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On Curvatures of Semi-invariant Submanifolds of Lorentzian Para-Sasakian Manifolds

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Abstract. A Lorentzian para-Sasakian (LP-Sasakian) space form is a kind of para-Sasakian manifold with constant $\varphi-$ holomorphic sectional curvature. The presented paper is on the curvatures of semi-invariant submanifolds of an LP-Sasakian space form. Firstly, the definition of a semi-invariant submanifold of LP-Sasakian space form is given and an example is presented. Then, some important results on Ricci and scalar curvatures have been obtained by using the Gauss equation related to curvatures. Furthermore, by suffering from these results, the conditions of a distribution is being Einstein have been examined. Finally, semi-invariant products of Lorentzian para-Sasakian manifolds have been considered and an important inequality for the second fundamental form is proved.

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1. Introduction

In differential geometry, a pseudo-Riemannian manifold is a generalization of a Riemannian manifold in which the metric tensor must be a non-degenerate bilinear form. A Lorentzian manifold in which one dimension has a sign opposite to the other three dimensions is an important notion of theoretical physics. In [15], Takashi studied on the Sasakian manifolds a class of contact manifolds with Lorentzian structure. By following Takashi's work, Duggal investigated contact structure and space-time manifolds [6]. In continuation, this notion has been studied by several authors [7, 8, 16].

In [14], Sato introduced almost para-contact metric manifold. The metric of an almost para-contact metric structure is a Riemann metric. Adati and Matsumoto [1] studied on para-Sasakian manifolds, which could be considered as a special case of almost para-contact manifolds described by Sato. In [10], Matsumoto defined Lorentzian almost para-contact (LAPC) manifolds and the Lorentzian para-Sasakian (LP-Sasakian) manifolds. The most basic feature is distinguished almost para-contact metric manifolds from LAPC manifolds is that metric of the LAPC structure is a Lorentzian metric and the characteristic vector field is a timelike vector field. Many authors have been studied on LP-Sasakian manifolds [3–5, 9, 11–13, 17]. It is well known that, a space form which a kind of manifold with constant sectional curvature is an important model of manifolds with special structures such as complex and contact. In contact geometry, an other notion of sectional curvature have been defined by using contact structure. This notion is referred as $\varphi-$ holomorphic sectional curvature and it is used for classifying contact manifolds. An LP-Sasakian space form N

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is an LP-Sasakian manifold with constant $\varphi-$holomorphic section curvature k. The curvature of an LP-Sasakian space form N is given in [2] by,

$$\begin{aligned} R(U,V,W,Z) &= \frac{k-3}{4}\{g(U,Z)g(V,W)-g(U,W)g(V,Z)\} \\ &+\frac{k+1}{4}\{g(\varphi U,Z)g(\varphi V,W)-g(U,\varphi W)g(\varphi V,Z) \\ &+2g(\varphi U,V)g(\varphi W,Z)+g(V,Z)\eta(U)\eta(W) \\ &-g(U,Z)\eta(V)\eta(W)+g(U,W)\eta(V)\eta(Z) \\ &-g(V,W)\eta(U)\eta(Z)\} \end{aligned}$$

for all $U,V,W,Z \in \Gamma(TN)$, where R is curvature tensor $N(k)$. A semi-invariant submanifold is a kind of submanifolds of contact structure with invariant and anti-invariant distributions. These distributions gives significant geometric properties about submanifolds.

In this paper, curvature properties of a semi-invariant submanifold of an LP-Sasakian space form are studied. Firstly, the definition of a semi-invariant submanifold of an LP-Sasakian space form is given and an example is presented. Then, using Gauss equation related to curvatures used for obtaining some important results on Ricci and scalar curvatures. Moreover, by suffering from these results conditions of distributions being Einstein have been examined. Finally, semi-invariant products of Lorentzian para-Sasakian manifolds have been considered and an important inequality for second fundamental form is proved.

2. Lorentzian Para-Sasakian Manifolds

This section is devoted to basic concepts on LPAC metric manifolds. A differentiable manifold N with a triple (φ,η,ξ) satisfying the following conditions is called a Lorentzian almost para-contact (LPAC) metric manifold

$$\varphi^2 = I+\eta\otimes\xi, \qquad \eta(\xi)=1,$$

and

$$g(\varphi X,\varphi Y)=g(X,Y)+\eta(X)\eta(Y), \qquad g(\xi,\xi)=-1,$$

where η is a $1-$form, φ is a tensor field of type $(1,1)$, ξ is a vector field and g is Lorentzian metric. The characteristic vector field ξ is never space-like and a light-like on N, i.e it is a time-like vector field.Then, for a local basis $\{e_1,...,e_{2n},\xi\}$ of TN,

$$g(e_i,e_j)=\delta_{ij} \text{ and } g(\xi,\xi)=-1,$$

that is, $e_1,...,e_{2n}$ are space-like and ξ is a time-like vector field.

An LPAC metric manifold is normal if

$$\mathcal{N}=[\varphi,\varphi]+2d\eta\otimes\xi=0,$$

where, $\mathcal{N}$ Nijenhuis tensor field. An LP-contact metric manifold is a normal LAPC metric manifold. LP-Sasakian manifolds are special classes of normal LPAC metric manifolds. An LP-contact metric manifold N is an LP-Sasakian manifold if and only if

$$(\nabla_X\varphi)Y=-g(X,Y)\xi-\eta(Y)X$$

for all $X,Y\in\Gamma(TN)$.

3. Semi-Invariant Submanifold of Lorentzian Para-Sasakian Manifold

In this section, a short introduction to semi-invariant submanifolds of an LP-Sasakian manifold is given.

A submanifold N of an LP-Sasakian manifold $\overline{N}$ is called a semi-invariant submanifold if the followings are satisfied:

(i) $\varphi D_x = D_x$ for any $x\in N$, that is D is invariant under φ,
(ii) $\varphi D_x^\perp \subseteq T_x^\perp N$ for any $x\in N$,that is, $D^\perp$ is anti-invariant under φ,
(iii) $TN = D\oplus D^\perp \oplus sp\{\xi\}$,

where D and $D^\perp$ are distributions on N and ξ is tangent to N. For a local orthonormal frame $\{e_1,...,e_{2p},e_{2p+1},...,e_{2m},\xi\}$,

$$D=sp\{e_1,...,e_{2p}\}, \qquad D^\perp = sp\{e_{2p+1},...,e_{2m}\}, \tag{3.1}$$

where $dimD=2p$ and $dimD^\perp=q$.

Example 3.1. $\mathbb{R}^{2n+1}$, with

$$\eta^{\alpha} = \frac{1}{2}\left(dz_{\alpha} - \sum_{i=1}^{n} y_i dx_i\right), \qquad \xi_{\alpha} = 2\frac{\partial}{\partial z_{\alpha}},$$

and

$$\varphi\left(\sum_{i=1}^{n}\left(X_i\frac{\partial}{\partial x_i} + Y_i\frac{\partial}{\partial y_i}\right) + Z\frac{\partial}{\partial z}\right) = \sum_{i=1}^{n}\left(Y_i\frac{\partial}{\partial x_i} - X_i\frac{\partial}{\partial y_i}\right) + \sum_{i=1}^{n} Y_i y_i \frac{\partial}{\partial z}$$

with the metric

$$g = \eta \otimes \eta + \frac{1}{4}\left(\sum_{i=1}^{n} dx_i \otimes dx_i + dy_i \otimes dy_i\right)$$

is an LP-Sasakian manifold which is a canonical example. Let N be a submanifold of $\mathbb{R}^9$ such that

$$X(x, y, u, l, s) = 2(x, u, 0, 0, y, 0, l, 0, s).$$

The local frame of TN from a basis of $T^{\perp}N$ is given by

$$\begin{aligned} e_1 &= 2\frac{\partial}{\partial x_1}, \qquad e_2 = 2\frac{\partial}{\partial y_1}, \\ e_3 &= 2\frac{\partial}{\partial x_2}, \qquad e_4 = 2\frac{\partial}{\partial y_3}, \qquad e_5 = 2\frac{\partial}{\partial z_1} = \xi, \end{aligned}$$

and

$$e_1^* = \frac{\partial}{\partial x_3}, \qquad e_2^* = \frac{\partial}{\partial y_2}.$$

Suppose that $D_1 = sp\{e_1, e_2\}$ and $D_2 = sp\{e_3, e_4\}$. Therefore, D_1 is invariant and D_2 is anti-invariant distribution. Thus, N is a semi invariant submanifold of $\mathbb{R}^9$.

Let $\overline{\nabla}$ and ∇ be Levi-Civita connections $\overline{N}$ and N, respectively. The Gauss and Weingarten equations are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$\overline{\nabla}_X V = \nabla_X^{\perp} V - A_V X$$

for all $X, Y \in \Gamma(TN)$ and $V \in \Gamma(T^{\perp}N)$, where A_V is the Weingarten endomorphism, h is the second fundamental from of $\overline{N}$ and $\nabla^{\perp}$ is the connection in the normal bundle. Therefore, it is obtained that

$$g(h(X, Y), V) = g(A_V X, Y).$$

A submanifold is said to be totally geodesic if second fundamental form vanishes.

The curvature tensors of $\overline{N}$ and N has the following relation,

$$\overline{R}(U, V, W, Z) = R(U, V, W, Z) + g(h(U, W), h(V, Z)) - g(h(U, Z), h(V, W)) \tag{3.2}$$

for all $U, V, W, Z \in \Gamma(TN)$, where $\overline{R}$ and R the Riemann curvature of $\overline{N}$ and N, respectively.

4. Semi-Invariant Submanifolds of a Lorentzian Para-Sasakian Space Forms

An LP-Sasakian space form is an LP-Sasakian manifold with constant $\varphi-$holomorphic section curvature k. The curvature of an LP-Sasakian space form is given in [2] by,

$$\begin{aligned} R(U, V, W, Z) = {} & \frac{k-3}{4}\{g(U, Z)g(V, W) - g(U, W)g(V, Z)\} + \frac{k+1}{4}\{g(\varphi U, Z)g(\varphi V, W) - g(U, \varphi W)g(\varphi V, Z) \\ & +2g(\varphi U, V)g(\varphi W, Z) + g(V, Z)\eta(U)\eta(W) - g(U, Z)\eta(V)\eta(W) + g(U, W)\eta(V)\eta(Z) \\ & -g(V, W)\eta(U)\eta(Z)\} \end{aligned} \tag{4.1}$$

for all $U, V, W, Z \in \Gamma(TN)$, where R is curvature tensor $N(k)$. In this section, curvature properties and Einstein conditions on distributions of a semi-invariant submanifold N of an LP-Sasakian space form $\overline{N}(k)$ are studied.

If (4.1) and (3.2) are used, N has constant φ-sectional curvature k if and only if

$$\begin{aligned} R(U,V,W,Z) \quad = \quad & \frac{k-3}{4}\{g(U,Z)g(V,W)-g(U,W)g(V,Z)\}+\frac{k+1}{4}\{g(\varphi U,Z)g(\varphi V,W)-g(U,\varphi W)g(\varphi V,Z) \\ & +2g(\varphi U,V)g(\varphi W,Z)+g(V,Z)\eta(U)\eta(W)-g(U,Z)\eta(V)\eta(W)+g(U,W)\eta(V)\eta(Z) \\ & -g(V,W)\eta(U)\eta(Z)\}+g(h(U,Z),h(V,W))-g(h(V,Z),g(U,W)), \end{aligned} \tag{4.2}$$

where R is the Riemannian curvature tensor. By using (4.2) the following result is stated;

Proposition 4.1. *Let N be semi-invariant submanifold of $\overline{N}(k)$ and $D^\perp$ is anti-invariant distribution. Therefore, for all $U,V,W,Z \in \Gamma(D^\perp)$*

$$R(U,V,W,Z)=\frac{k-3}{4}\{g(U,Z)g(V,W)-g(U,W)g(V,Z)\}+g(h(U,Z),h(V,W))-g(h(V,Z),g(U,W)). \tag{4.3}$$

Theorem 4.2. *Let N be semi-invariant submanifold of $\overline{N}(k)$ with totally geodesic anti-invariant distribution. Then, invariant distribution is an Einstein manifold with scalar curvature $\tau_{D^\perp} = \frac{k-3}{4}q(q-1)$.*

Proof. From (4.2) the Ricci curvature is given by

$$S(W,Z)=\sum_{i=2p+1}^{2m} R(E_i,W,Z,E_i)$$

for all $W,Z \in \Gamma(D^\perp)$. Hence, by using (4.3) it is obtained that

$$S(W,Z)=\frac{k-3}{4}\sum_{i=2p+1}^{2m}\{g(E_i,E_i)g(W,Z)-g(E_i,Z)g(W,E_i)\}.$$

Thus, $D^\perp$ is Einstein and $\tau_{D^\perp} = \frac{k-3}{4}q(q-1)$. □

Now, by using (4.2) for invariant distribution, it is obtained that;

Proposition 4.3. *Let N be semi-invariant submanifold of $\overline{N}(k)$ and D is invariant distribution. The curvature of D is given by*

$$\begin{aligned} R(U,V,W,Z) \quad = \quad & \frac{k-3}{4}\{g(U,Z)g(V,W)-g(U,W)g(V,Z)\}+\frac{k+1}{4}\{g(\varphi U,Z)g(\varphi V,W)-g(U,\varphi W)g(\varphi V,Z) \\ & +2g(\varphi U,V)g(\varphi W,Z)\}+g(h(U,Z),h(V,W))-g(h(V,Z),g(U,W)) \end{aligned} \tag{4.4}$$

for all $U,V,W,Z \in \Gamma(D)$.

Theorem 4.4. *Let $\overline{N}(k)$ be an LP-Sasakian space form and N is semi-invariant submanifold of $\overline{N}(k)$. If the invariant distribution D is totally geodesic then it is Einstein.*

Proof. The proof follows from (4.4). □

Theorem 4.5. *Let $\overline{N}(k)$ be an LP-Sasakian space form and N is a semi-invariant submanifold of $\overline{N}(k)$. If the invariant distribution D is totally geodesic then scalar curvature τ_D of D is given by*

$$\tau_D = p\frac{3(k+1)+(k-3)(2p-1)}{2}.$$

Proof. For all $X,Y \in \Gamma(D)$, by using (4.4), the Ricci curvature of D is given by

$$S(X,Y)=\frac{3(k+1)+(k-3)(2p-1)}{4}g(Y,Z),$$

which gives proof. □

Theorem 4.6. *Let $\overline{N}(k)$ be an LP-Sasakian space form and N is semi-invariant submanifold of $\overline{N}(k)$. If φ-sectional curvature of invariant distribution D is $-k$ then it is totally geodesic.*

Proof. From (4.2), curvature of D is given by

$$\begin{aligned} R(W,\varphi W,W,\varphi W) &= \frac{k-3}{4}\{g(W,\varphi W)g(\varphi W,W)-g(W,W)g(\varphi W,\varphi W)\} \\ &+\frac{k+1}{4}\big\{g(\varphi W,\varphi W)g(\varphi^2 W,W)-g(\varphi W,W)g(\varphi^2 W,\varphi W) \\ &-2g(\varphi W,\varphi W)g(\varphi W,\varphi W)\}+g(h(W,\varphi W),h(\varphi W,W))-g(h(\varphi W,\varphi W),h(W,W)) \end{aligned}$$

for all $W \in \Gamma(D)$. Then, it is obtained that

$$R(W,\varphi W,W,\varphi W) = -k-2\,\|h(W,W)\|^2\,,$$

which proves assertion. □

5. Semi-Invariant Product in a Lorentzian para-Sasakian Space Form

A semi-invariant submanifold N in a Lorentzian para-Sasakian space form $\overline{N}(k)$ is called a semi-invariant product if the distribution $D \oplus sp\{\xi\}$ is integrable and N is a locally Riemannian product $N_1 \times N_2$, where N_1 and N_2 are leaf of $D \oplus sp\{\xi\}$ and $D^\perp$, respectively. Moreover, N is proper semi-invariant product if $pq \neq 0$. In this section, it is assumed that the dimension of distributions are $pq \neq 0$.

For any $U \in \Gamma D$ and $V \in \Gamma D^\perp$, by using (4.2) it is obtained;

Proposition 5.1. *Let $\overline{N}(k)$ be an LP-Sasakian space form. If N is a semi-invariant product of $\overline{N}(k)$, then*

$$R(U,\varphi U,V,\varphi V) = 2\left(\|h(U,V)\|^2 - \frac{k-1}{4}\right)$$

for any unit vector fields $U \in \Gamma(D)$ and $V \in \Gamma(D^\perp)$.

Theorem 5.2. *Let $\overline{N}(k)$ be a Lorentzian para-Sasakian space form. If N is a semi-invariant product of $\overline{N}(k)$, then h is satisfied the following inequality,*

$$\|h\|^2 \geq pq(1-k)+2qp.$$

Proof. It is obtained that

$$\|h\|^2 = \sum_{i,j=1}^{2p}\left\|h(E_i,E_j)\right\|^2 + \sum_{k,l=2p+1}^{2m}\|h(E_k,E_l)\|^2 + 2\sum_{i=1}^{2p}\sum_{k=2p+1}^{2m}\|h(E_i,E_k)\|^2 + 2\sum_{k=2p+1}^{2m}\|h(E_k,\xi)\|^2\,.$$

By virtue of (3.1) it implies that,

$$\|h\|^2 = pq(1-k)+2qp+\sum_{i,j=1}^{2p}\left\|h(E_i,E_j)\right\|^2 + \sum_{k,l=2p+1}^{2m}\|h(E_k,E_l)\|^2\,,$$

which gives the assertion. □

Proposition 5.3. *Let $\overline{N}(k)$ be an LP-Sasakian space form. If N is a semi-invariant product of $\overline{N}(k)$, then*

$$R(U,V,W,Z) = 0$$

for all $U,V \in \Gamma(D \oplus sp\{\xi\})$ and $W,Z \in \Gamma(D^\perp)$.

Proof. The proof follows from (4.2). □

Corollary 5.4. *Let $\overline{N}(k)$ be an LP-Sasakian space form. If N is a semi-invariant product of $\overline{N}(k)$, then*

$$R(U,V,W,Z) = 0$$

for all $U,V \in \Gamma(D)$ and $W,Z \in \Gamma(D^\perp \oplus sp\{\xi\})$.

Conclusion

Lorentzian manifolds have potential for applications in many fields of mathematics and physics. In particular it is applicable to the theory of relativity. Manifolds with structures such as complex and contact are also significant working area for theoretical physics and mathematics. LP-Sasakian space forms has Lorentzian metric and para-contact structure with constant sectional curvature. In this manner, LP-Sasakian space forms are interesting subject. This study is on LP-Sasakian space forms. Semi-invariant manifolds of such manifolds have been investigating. Einstein conditions and curvature properties of invariant and anti-invariant distributions have been studied. The results obtained will be a source for future studies on the subject.

Authors Contribution Statement

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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