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Research Article

Deferred statistical order convergence in Riesz spaces

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Abstract

In recent years, researchers have focused on exploring different forms of statistical convergence in Riesz spaces, such as statistical order convergence and statistical unbounded order convergence. This study aims to present the concept of deferred statistical convergence within Riesz spaces, specifically concerning its relationship with order convergence. Furthermore, we delve into the interconnections between deferred statistical order convergence and various other types of statistical convergence. Moreover, we explore in depth the intricate connections between deferred statistical order convergence and other notable forms of statistical convergence. We provide valuable insights into the broader framework of statistical convergence theory in Riesz spaces.

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1. Introduction and introductory facts

Statistical convergence, originally introduced by Steinhaus in [19], serves as a broader extension of the conventional convergence seen in real or complex sequences. Its applicability has been explored by Maddox in more generalized abstract spaces, including locally convex spaces [17]. Moreover, Küçükaslan and Ylmaztürk delved into the concept of deferred statistical convergence in [14], paving the way for further investigations in this area. Several other works have also contributed to the theory of statistical convergence [1, 10–13, 15].

On a separate note, Riesz space, also known as a vector lattice, emerged from the realm of functional analysis through Riesz's contributions [18]. Over time, this ordered vector space has found applications in diverse fields such as measure theory, Banach spaces, operator theory, and economics [2–4, 8, 16, 21]. The current paper endeavors to combine the ideas of deferred statistical convergence of real sequences with order convergence in Riesz spaces.

Recall the definition of an ordered vector space, denoted by E, which is a real-valued vector space equipped with an order relation. In this setting, for any x and y in E, with

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x less than or equal to $y, (x \leq y)$, we have the property that $x+z$ is less than or equal to $y+z$ for all z in E, and αx is less than or equal to αy for all positive real numbers α. When an ordered vector space E satisfies an additional property, namely that the infimum and supremum operators exist for any two vectors x and y in E, denoted as $x \wedge y = \inf(x,y)$ and $x \vee y = \sup(x,y)$ respectively, it is referred to as a Riesz space or a vector lattice. For a given element x in a vector lattice E, the positive part denoted as x^+ is defined as $x \vee \theta$, the negative part denoted as x^- is defined as $(-x) \vee \theta$, and the modulus of x denoted as $|x|$ is defined as $x \vee (-x)$. Thus, in this paper, the symbol $|\cdot|$ represents the modulus of elements in vector lattices. Moreover, in a vector lattice E, a subset A is called solid if for every x in A and y in E such that the modulus of y is less than or equal to the modulus of x, it implies that y is in A. An ideal is the term used to describe a solid vector subspace within a vector lattice. Additionally, a vector lattice is considered to be σ-order complete when every countable subset, which is both nonempty and bounded above, has a supremum. Similarly, it can also be described as σ-order complete if every nonempty bounded below countable subset has an infimum.

Next, let's define the terms *increasing sequence* and *decreasing sequence* in the context of a Riesz space E. A sequence (x_n) in a Riesz space E is said to be increasing whenever $x_1 \leq x_2 \leq \cdots$ and is decreasing if $x_1 \geq x_2 \geq \cdots$ holds. Then, we denote them by $x_n \uparrow$ and $x_n \downarrow$, respectively. Moreover, if $x_n \uparrow$ and $\sup x_n = x$, then we write $x_n \uparrow x$. Similarly, if $x_n \downarrow$ and $\inf x_n = x$, then we write $x_n \downarrow x$. A sequence is said to be monotonic if it is either increasing or decreasing. In the theory of Riesz spaces, order convergence plays a crucial role, and thus we proceed with its definition.

Definition 1.1. Let (x_n) be a sequence in a vector lattice E. Then, it is called *order convergent* to $x \in E$ if there exists another sequence $y_n \downarrow \theta$ (i.e., $\inf y_n = \theta$ and $y_n \downarrow$) such that $|x_n - x| \leq y_n$ holds for all $n \in \mathbb{N}$, and abbreviated it as $x_n \xrightarrow{o} x$.

Now, let's move on to discussing the concept of statistical convergence, specifically focusing on the natural density of subsets of natural numbers. The density of a subset K of the set of natural numbers is defined as the limit (whenever it exists)

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : k \in K\}\right|.$$

We denote this density as $\delta(K)$, where $|\{k \leq n : k \in K\}|$ represents the count of elements in K that do not exceed n. On the other hand, a sequence (x_n) of real numbers is statistically convergent to a real number l if, for every positive value ε, the following condition is satisfied:

$$\lim_{n\to\infty} \frac{1}{n}\left|\{k : n \geq k,\ |x_n - l| > \varepsilon\}\right| = 0.$$

In the subsequent discussion, we consider a sequence x denoted as (x_k), and we introduce two sequences (p_n) and (q_n) consisting of non-negative integers such that for each n, p_n is less than q_n, and q_n diverges to infinity. Based on these sequences, we define a new sequence as follows:

$$(D_{p,q}x)_n := \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} x_k,$$

where $n \in \mathbb{N}$. The sequence $(D_{p,q}x)_n$ is referred to as the *deferred Cesàro mean*, which generalizes the concept of the Cesàro mean for real (or complex) valued sequences. Moreover, we say that x is *strong $D_{p,q}$-convergent* to l if the following limit exists:

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \sum_{k=p_n+1}^{q_n} |x_k - l| = 0.$$

We use the notation $x_k \xrightarrow{D[p,q]} l$ to represent this convergence. In the context of this article, whenever we mention p and q sequences, they always satisfy the above properties, and these properties are referred to as *the deferred property.* A sequence $x = (x_k)$ is said to be *deferred statistical convergent* to $l \in \mathbb{R}$ if, for all $\varepsilon > 0$, the following condition holds:

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} |\{p_n < k \le q_n : |x_k - l| \ge \varepsilon\}| = 0$$

We denote this convergence as $x_k \xrightarrow{DS[p,q]} l$.

In [9], Ercan presented a characterization of statistical convergence on vector lattices. Then, Aydın explored different forms of statistical convergence in Riesz spaces in [5–7]. In this paper, we remind the following two definitions of statistical convergence in a Riesz space.

Definition 1.2. Let (x_n) be a sequence in a Riesz space E. Then, (x_n) is called

- *statistical order decreasing* to θ if there exists a set $K = k_1 < k_2 < \cdots \subset \mathbb{N}$ with density $\delta(K) = 1$ such that (x_{k_n}) is decreasing and $\inf_{n\in K}(x_{k_n}) = \theta$. We abbreviate this as $x_n \downarrow^{\mathrm{st_o}} \theta$.
- *statistical order convergent* to $x \in E$ if there exists a sequence $q_n \downarrow^{\mathrm{st_o}} \theta$ with an index set $K = k_1 < k_2 < \cdots \subset \mathbb{N}$ such that $\delta(K) = 1$ and $|x_{k_n} - x| \le q_{k_n}$ for every $k_n \in K$. We write this as $x_n \xrightarrow{\mathrm{st_o}} x$.

It is clear that every order convergent sequence is statistical order convergent to the same point.

2. Deferred statistical decreasing

Tripathy introduced the concept of statistical monotonicity for real sequences [20], and the study of statistically monotone sequences in Riesz spaces was also conducted (see for example [7]). In this paper, we propose an extension of this concept to deferred statistical decreasing in Riesz spaces.

Definition 2.1. Consider sequences (p_n) and (q_n) consisting of nonnegative integers that satisfy the deferred property. Let (z_n) be a sequence in a Riesz space E. We say that (z_n) is a *deferred statistical order decreasing* sequence to zero if there exists a set $K \subseteq \mathbb{N}$ such that the deferred density of K is equal to one, given by the formula:

$$\delta_{p,q}(K) := \lim_{n\to\infty} \frac{1}{q_n - p_n} |\{p_n < k \le q_n : k \in K\}| = 1$$

Furthermore, the subsequence $(z_{k_n})_{k_n\in K}$ must decreasing to zero on the set K, which can be denoted as $(z_{k_n})_{k_n\in K} \downarrow \theta$. To simplify notation, we use the abbreviation $z_n \downarrow_{p,q}^{D_{st_o}} \theta$ to represent the property of a sequence (z_n) being a deferred statistical order decreasing sequence to θ with respect to p and q.

Remark 2.2.

(i) When the sequences $q(n) = n$ and $p(n) = \theta$ are used in Definition 2.1, it coincides with the definition of statistical order decreasing.

(ii) If a sequence (z_n) is monotonically decreasing and converges to zero, then it is also deferred statistical order decreasing to zero. However, the converse does not hold in general. To illustrate this, consider the Euclidean space $\mathbb{R}^2$ equipped with the coordinatewise ordering. Let $q(n) = n$ and $p(n) = 0$, and define the sequence (z_n) as follows:

$$z_n := \begin{cases} (0, n^2) & \text{if } n = k^3 \\ (0, \frac{1}{n^2}) & \text{if } n \ne k^3 \end{cases},$$

where k is a natural number. In this case, we have $z_n \downarrow_{p,q}^{D_{st_o}} (0,0)$. However, it is important to note that the entire sequence (z_n) is not monotonically decreasing.

(iii) A deferred statistical order decreasing to zero sequence may contain a subsequence with elements that either decrease or are incomparable in the Riesz space E. However, the index set corresponding to such a subsequence has deferred density zero.

(iv) In Riesz spaces, it is a well-known fact that if $z_n \downarrow \theta$, then $z_{k_n} \downarrow \theta$ for every subsequence (z_{k_n}) of (z_n). However, this property may not hold for deferred statistical monotone decreasing sequences. For instance, consider the sequences described in (ii), and take a subsequence (z_{k_n}) where $k_n = j^3$ for some $j \in \mathbb{N}$. In this case, the subsequence (z_{k_n}) does not have a supremum.

In the general, Remark 2.2(iv) provides an example illustrating that a subsequence of a deferred statistical monotone decreasing sequence may not necessarily be deferred statistical monotone decreasing. However, we present a positive result in the following theorem.

Theorem 2.3. *Consider a sequence (z_n) in a Riesz space E. If $z_n \downarrow_{p,q}^{D_{st_o}} \theta$ holds, then any subsequence (z_{k_n}) of (z_n), with an index set K satisfying $\delta_{p,q}(K) = 1$ and such that (z_{k_n}) is decreasing on K, is also deferred statistical order decreasing to zero.*

Proof. Assume that $z_n \downarrow_{p,q}^{D_{st_o}} \theta$ holds in E. Thus, there exists a set $K \subset \mathbb{N}$ such that $\delta p, q(K) = 1$, and $(z_{k_n})_{k_n \in K} \downarrow \theta$ on K. Let us consider an arbitrary index set $M \subseteq \mathbb{N}$ satisfying $K \neq M$, $\delta p, q(M) = 1$, and (z_n) is decreasing on M. It can be observed that if no such set M exists, then the proof is complete. Since $z_{k_n} \downarrow \theta$, we have $\theta \leq z_{k_n}$ for all $k_n \in K$. Additionally, we find that $\delta_{p,q}(K \cap M) = 1$. Consequently, for some $k_m \in K$ and $m_n \in M$, we have $k_n = m_n$. Hence, we have $z_{m_1} \geq z_{m_2} \geq \cdots \geq z_{m_n} = z_{k_n} \geq \theta$. We can find infinitely many such pairs of indices. By continuing this process, we obtain $z_{m_n} \geq \theta$ for every $m_n \in M$, i.e., zero is a lower bound of (z_{m_n}). Now, let's take another lower bound u of (z_{m_n}). Therefore, we have $u \leq z_{m_n}$ for every $m_n \in M$. Then, we can find some $z_{n_{k_t}}$ such that $z_{n_{k_t}} = z_{m_k} \geq u$ for some $m_k \in M$. By following this approach, we can construct a subsequence $(z_{n_{k_1}}, z_{n_{k_2}}, \cdots)$ of (z_{k_n}) such that u is a lower bound of $(z_{n_{k_t}})$ for $t \in \mathbb{N}$. As $z_{k_n} \downarrow \theta$, the infimum of every subsequence of (z_{k_n}) is zero. Hence, we obtain $u = \theta$. Therefore, we conclude that $z_{m_n} \downarrow_{p,q}^{D_{st_o}} \theta$ as desired. ☐

In the next results without proof, we give the linear property of deferred statistical order decreasing sequences.

Proposition 2.4. *Consider sequences $x_n \downarrow_{p,q}^{D_{st_o}} \theta$ and $y_n \downarrow_{p,q}^{D_{st_o}} \theta$ in a Riesz space E, where $\lambda \in \mathbb{R}$. Then, the following properties hold:*

(i) The sequence $(x_n + y_n) \downarrow_{p,q}^{D_{st_o}} \theta$.

(ii) The sequence $\lambda x_n \downarrow_{p,q}^{D_{st_o}} \theta$.

3. Deferred statistical order convergence

Definition 3.1. Consider sequences p and q of positive integers satisfying the deferred property. Then, a sequence (x_n) in a Riesz space E is referred to as *deferred statistical order convergent* to x if there exists a sequence $z_n \downarrow_{p,q}^{D_{st_o}} \theta$ with an index set $K \subseteq \mathbb{N}$ such that $\delta_{p,q}(K) = 1$, and the following inequality holds for all $k_n \in K$:

$$|x_{k_n} - x| \leq z_{k_n}.$$

We denote this as $x_n \xrightarrow{D_{st_o}(p,q)} x$.

Remark 3.2. It can be observed that when $x_n \xrightarrow{D_{st_o}(p,q)} x$ holds, the following statement is true:

$$\delta_{p,q}(\{n \in \mathbb{N} : |x_n - x| \nleq z_n\}) = 0.$$

Remark 3.3. It can be observed that the deferred statistical order convergence of the sequence (x_n) in Definition 3.1 with sequence (z_n) to x implies that $x_{k_n} \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$ with the same sequence (z_n). The converse is also true, i.e., if there exists a subsequence $(x_{k_n}) \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$ of a sequence (x_n) with a sequence $z_n \downarrow_{p,q}^{D_{st_o}} \theta$, then $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$ with the same sequence (z_n).

It is obvious that a deferred statistical order decreasing sequence is deferred statistical order convergent. However, the converse does not hold in general.

Remark 3.4. Consider $q(n) = n$ and $p(n) = 0$. The following statements can be observed:

(i) An order convergent sequence is also deferred statistical order convergent to its order limit.

(ii) Statistical order convergence and deferred statistical order convergence are equivalent.

It should be noted that a subsequence of a deferred statistical order convergent sequence may not necessarily be deferred statistical order convergent.

Proposition 3.5. *Let (x_n) be a sequence in a Riesz space E. Then, $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$ satisfies if and only if there exists another sequence (y_n) in E satisfying $\delta_{p,q}(\{n \in \mathbb{N} : x_n = y_n\}) = 1$ and $y_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$.*

Proof. Assume that there exists a sequence (y_n) in E such that $\delta_{p,q}(\{n \in \mathbb{N} : x_n = y_n\}) = 1$ and $y_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$. This implies the existence of another sequence $z_n \downarrow_{p,q}^{D_{st_o}} \theta$ in E with $\delta_{p,q}(K) = 1$, where $|x_{k_n} - x| \leq z_{k_n}$ holds for each $k_n \in K$. Thus, it follows from the including

$$\{p_n + 1 \leq m \leq q_n : |x_m - x| \nleq z_m\} \subseteq \{p_n + 1 \leq m \leq q_n : x_m \neq y_m\} \cup \{p_n + 1 \leq m \leq q_n : |y_m - x| \nleq z_m\}$$

that we can observe that

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} |\{p_n+1 \leq m \leq q_n : |x_m - x| \nleq z_m\}| \leq \lim_{n\to\infty} \frac{1}{q_n - p_n} |\{p_n+1 \leq m \leq q_n : x_m \neq y_m\}|$$

due to $\delta_{p,q}(\{p_n + 1 \leq m \leq q_n : |y_m - x| \nleq z_m\}) = 0$. As a result, we have

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} |\{p_n + 1 \leq m \leq q_n : |x_m - x| \nleq z_m\}| = 0.$$

Thus, we obtain the desired result, $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$. The proof for the other part is straightforward and therefore omitted. □

Proposition 3.6. *The deferred statistical order limit is a linear operator and uniquely determined.*

Proof. Suppose that $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$ and $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} y$ in a Riesz space E. This means that there exist sequences $z_n \downarrow_{p,q}^{D_{st_o}} \theta$ with $\delta_{p,q}(K) = 1$ and $t_n \downarrow_{p,q}^{D_{st_o}} \theta$ with $\delta_{p,q}(M) = 1$, where K and M are index sets, such that $|x_{k_n} - x| \leq z_{k_n}$ and $|x_{m_n} - y| \leq t_{m_n}$ for all $k_n \in K$ and $m_n \in M$. Consequently, we have

$$|x - y| \leq |x - x_{j_n}| + |x_{j_n} - y| \leq z_{j_n} + t_{j_n}$$

for every $j_n \in J := K \cap M$. By utilizing the fact that $(z_{j_n} + t_{j_n})_{j_n \in J} \downarrow \theta$, we deduce that $|x - y| = \theta$, which implies that $x = y$. Hence, x and y are equal.

Now, to prove the linearity of the deferred statistical order limit, consider sequences $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$ and $y_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} y$ in a Riesz space E. This means that there exist sequences

$z_n \downarrow_{p,q}^{D_{sto}} \theta$ and $t_n \downarrow^{D_{sto}} \theta$ such that $\delta_{p,q}(\{n \in \mathbb{N} : |x_n - x| \nleq z_n\}) = 0$ and $\delta_{p,q}(\{n \in \mathbb{N} : |y_n - y| \nleq t_n\}) = 0$. By using the triangle inequality in Riesz spaces, we can conclude that $\{n \in \mathbb{N} : |(x_n + y_n) - (x + y)| \nleq z_n + t_n\} \subseteq \{n \in \mathbb{N} : |x_n - x| \nleq z_n\} \cup \{n \in \mathbb{N} : |y_n - y| \nleq t_n\}$. Therefore, we can deduce that $\delta_{p,q}(\{n \in \mathbb{N} : |(x_n + y_n) - (x + y)| \nleq z_n + t_n\}) = 0$, which means that $x_n + y_n$ converges to $x + y$. □

The following theorem presents various relationships between deferred statistical order convergence and lattice properties.

Theorem 3.7. *Let $x_n \xrightarrow{D_{sto}(p,q)} x$ and $y_n \xrightarrow{D_{sto}(p,q)} y$ in a Riesz space E. Then the following statements hold:*

(i) $x_n \vee y_n$ deferred statistical order convergent to $x \vee y$;
(ii) $x_n \wedge y_n$ deferred statistical order convergent to $x \wedge y$;
(iii) x_n^+ deferred statistical order convergent to x^+;
(iv) x_n^- deferred statistical order convergent to x^-;
(v) $|x_n|$ deferred statistical order convergent to $|x|$.

Proof. It suffices to prove the first statement since the other cases can be obtained by applying Theorem 1.7 from [3] and the previously mentioned proposition. From (x_n) and (y_n) converging to x and y respectively, we can find sequences $z_n \downarrow_{p,q}^{D_{sto}} \theta$ and $t_n \downarrow_{p,q}^{D_{sto}} \theta$, along with index sets K and M such that $\delta_{p,q}(K) = \delta_{p,q}(M) = 1$, and $|x_{k_n} - x| \leq z_{k_n}$ and $|y_{m_n} - y| \leq t_{m_n}$ hold for all $k_n \in K$ and $m_n \in M$. By utilizing Theorem 1.9 from [3] and taking $J := N \cap M$, we can deduce that

$$|x_{j_n} \vee y_{j_n} - x \vee y| \leq |x_{j_n} - x| + |y_{j_n} - y| \leq z_{j_n} + t_{j_n}$$

for every $j_n \in J$. Therefore, we obtain

$$\delta_{p,q}(\{n \in \mathbb{N} : |x_{j_n} \vee y_{j_n} - x \vee y| \nleq z_{j_n} + t_{j_n}\}) = 0.$$

Consequently, we have proven that $x_n \vee y_n$ converges to $x \vee y$. □

Corollary 3.8. *The positive cone $E_+ = \{x \in E : \theta \leq x\}$ of a Riesz space E remains closed under the deferred statistical order convergence.*

Proposition 3.9. *If the sequences $x_n \xrightarrow{D_{sto}(p,q)} x$, $y_n \xrightarrow{D_{sto}(p,q)} y$, and $x_n \geq y_n$ hold for every $n \in \mathbb{N}$ in a Riesz space, then $x \geq y$.*

Proof. Assuming that $y_n \leq x_n$ for each $n \in \mathbb{N}$, we can deduce that $\theta \leq x_n - y_n \in E_+$ for every $n \in \mathbb{N}$. By utilizing the previously stated corollary, we have $x_n - y_n \xrightarrow{D_{sto}(p,q)} x - y \in E_+$ due to the fact that $(x_n - y_n) \in E_+$. Consequently, we get $x - y \geq \theta$, which implies $x \geq y$. □

Theorem 3.10. *If the sequence (x_n) is both monotone and deferred statistical order convergent in a Riesz space, then it is order convergent.*

Proof. Let $(x_n) \downarrow$ and $x_n \xrightarrow{D_{sto}(p,q)} x$ in a Riesz space E. Fix any $k \in \mathbb{N}$. It follows that $x_k - x_n \geq \theta$ for all $n \geq k$, meaning $x_k - x_n \xrightarrow{D_{sto}(p,q)} x_k - x \geq \theta$, which further gives $x_k \geq x$. Thus, x is a lower bound of (x_n) since k is arbitrary. Now, choose another lower bound z of (x_n). We then have $x_n - z \xrightarrow{D_{sto}(p,q)} x - z \geq \theta$, leading to $x \geq z$. As a result, we conclude that $x_n \downarrow x$. □

Remark 3.11. Let A be an ideal in a vector lattice E and (a_n) be a sequence in A. One can observe that if $a_n \xrightarrow{o} \theta$ in A, then $a_n \xrightarrow{o} \theta$ in E. Hence, it is clear that $a_n \downarrow_{p,q}^{D_{sto}} \theta$ in A implies $a_n \downarrow_{p,q}^{D_{sto}} \theta$ in E. For the converse, if $a_n \xrightarrow{o} \theta$ in E and order bounded, then $a_n \xrightarrow{o} \theta$ in A, and so, $a_n \downarrow_{p,q}^{D_{sto}} \theta$ in E implies $a_n \downarrow_{p,q}^{D_{sto}} \theta$ in A for order bounded sequences.

Using Remark 3.11, we can establish the following two results.

Theorem 3.12. *Let A be an ideal in an σ-order complete vector lattice and (x_n) be a sequence in A. Then, $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} \theta$ in A if and only if $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} \theta$ in E.*

Proof. Assume that $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} \theta$ in A. Then, there exists a sequence $z_n \downarrow_{p,q}^{D_{st_o}} \theta$ in A with index set $\delta_{p,q}(K) = 1$ such that $|x_{k_n}| \leq z_{k_n}$ for all $k_n \in K$. Now, by using Remark 3.11, it follows from $(z_{k_n})_{k_n \in K} \downarrow \theta$ in A that $(z_{k_n})_{k_n \in K} \downarrow \theta$ in E, i.e., we get $z_n \downarrow_{p,q}^{D_{st_o}} \theta$ in E. Therefore, we have $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} \theta$ in E.

Conversely, assume $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} \theta$ in E. Then, there is a sequence $z_n \downarrow_{p,q}^{D_{st_o}} \theta$ in E with index set $\delta_{p,q}(K) = 1$ such that $|x_{k_n}| \leq z_{k_n}$ for all $k_n \in K$. Thus, Remark 3.11 implies that $z_n \downarrow_{p,q}^{D_{st_o}} \theta$ in A. Therefore, we get $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} \theta$ in A. □

The theorem stated as [15, Thm.3.1.] yields a similar result, which can be summarized as the following theorem.

Theorem 3.13. *Let (x_n) be a sequence in a Riesz space E and $(x_{k_n})_{k_n \in K}$ be a subsequence of (x_n). If the limit*

$$\liminf_{n\to\infty} \frac{1}{q_n - p_n} \big|\{p_n < k_n \leq q_n : k_n \in K\}\big| > 0$$

holds and $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$ for some sequences p and q satisfying the deferred property, then $x_{k_n} \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$.

Proof. Let's assume that $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$ holds true in the Riesz space E. Thus, there exists a sequence $z_n \downarrow_{p,q}^{D_{st_o}} \theta$ in E such that $\delta_{p,q}(\{n \in \mathbb{N} : |x_n - x| \nleq z_n\}) = 0$. It can be observed that

$$\{p_n < k_n \leq q_n : k_n \in K,\ |x_{k_n} - x| \nleq z_n\} \subseteq \{p_n < n \leq q_n : |x_n - x| \nleq z_n\}.$$

By defining $H_n := \{p_n < k_n \leq q_n : k_n \in K\}$ for all n, we can express the inequality as follows:

$$\frac{1}{|H_n|}\big|\{p_n < k_n \leq q_n : k_n \in K,\ |x_{k_n} - x| \nleq z_n\}\big| \leq \frac{1}{|H_n|}\{p_n < n \leq q_n : |x_n - x| \nleq z_n\}.$$

Therefore, to prove the convergence $x_{k_n} \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$, it is sufficient to show that

$$\limsup_{n\to\infty} \frac{1}{|H_n|}\big|\{p_n < n \leq q_n : |x_n - x| \nleq z_n\}\big| = 0.$$

We observe the following inequality:

$$\begin{aligned}\liminf_{n\to\infty} \frac{|H_n|}{q_n - p_n} &= \limsup_{n\to\infty} \frac{|\{p_n < n \leq q_n : |x_n - x| \nleq z_n\}|}{|H_n|} \\ &\leq \limsup_{n\to\infty} \frac{|\{p_n < n \leq q_n : |x_n - x| \nleq z_n\}|}{q_n - p_n}.\end{aligned}$$

Thus, we have

$$\limsup_{n\to\infty} \frac{1}{|H_n|}\big|\{p_n < n \leq q_n : |x_n - x| \nleq z_n\}\big| = 0$$

due to $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$. This establishes the desired result. □

In Remark 3.4, we establish a connection between statistical order convergence and deferred statistical order convergence by selecting $q(n) = n$ and $p(n) = 0$. We introduce another relationship in the subsequent theorem, subject to a new condition.

Theorem 3.14. *If the sequence* $(\frac{p_n}{q_n - p_n})$ *is bounded for any sequences p and q that satisfy the deferred property, then statistical order convergence implies deferred statistical order convergence.*

Proof. Let us assume that $x_n \xrightarrow{\mathrm{st_o}} x$ in a Riesz space E, and $(\frac{p_n}{q_n - p_n})$ is a bounded sequence for certain sequences p and q that fulfill the deferred property. Consequently, there exists a sequence $z_n \downarrow^{\mathrm{st_o}} \theta$ such that:

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - x| \nleq z_k\}| = 0.$$

By utilizing the deferred property of (q_n), we derive:

$$\lim_{n\to\infty} \frac{1}{q_n} |\{k \leq q_n : |x_k - x| \nleq z_k\}| = 0.$$

Thus, through the inclusion:

$$\{p_n < k \leq q_n : |x_k - x| \nleq z_k\} \subseteq \{k \leq q_n : |x_k - x| \nleq z_k\},$$

we can deduce:

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} |\{p_n < k \leq q_n : |x_k - x| \nleq z_k\}| \leq \lim_{n\to\infty} \frac{1}{q_n} (1 + \frac{p_n}{q_n - p_n}) |\{k \leq q_n : |x_k - x| \nleq z_k\}|.$$

Hence, we obtain the desired result, $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$. □

The converse of Theorem 3.14 does not necessarily hold in general. An example is provided to illustrate this point.

Example 3.15. Consider the Riesz space $E = \mathbb{R}^2$ with the coordinatewise ordering. Let (x_n) be a sequence in E defined as follows:

$$x_n := \begin{cases} (0, \frac{n+1}{2}), & \text{n is odd} \\ (0, -\frac{n}{2}), & \text{n is even} \end{cases}$$

for all n. Additionally, consider the sequences $(q_n) = (2n)$ and $(p_n) = (4n)$. It is clear that the assumption of Theorem 3.14 is satisfied, and $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} (0,0)$. However, it is not statistically order convergent.

The following theorem and its proof demonstrate a result related to convergence in Riesz spaces.

Theorem 3.16. *Let* p', q'*, and* p, q *be pairs of sequences satisfying the deferred property, where* $p_n \leq p'_n$ *and* $q'_n \leq q_n$ *for each* $n \in \mathbb{N}$*. Let* (x_n) *be a sequence in a Riesz space E. If* $x_n \xrightarrow{Dst_o(p',q')} x$*, and the sets* $\{k : p_n < k \leq p'_n\}$ *and* $\{k : q'_n < k \leq q_n\}$ *are finite for every* $n \in \mathbb{N}$*, then* $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$ *in E.*

Proof. Assume that $x_n \xrightarrow{D_{st_o}(p',q')} x$ holds in E. Then, there exists a sequence $z_n \downarrow_{p,q}^{D_{st_o}} \theta$ such that $\delta_{p,q}(n \in \mathbb{N} : |x_n - x| \nleq z_n) = 0$. By considering the equality

$$\{k : p_n < k \leq q_n,\ |x_n - x| \nleq z_n\} = \{k : p_n < k \leq p'_n,\ |x_n - x| \nleq z_n\}$$
$$\cup \{k : p'_n < k \leq q'_n,\ |x_n - x| \nleq z_n\} \cup \{k : q'_n < k \leq q_n,\ |x_n - x| \nleq z_n\},$$

we obtain

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} |\{k : p_n < k \leq q_n,\ |x_n - x| \nleq z_n\}| = 0.$$

Hence, $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$. □

Corollary 3.17. *Let p', q', and p, q be pairs of sequences satisfying the deferred property, where $\lim_{n\to\infty} \frac{q_n - p_n}{q'_n - p'n} = t > 0$ and (x_n) be a sequence in a Riesz space E. If $x_n \xrightarrow{Dst_o(p',q')} x$, then $x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} x$ in E.*

Now, consider the set $C^{p,q}_{(z_n)} = \{(x_n) : \exists x \in E, x_n \xrightarrow{\mathrm{D_{st_o}(p,q)}} x\}$ with (z_n) for a fixed sequence $z_n \downarrow^{Dst_o} \theta$. It is evident that $C^{p,q}_{(z_n)} \subseteq C^{p,q}_{(w_n)}$ whenever $z_n \leq w_n$ for all $n \in \mathbb{N}$.

Proposition 3.18. *If $\delta_{p,q}(\{n \in \mathbb{N} : z_n \neq w_n\}) = 0$, then $C^{p,q}_{(z_n)} = C^{p,q}_{(w_n)}$.*

Proof. Suppose that $\delta_{p,q}(\{n \in \mathbb{N} : z_n \neq w_n\}) = 0$ holds for some sequences $z_n \downarrow^{D_{st_o}}_{p,q} \theta$ and $w_n \downarrow^{D_{st_o}} \theta$. Take any element $(x_n) \in C^{p,q}_{(z_n)}$. Then, there exists $x \in E$ and an index set $\delta p, q(M) = 1$ such that $|x_{m_n} - x| \leq z_{m_n}$ for all $m_n \in M$. By considering the inclusion

$$\{n : |x_n - x| \nleq w_n\} \subseteq \{n : |x_n - x| \nleq z_n\} \cup \{n : z_n \neq w_n\},$$

we can conclude that $(x_n) \in C^{p,q}(w_n)$. Similarly, $(x_n) \in C^{p,q}(w_n)$ implies $(x_n) \in C^{p,q}(z_n)$. Therefore, $C^{p,q}_{(z_n)} = C^{p,q}_{(w_n)}$. □

It is apparent that $C^{p,q}_{(zk_n)} \subseteq C^{p,q}_{(z_n)}$ for any subsequence $z_{k_n} \downarrow^{D_{st_o}}_{p,q} \theta$ of sequence $z_n \downarrow^{D_{st_o}}_{p,q} \theta$. The following proposition provides a result for the converse.

Proposition 3.19. *If $z_n \downarrow \theta$, then $C^{p,q}_{(z_n)} \subseteq C^{p,q}_{(z_{k_n})}$ holds for each subsequence $(z_{k_n})_{k_n \in K}$ of (z_n) with $\delta_{p,q}(K) = 1$.*

Proof. Assume that $z_n \downarrow \theta$ and $(z_{k_n})_{k_n \in K}$ is a subsequence of (z_n) with $\delta p, q(K) = 1$. Consider any element $(x_n) \in C^{p,q}(z_n)$. Then, there exists $x \in E$ and an index set $\delta p, q(M) = 1$ such that $|x_{m_n} - x| \leq z_{m_n}$ for all $m_n \in M$. By defining $J := M \cap K$, it follows that $|x_{j_n} - x| \leq z_{j_n}$ for each $j_n \in J$. Since (z_{j_n}) is a subsequence of (z_{k_n}), we have $(x_n) \in C^{p,q}_{(z_{k_n})}$. □

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