = (x_k) \in w$, the $F^{u,v}$-transform of x is defined as follows:

$$y_k = (F^{u,v}x)_k = \frac{1}{(v+u)^k}\sum_{i=0}^{k}\binom{k}{i}v^{k-i}u^i\left(\frac{1}{i+1}\right)x_i,$$

where $\forall k \in \mathbb{N}$.

Theorem 2.1. *Let $X \in \{b_c^{u,v}, b_0^{u,v}, b_\infty^{u,v}\}$. The sequence spaces $\int X$ and dX are BK-spaces with their norms defined as follows;*

$$\|x\|_{\int X} = \|G^{u,v}x\|_\infty = \sup_{k\in\mathbb{N}}|(G^{u,v}x)_k| \tag{2.3}$$

and

$$\|x\|_{dX} = \|F^{u,v}x\|_\infty = \sup_{k\in\mathbb{N}}|(F^{u,v}x)_k|.$$

Proof. The sequence spaces c,c_0 and ℓ_∞ are already known that BK-spaces with the norm $\|x\|_\infty = \sup_{k\in\mathbb{N}}|x_k|$, $G^{u,v} = (g_{nk}^{u,v})$ and $F^{u,v} = (f_{nk}^{u,v})$ are two triangular matrices and the states (2.1) and (2.2) hold. If we combining these results along with Wilansky's Theorem 4.3.12 of [22], we obtain that the spaces $\int X$and dX are BK-spaces. Thus, the proof is complete. □

Theorem 2.2. *The sequence spaces $\int b_c^{u,v}$ and $db_c^{u,v}$,$\int b_0^{u,v}$ and $db_0^{u,v}$,$\int b_\infty^{u,v}$ and $db_\infty^{u,v}$ are linearly isomorphic to the sequence spaces c, c_0 and ℓ_∞ , respectively.*

Proof.
i) We provide the proof of theorem for the sequence space $\int b_0^{u,v}$. Let L is a transformation. This transformation is defined as $L : \int b_0^{u,v} \to c_0$ such that $L(x) = G^{u,v}x$. The linearity of L is clear. Also, it is obvious that $x = \theta$ whenever $G^{u,v} = \theta$. Therefore, L is injective.

Now, let us define a sequence $x = (x_n)$ for a given sequence $y = (y_n) \in c_0$ as follows:

$$x_n = u^{-n}\frac{1}{n+1}\sum_{k=0}^{n}\binom{n}{k}(-v)^{n-k}(u+v)^k y_k \tag{2.4}$$

for all $k \in \mathbb{N}$.

$$\Big(\big((k+1)\mathrm{I}\big)x\Big)_k = (k+1)x_k = u^{-k}\sum_{l=0}^{k}\binom{k}{l}(-v)^{k-l}(u+v)^l y_l.$$

Then, we obtain

$$\begin{aligned}\lim_{n\to\infty}(G^{u,v}x)_n &= \lim_{n\to\infty}\frac{1}{(v+u)^n}\sum_{k=0}^{n}\binom{n}{k}v^{n-k}u^k(k+1)x_k\\ &= \lim_{n\to\infty}\frac{1}{(v+u)^n}\sum_{k=0}^{n}\binom{n}{k}v^{n-k}u^k\left(u^{-k}\sum_{l=0}^{k}\binom{k}{l}(-v)^{k-l}(u+v)^l y_l\right)\\ &= \lim_{n\to\infty} y_n\\ &= 0.\end{aligned}$$

This implies that $x = (x_n) \in \int b_0^{u,v}$ and $L(x) = y$. Consequently, L is surjective and norm preserving from (2.3). Thus, $\int b_0^{u,v} \cong c_0$.

ii) Let T be a transformation such that $T : db_0^{u,v} \to c_0$, $T(x) = F^{u,v}x$. By using;

$$x_n = u^{-n}(n+1)\sum_{k=0}^{n}\binom{n}{k}(-v)^{n-k}(u+v)^k y_k \tag{2.5}$$

in proof (i), similarly processes are carried out. Thus, the proof of (ii) is complete.

□

Theorem 2.3. *The inclusion $c \subset db_0^{u,v}$ is strict.*

Proof. Suppose that $x = (x_k) \in c$, namely $\lim_{k\to\infty} x_k = \ell$. Then, we have; $\lim_{k\to\infty}\left(\left(\frac{1}{k+1}\mathrm{I}\right)x\right)_k = 0$. Furthermore, the binomial matrix is regular whenever $u \cdot v > 0$. By combining these facts, we conclude that for $x \in c$, the sequence $B^{u,v}\left(\frac{1}{k+1}\mathrm{I}\right)x \in c_0$. Therefore, $x \in db_0^{u,v}$ whenever $x \in c$. So, the inclusion $c \subset db_0^{u,v}$ holds.

Now, we determine a sequence $x = (x_k)$ defined as $x_k = (-1)^k$ for all $k \in \mathbb{N}$. Then, we know that $x_k = (-1)^k \notin c$ but $\left(\frac{1}{k+1}\right)\mathrm{I}(-1)^k \in c_0$. Also, since the binomial matrix is regular;

$$B^{u,v}\left(\left(\frac{1}{k+1}\right)\mathrm{I}\right)x = F^{u,v}x = \frac{1}{(v+u)^k}\sum_{i=0}^{k}\binom{k}{i}v^{k-i}u^i\frac{1}{i+1}(-1)^i \in c_0$$

and then $x \in db_0^{u,v}$. This result shows that the inclusion $c \subset db_0^{u,v}$ is strict. □

Theorem 2.4. *The inclusions $db_0^{u,v} \subset db_c^{u,v} \subset db_\infty^{u,v}$ strictly hold.*

Proof. We know that every null sequence is convergent and every convergent sequence is bounded. So, the inclusions $db_0^{u,v} \subset db_c^{u,v} \subset db_\infty^{u,v}$ hold.

Now, we define two sequences $y = (y_k)$ and $z = (z_k)$ such that;

$$y_k = k+1$$

and

$$z_k = (k+1)\left(-\frac{2v+u}{u}\right)^k$$

for all $k \in \mathbb{N}$. Then, we can observe that;

$$F^{u,v}y = e \in c \setminus c_0$$

and

$$F^{u,v}z = (-1)^k \in \ell_\infty \setminus c.$$

Namely $y = (y_k) \in db_c^{u,v} \setminus db_0^{u,v}$ and $z = (z_k) \in db_\infty^{u,v} \setminus db_c^{u,v}$. These two facts shows that the inclusions $db_0^{u,v} \subset db_c^{u,v} \subset db_\infty^{u,v}$ are strict. The proof is complete. □

3. Dual Spaces

In this section, we establish $\alpha-$, $\beta-$ and $\gamma-$ duals of the integrated and differentiated sequence spaces $\int Y$ and dY, where $Y \in \{b_c^{u,v}, b_0^{u,v}, b_\infty^{u,v}\}$.

Given two sequence spaces T and U, the multiplier space $M(T, U)$ is defined as follows;

$$M(T,U) = \{a = (a_k) \in w : at = (a_k t_k) \in U \;\; for \;\; all \;\; t = (t_k) \in T\}.$$

The following are called $\alpha-$ dual, $\beta-$dual and $\gamma-$dual of T, respectively;

$$T^\alpha = M(T, \ell_1), \quad T^\beta = M(T, cs) \quad and \quad T^\gamma = M(T, bs).$$

Now, let us give some properties that will be useful to us in the subsequent lemma.

$$\sup_{K\in\mathcal{F}}\sum_n\left|\sum_{k\in K}a_{nk}\right|<\infty \quad (\mathcal{F}\ is\ the\ collection\ of\ all\ finite\ subsets\ of\ \mathbb{N}), \tag{3.1}$$

$$\sup_{n\in\mathbb{N}}\sum_k|a_{nk}|<\infty, \tag{3.2}$$

$$\lim_{n\to\infty}\sum_k|a_{nk}|=\sum_k\left|\lim_{n\to\infty}a_{nk}\right|, \tag{3.3}$$

$$\lim_{n\to\infty}a_{nk}=\mu_k\ \ for\ each\ \ k\in\mathbb{N}, \tag{3.4}$$

$$\lim_{n\to\infty}\sum_k a_{nk}=\mu. \tag{3.5}$$

Lemma 3.1 ([20]). *Let $A=(a_{nk})$ be an infinite matrix. In that case, the following statements hold;*

(i) $A=(a_{nk})\in(c_0:\ell_1)=(c:\ell_1)=(\ell_\infty:\ell_1)\Leftrightarrow$ (3.1)

(ii) $A=(a_{nk})\in(c_0:\ell_\infty)=(c:\ell_\infty)=(\ell_\infty:\ell_\infty)\Leftrightarrow$ (3.2)

(iii) $A=(a_{nk})\in(c_0:c)\Leftrightarrow$ (3.2) *and* (3.4)

(iv) $A=(a_{nk})\in(c:c)\Leftrightarrow$ (3.2), (3.4) *and* (3.5)

(v) $A=(a_{nk})\in(\ell_\infty:c)\Leftrightarrow$ (3.3) *and* (3.4)

(vi) $A=(a_{nk})\in(c:c_0)\Leftrightarrow$ (3.2), (3.4) *and* (3.5) *with* $\mu_k=0$ *for all* $k\in\mathbb{N}$ *and* $\mu=0$

Theorem 3.2. *The α-dual of the intagrated sequence spaces $\int X$ is the set;*

$$\xi_1^{u,v}=\left\{a=(a_k)\in w:\sup_{K\in\mathcal{F}}\sum_n\left|\frac{1}{n+1}\sum_{k\in K}\frac{1}{u^n}\binom{n}{k}(-v)^{n-k}(u+v)^k a_n\right|<\infty\right\},$$

where $X\in\{b_c^{u,v},b_0^{u,v},b_\infty^{u,v}\}$.

Proof. Let any $a=(a_n)\in w$ be given. Then, by considering the sequence $x=(x_n)$ defined by (2.4). We have;

$$a_nx_n=\sum_{k=0}^{n}\frac{1}{u^n}\binom{n}{k}(-v)^{n-k}(u+v)^k\left(\frac{1}{n+1}\right)a_ny_k=(M^{u,v}y)_n$$

for all $n\in\mathbb{N}$.

By considering the result above, we conclude that $ax=(a_nx_n)\in\ell_1$ whenever $x=(x_k)\in X$ if and only if $M^{u,v}y\in\ell_1$ whenever $y=(y_k)\in c_0,c$ or ℓ_∞. This demonstrates that $a=(a_n)\in\left\{\int X\right\}^\alpha$ if and only if $M^{u,v}\in(c_0:\ell_1)=(c:\ell_1)=(\ell_\infty:\ell_1)$. By combining above result with Lemma 3.1 (i) we obtain;

$$a=(a_n)\in\left\{\int X\right\}^\alpha\Leftrightarrow\sup_{K\in\mathcal{F}}\sum_n\left|\sum_{k\in K}\frac{1}{u^n}\binom{n}{k}(-v)^{n-k}(u+v)^k\left(\frac{1}{n+1}\right)a_n\right|<\infty.$$

Hence, $\left\{\int X\right\}^\alpha=\xi_1^{u,v}$. So, the proof is complete. □

Theorem 3.3. *The α-dual of the differentiated sequence spaces dX is the set;*

$$\xi_2^{u,v}=\left\{a=(a_k)\in w:\sup_{K\in\mathcal{F}}\sum_n\left|\sum_{k\in K}\frac{1}{u^n}\binom{n}{k}(-v)^{n-k}(u+v)^k(n+1)a_n\right|<\infty\right\},$$

where $X\in\{b_c^{u,v},b_0^{u,v},b_\infty^{u,v}\}$.

Proof. The proof of the theorem is carried out using a similar way as that used in the proof of Theorem 3.2, where x be defined by (2.5) instead of (2.4). □

Theorem 3.4. *Sets $\xi_3^{u,v}, \xi_4^{u,v}, \xi_5^{u,v}$ and $\xi_6^{u,v}$ are defined by;*

$$\xi_3^{u,v} = \left\{ a = (a_k) \in w : \sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{n} \frac{1}{u^j} \binom{j}{k} (-v)^{j-k} (u+v)^k \left(\frac{1}{j+1}\right) a_j \right| < \infty \right\},$$

$$\xi_4^{u,v} = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_k \left| \sum_{j=k}^{n} \frac{1}{u^j} \binom{j}{k} (-v)^{j-k} (u+v)^k \left(\frac{1}{j+1}\right) a_j \right| = \sum_k \left| \lim_{n\to\infty} \sum_{j=k}^{n} \frac{1}{u^j} \binom{j}{k} (-v)^{j-k} (u+v)^k \left(\frac{1}{j+1}\right) a_j \right| \right\},$$

$$\xi_5^{u,v} = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_{j=k}^{n} \frac{1}{u^j} \binom{j}{k} (-v)^{j-k} (u+v)^k \left(\frac{1}{j+1}\right) a_j \ \text{exists for each}\ k \in \mathbb{N} \right\}$$

and

$$\xi_6^{u,v} = \left\{ a = (a_k) \in w : \lim_{n\to\infty} \sum_k \sum_{j=k}^{n} \frac{1}{u^j} \binom{j}{k} (-v)^{j-k} (u+v)^k \left(\frac{1}{j+1}\right) a_j \ \text{exists} \right\}.$$

Afterwards, the following statements hold;

i) $\left\{\int b_0^{u,v}\right\}^\beta = \xi_3^{u,v} \cap \xi_5^{u,v}$,

ii) $\left\{\int b_c^{u,v}\right\}^\beta = \xi_3^{u,v} \cap \xi_5^{u,v} \cap \xi_6^{u,v}$,

iii) $\left\{\int b_\infty^{u,v}\right\}^\beta = \xi_4^{u,v} \cap \xi_5^{u,v}$,

iv) $\left\{\int b_0^{u,v}\right\}^\gamma = \left\{\int b_c^{u,v}\right\}^\gamma = \left\{\int b_\infty^{u,v}\right\}^\gamma = \xi_3^{u,v}$.

Proof. We only prove the part (i) and (iv). By using related methods and Lemma 3.1, the remaining parts of the theorem can be demonstrated.

Consider the sequence $x = (x_k)$ defined by relation (2.4) and an arbitrary sequence $a = (a_k) \in w$. Then, we have

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} \left[\sum_{j=0}^{k} \frac{1}{u^k} \binom{k}{j} (-v)^{k-j} (u+v)^j \left(\frac{1}{k+1}\right) y_j \right] a_k \\ &= \sum_{k=0}^{n} \left[\sum_{j=k}^{n} \frac{1}{u^j} \binom{j}{k} (-v)^{j-k} (u+v)^k \left(\frac{1}{j+1}\right) a_j \right] y_k \\ &= (Z^{u,v} y)_n \end{aligned}$$

for all $n \in \mathbb{N}$, where the matrix $Z^{u,v} = (z_{nk}^{u,v})$ is defined by,

$$z_{nk}^{u,v} = \begin{cases} \sum_{j=k}^{n} \frac{1}{u^j} \binom{j}{k} (-v)^{j-k} (u+v)^k \left(\frac{1}{j+1}\right) a_j & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases}$$

for all $n, k \in \mathbb{N}$.

i. The above calculation shows that $ax = (a_k x_k) \in cs$, whenever $x = (x_k) \in \int b_0^{u,v}$ if and only if $Z^{u,v} y \in c$ whenever $y = (y_k) \in c_0$. This outcome makes clear that $a = (a_k) \in \left\{\int b_0^{u,v}\right\}^\beta$ if and only if $Z^{u,v} \in (c_0 : c)$. By combining this conclusion with Lemma 3.1 (iii), we conclude that $a = (a_k) \in \left\{\int b_0^{u,v}\right\}^\beta$ if and only if

$$\sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{n} \frac{1}{u^j} \binom{j}{k} (-v)^{j-k} (u+v)^k \left(\frac{1}{j+1}\right) a_j \right| < \infty \tag{3.6}$$

and

$$\lim_{n\to\infty}\sum_{j=k}^{n}\frac{1}{u^j}\binom{j}{k}(-v)^{j-k}(u+v)^k\left(\frac{1}{j+1}\right)a_j \text{ exists for each } k\in\mathbb{N}.$$

As a result, $\left\{\int b_0^{u,v}\right\}^\beta = \xi_3^{u,v}\cap\xi_5^{u,v}$.

ii. $ax=(a_kx_k)\in bs$, whenever $x=(x_k)\in\int b_0^{u,v}, \int b_c^{u,v}$ or $\int b_\infty^{u,v}$ if and only if $Z^{u,v}y\in\ell_\infty$, whenever $y=(y_k)\in c_0, c$ or ℓ_∞. This shows that, $a=(a_k)\in\left\{\int b_0^{u,v}\right\}^\gamma=\left\{\int b_c^{u,v}\right\}^\gamma=\left\{\int b_\infty^{u,v}\right\}^\gamma$ if and only if $Z^{u,v}\in(c_0:\ell_\infty)=(c:\ell_\infty)=(\ell_\infty:\ell_\infty)$. By combining this fact and Lemma 3.1 (ii), we conclude that (3.6) holds. Hence, $\left\{\int b_0^{u,v}\right\}^\gamma=\left\{\int b_c^{u,v}\right\}^\gamma=\left\{\int b_\infty^{u,v}\right\}^\gamma=\xi_3^{u,v}$. So, the proof is complete.

□

Theorem 3.5. *Sets $\xi_7^{u,v}, \xi_8^{u,v}, \xi_9^{u,v}$ and $\xi_{10}^{u,v}$ are defined by*

$$\xi_7^{u,v}=\left\{a=(a_k)\in w:\sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=k}^{n}\frac{1}{u^j}\binom{j}{k}(-v)^{j-k}(u+v)^k(j+1)a_j\right|<\infty\right\},$$

$$\xi_8^{u,v}=\left\{a=(a_k)\in w:\lim_{n\to\infty}\sum_k\left|\sum_{j=k}^{n}\frac{1}{u^j}\binom{j}{k}(-v)^{j-k}(u+v)^k(j+1)a_j\right|=\sum_k\left|\lim_{n\to\infty}\sum_{j=k}^{n}\frac{1}{u^j}\binom{j}{k}(-v)^{j-k}(u+v)^k(j+1)a_j\right|\right\},$$

$$\xi_9^{u,v}=\left\{a=(a_k)\in w:\lim_{n\to\infty}\sum_{j=k}^{n}\frac{1}{u^j}\binom{j}{k}(-v)^{j-k}(u+v)^k(j+1)a_j \text{ exists for each } k\in\mathbb{N}\right\}$$

and

$$\xi_{10}^{u,v}=\left\{a=(a_k)\in w:\lim_{n\to\infty}\sum_k\sum_{j=k}^{n}\frac{1}{u^j}\binom{j}{k}(-v)^{j-k}(u+v)^k(j+1)a_j \text{ exists}\right\}.$$

Afterwards, the following statements hold;

i) $\left\{db_0^{u,v}\right\}^\beta=\xi_7^{u,v}\cap\xi_9^{u,v}$,

ii) $\{db_c^{u,v}\}^\beta=\xi_7^{u,v}\cap\xi_9^{u,v}\cap\xi_{10}^{u,v}$,

iii) $\{db_\infty^{u,v}\}^\beta=\xi_8^{u,v}\cap\xi_9^{u,v}$,

iv) $\left\{db_0^{u,v}\right\}^\gamma=\{db_c^{u,v}\}^\gamma=\{db_\infty^{u,v}\}^\gamma=\xi_7^{u,v}$.

Proof. The proof of the theorem is carried out using a similar way as that used in the proof of Theorem 3.4. Where, $x=(x_k)$ be defined by (2.5) instead of (2.4). □

4. Some Matrix Classes

In this part, we provide a characterization of certain matrix classes that are associated with the spaces $\int b_c^{r,s}$ and $db_c^{r,s}$.

Lemma 4.1 ([1]). *Let us consider X and Y as arbitrary sequence spaces, B as an infinite matrix and C as a triangle matrix. In that case, $B\in(X:Y_C)\Leftrightarrow CB\in(X:Y)$.*

Here and in the subsequent notations, we prefer to use,

$$\eta_{nk}^{u,v}=\sum_{j=k}^{n}\frac{1}{u^j}\binom{j}{k}(-v)^{j-k}(u+v)^k\left(\frac{1}{j+1}\right)a_{nj}$$

and

$$\theta_{nk}^{u,v}=\sum_{j=k}^{n}\frac{1}{u^j}\binom{j}{k}(-v)^{j-k}(u+v)^k(j+1)a_{nj}$$

for all $n, k \in \mathbb{N}$.

Theorem 4.2. *$A \in (db_c^{u,v} : \ell_\infty)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |\eta_{nk}^{u,v}| < \infty, \tag{4.1}$$

$$\eta_{nk}^{u,v} \ \ exist \ \ for \ \ all \ \ n, k \in \mathbb{N}, \tag{4.2}$$

$$\sup_{m\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{m} \frac{1}{u^j} \binom{j}{k} (-v)^{j-k} (u+v)^k \left(\frac{1}{j+1}\right) a_{nj} \right| < \infty \quad (m \in \mathbb{N}), \tag{4.3}$$

$$\lim_{m\to\infty} \sum_{j=k}^{m} \frac{1}{u^j} \binom{j}{k} (-v)^{j-k} (u+v)^k \left(\frac{1}{j+1}\right) a_{nj} \ \ exists \ for \ each \ \ m \in \mathbb{N}. \tag{4.4}$$

Proof. Consider that $A \in (db_c^{u,v} : \ell_\infty)$. When $x = (x_k) \in db_c^{u,v}$, it is evident that Ax exists and belong to ℓ_∞. As a result, $\{a_{nk}\}_{k\in\mathbb{N}} \in \{db_c^{u,v}\}^\beta$ for all $n \in \mathbb{N}$. Now, let us combine this conclusion with Theorem 3.5 (ii). Consequently, we conclude that the conditions (4.2), (4.3) and (4.4) are satisfied. Considering the fact that $x = (k+1) \in db_c^{u,v}$ and $Ax \in \ell_\infty$ for all $x \in db_c^{u,v}$, one can see that the condition (4.1) holds. On the other hand, suppose that (4.1)-(4.4) hold. Take an arbitrary $x = (x_k) \in db_c^{u,v}$ and consider the equivalence;

$$\begin{aligned} \textstyle\sum_{k=0}^{m} a_{nk} x_k &= \textstyle\sum_{k=0}^{m} \left[\sum_{j=0}^{k} \frac{1}{u^k} \binom{k}{j} (-v)^{k-j} (u+v)^j (k+1) y_j \right] a_{nk} \\ &= \textstyle\sum_{k=0}^{m} \left[\sum_{j=k}^{m} \frac{1}{u^j} \binom{j}{k} (-v)^{j-k} (u+v)^k (j+1) \right] a_{nj} y_k \end{aligned} \tag{4.5}$$

for all $m, n \in \mathbb{N}$. If we take the limit of both sides of (4.5) and assume that $m \to \infty$, then we arrive with the conclusion that

$$\sum_k a_{nk} x_k = \sum_k \theta_{nk}^{u,v} y_k \tag{4.6}$$

for all $n \in \mathbb{N}$. Furthermore, by taking sup-norm of both sides (4.6), we obtain

$$\|Ax\|_\infty \le \sup_{n\in\mathbb{N}} \sum_k \left|\theta_{nk}^{u,v}\right| |y_k| \le \|y\|_\infty \sup_{n\in\mathbb{N}} \sum_k \left|\theta_{nk}^{u,v}\right| < \infty.$$

As a result $Ax \in \ell_\infty$, namely $A \in (db_c^{u,v} : \ell_\infty)$. Completes the proof. □

Theorem 4.3. *$A \in \left(\int b_c^{u,v} : c\right)$ if and only if the conditions (4.1)-(4.4) are satisfied and the following conditions hold;*

$$\lim_{n\to\infty} \sum_k \eta_{nk}^{u,v} = \lambda, \tag{4.7}$$

$$\lim_{n\to\infty} \eta_{nk}^{u,v} = \lambda_k \ \ for \ all \ \ k \in \mathbb{N}. \tag{4.8}$$

Proof. Suppose that $A \in \left(\int b_c^{u,v} : c\right)$. The inclusion $c \subset \ell_\infty$ holds, as is known. We determine that the conditions (4.1)-(4.4) by combining the fact and Theorem 4.2. Furthermore, it is clear that Ax exists and belongs to c for all $x = (x_k) \in \int b_c^{u,v}$. Given this fact, if we choose two sequences $x = k+1$ and $x = \eta^{(k)}$ is defined by

$$\eta_n^{(k)} = \begin{cases} 0 & , \quad 0 \le n \le k \\ \frac{1}{u^n} \binom{n}{k} (-v)^{n-k} (u+v)^k \left(\frac{1}{n+1}\right) & , \quad k \le n \end{cases}$$

Conversely, for a given $x = (x_k) \in \int b_c^{u,v}$, if we assume that the conditions (4.1)-(4.4), (4.7) and (4.8) are satisfied, we deduce from Theorem 3.4 (ii), that $\{a_{nk}\}_{k\in\mathbb{N}} \in \{db_c^{u,v}\}^\beta$ for all $n \in \mathbb{N}$. This implies the existence of Ax. Moreover, based on the conditions (4.1) and (4.8), we conclude that

$$\sum_{k=0}^{m} |\lambda_k| \le \sup_{n\in\mathbb{N}} \sum_k |\eta_{nk}^{u,v}| < \infty$$

holds for all $m \in \mathbb{N}$. This demonstrates that $\lambda_k \in \ell_1$. So the series $\sum_k \lambda_k y_k$ absolute convergens. If, $a_{nk} - \lambda_k$ is used in place of a_{nk} in condition (4.6) we have,

$$\sum_k (a_{nk} - \lambda_k)x_k = \sum_k \sum_{j=0}^{k} \frac{1}{u^k}\binom{k}{j}(-v)^{k-j}(u+v)^j \left(\frac{1}{k+1}\right)(a_{nj} - \lambda_j)y_k \tag{4.9}$$

for all $n \in \mathbb{N}$. By combining (4.9) with Lemma 3.1 (vi), we have

$$\lim_{n\to\infty} \sum_k (a_{nk} - \lambda_k)x_k = 0. \tag{4.10}$$

Finally, if we combine the circumstance (4.10) with the fact $(\lambda_k y_k) \in \ell_1$, we conclude that $Ax \in c$, that is $A \in \left(\int b_c^{u,v} : c\right)$. Then, the proof is complete. □

5. Some Geometric Properties

In this part, we investigate some geometric properties of the sequence spaces $\int X$, where $X \in \{b_c^{u,v}, b_0^{u,v}, b_\infty^{u,v}\}$.

Given a Banach space $(Z, \|.\|)$. Denote by $S(Z)$ the unit sphere of Z and by $B(Z)$ the unit ball of Z. Clarkson's modulus of convexity is defined as follows [8,9]:

$$\delta_Z(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in S(Z), \|x-y\| = \epsilon\right\},$$

where $0 \leq \epsilon \leq 2$. The inequality $\delta_Z(\epsilon) > 0$ for every $\epsilon \in (0, 2]$ characterizes the uniformly convex spaces.

Gurari's modulus of convexity is defined as [13]:

$$\beta_Z(\epsilon) = \inf\left\{1 - \inf_{\lambda\in[0,1]} \|\lambda x + (1-\lambda)y\| : x, y \in S(Z), \|x-y\| = \epsilon\right\},$$

where $0 \leq \epsilon \leq 2$. It is clear that, $\delta_Z(\epsilon) \leq \beta_Z(\epsilon) \leq 2\delta_Z(\epsilon)$ for any $0 \leq \epsilon \leq 2$. Therefore, Z is uniformly convex for $0 < \beta_Z(\epsilon) < 1$ and Z is strictly convex for $\beta_Z(\epsilon) < 1$.

If provided $\limsup_{n\to\infty} \|x_n\| < \limsup_{n\to\infty} \|x_n - x\|$ for $\forall x \in Z, x \neq 0$ when $x_n \xrightarrow{w} 0$ then Z is said to have Opial property [18].

In [10], Garcia-Falset defined the coefficient $R(Z)$ as follows:

$$R(Z) := \sup\left\{\liminf_{n\to\infty} \|x_n - x\| : x_n \xrightarrow{w} 0, \|x_n\| \leq 1 \forall n \in \mathbb{N}, \|x\| \leq 1\right\}.$$

Also, a Banach space Z with $R(Z) < 2$ has the weak fixed point property [11].

Theorem 5.1. *Let $Z \in \left\{b_c^{u,v}, b_0^{u,v}, b_\infty^{u,v}\right\}$. Gurari's modulus of convexity for the $\int Z$ as follows:*

$$\beta_{\int Z} \leq 1 - |1 - \epsilon|,$$

where $0 \leq \epsilon \leq 2$.

Proof. Let $x \in \int Z$. Then, we write;

$$\|x\|_{\int Z} = \|G^{u,v}x\|_\infty = \sup_{k\in\mathbb{N}} |(G^{u,v}x)_k|.$$

We assume that $0 \leq \epsilon \leq 2$ and the sequences $x = (x_n)$ and $y = (y_n)$ as follows:

$$x = (x_n) = \left(H^{u,v}(1), H^{u,v}\left(-\frac{\epsilon}{2}\right), H^{u,v}\left(\frac{\epsilon}{2}\right), 0, 0, ...\right)$$

and

$$y = (y_n) = \left(H^{u,v}(1), H^{u,v}(0), H^{u,v}\left(-\frac{\epsilon}{2}\right), H^{u,v}\left(-\frac{\epsilon}{2}\right), 0, ...\right),$$

where $H^{u,v}$ is the inverse of matrix $G^{u,v}$. Thus, $\dot{x}_n = (G^{u,v}x)_n$ and $\dot{y}_n = (G^{u,v}y)_n$ we obtain;

$$\dot{x} = \left(\dot{x}_n\right) = \left(1, -\frac{\epsilon}{2}, \frac{\epsilon}{2}, 0, 0, ...\right)$$

and

$$\dot{y} = \left(\dot{y_n}\right) = \left(1, 0, -\frac{\epsilon}{2}, -\frac{\epsilon}{2}, 0, ...\right).$$

Considering these sequences we get;

$$\|x\|_{\int Z} = \|G^{u,v}x\|_\infty = \sup_{k\in\mathbb{N}} |(G^{u,v}x)_k| = 1,$$

$$\|y\|_{\int Z} = \|G^{u,v}y\|_\infty = \sup_{k\in\mathbb{N}} |(G^{u,v}y)_k| = 1$$

and

$$\|x-y\|_{\int Z} = \|G^{u,v}x - G^{u,v}y\|_\infty = \sup_{k\in\mathbb{N}} |(G^{u,v}x)_k - (G^{u,v}y)_k| = \epsilon.$$

Additionally, we have

$$\begin{aligned}\inf_{0\le\lambda\le 1} \|\lambda x + (1-\lambda)y\|_{\int Z} &= \inf_{0\le\lambda\le 1} \|\lambda G^{u,v}x + (1-\lambda)G^{u,v}y\|_\infty \\ &= \inf_{0\le\lambda\le 1} \left\|\left(1, -\frac{\lambda\epsilon}{2}, \frac{\epsilon(2\lambda-1)}{2}, -\frac{\epsilon(1-\lambda)}{2}, 0, 0, ...\right)\right\|_\infty \\ &= \inf_{0\le\lambda\le 1} |1-\epsilon| = |1-\epsilon|.\end{aligned}$$

As a result of this, we obtain

$$\beta_{\int Z} \le 1 - |1-\epsilon|.$$

Thus, the proof is completed. □

Corollary 5.2. *Let $Z \in \left\{b_c^{u,v}, b_0^{u,v}, b_\infty^{u,v}\right\}$. In this case, $\delta_{\int Z}(\epsilon) = 0$, for $0 \le \epsilon \le 2$. Additionally, since $\delta_{\int Z}(\epsilon) \le \beta_{\int Z}(\epsilon) \le 2\delta_{\int Z}(\epsilon)$ we can write $\beta_{\int Z}(\epsilon) = 0 < 1$, for $0 \le \epsilon \le 2$. Hence, $\int Z$ are strictly convex.*

Theorem 5.3. *If $Z \in \left\{\int b_c^{u,v}, \int b_o^{u,v}\right\}$, then $R(Z) = 1$.*

Proof. For $Z \in \left\{\int b_c^{u,v}, \int b_o^{u,v}\right\}$, $e^{(k)} \in S(Z)$ and $e^{(k)} \xrightarrow{w} 0$. Hence, $e^{(1)} \in S(Z)$, we obtain;

$$\left\|e^{(k)} - e^{(1)}\right\|_Z \xrightarrow{n\to\infty} 1.$$

As a result of this, $R(Z) = 1$. □

Corollary 5.4. *Let $Z \in \left\{\int b_c^{u,v}, \int b_o^{u,v}\right\}$. Since $R(Z) = 1$ the sequence spaces $\int b_c^{u,v}$ and $\int b_0^{u,v}$ have the weak fixed point property.*

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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The author has read and agreed the published version of the manuscript.

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