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The Relationship Between Generalized Fibonacci Polynomials

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Abstract. In this study, we obtain the relationship between two different generalized Fibonacci polynomials ($F_{k,n}(t)$ and $F_{k,n}(s)$). We discuss some of the special cases of $F_{k,n}(t)$ and $F_{k,n}(s)$, and we show that the obtained results are valid in these special cases. Since $F_{k,n}(s)$ is a new polynomial obtained by a different selection of the coefficients of the core polynomial used to define $F_{k,n}(t)$, our results will provide a new perspective on this issue. This perspective allows us to generalize classical results, such as the relationship between number sequences, the connection between this relationship and the coefficients of the core polynomial, and the method of obtaining these sequences using matrices.

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1. Introduction

Some of the research on Fibonacci-like numbers and polynomials focuses on the determinant, permanent, and matrix representations of these sequences. In recent years, it has become more important to reveal the relationship between these sequences and to obtain new approaches with the help of this relationship. Among the important arguments used to derive these relationships are generalizations of these sequences. Considering that the most famous of these sequences is the Fibonacci sequence, it is obvious that the generalizations of this sequence will also be the focus of attention. In 1999, Machenry [13] gave one of the most important generalizations of the Fibonacci sequence and obtained important properties of this generalization in [14, 15]. MacHenry [13] defined generalized Fibonacci polynomials, where t_i $(1 \leq i \leq k)$ are constant coefficients of the core polynomial

$$P(x; t_1, t_2, ..., t_k) = x^k - t_1 x^{k-1} - ... - t_k,$$

which is denoted by the vector $t = (t_1, t_2, \ldots, t_k)$. $F_{k,n}(t)$ is defined inductively by

$$\begin{aligned} F_{k,n}(t) &= 0, n < 0, \\ F_{k,0}(t) &= 1, \\ F_{k,n}(t) &= t_1 F_{k,n-1}(t) + \ldots + t_k F_{k,n-k}(t). \end{aligned}$$

In order to understand the importance of the generalized Fibonacci polynomials, the literature should be examined carefully. Many of the important sequences that have a large area of research are special cases of this polynomial sequence. Some of these sequences are the Fibonacci sequence, Pell sequence, Tribonacci sequence, Tripell sequence, Triangular numbers, Padovan sequence, Tetranacci sequence, Pentanacci sequence, etc. Therefore, the properties

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obtained for generalized Fibonacci polynomial will also be valid for many sequences and polynomials. Although the results obtained in this study are valid for all these sequences and polynomials, we will only give some of them as conclusions. For this reason, we find it useful to remind the definition of some sequences.

The Fibonacci(F_n), Pell(P_n), Tribonacci(T_n) and Tripell(tp_n) sequences are defined by the recurrence formulas

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2}, F_0 = 1, F_1 = 1; \\ P_n &= 2P_{n-1} + P_{n-2}, P_0 = 1, P_1 = 2; \\ T_n &= T_{n-1} + T_{n-2} + T_{n-3}, T_0 = 1, T_1 = 1, T_2 = 2; \\ tp_n &= 2tp_{n-1} + tp_{n-2} + tp_{n-3}, tp_0 = 1, tp_1 = 2, tp_2 = 5. \end{aligned}$$

When the literature is examined, it is seen that there are many studies on such sequences. Several studies have focused on determining the relations between sequences [2–4, 8, 18, 19] and as well as deriving the terms of these sequences using determinant [1, 5, 7, 9–12, 17]. Providing a new perspective that encompasses both areas of study will make a significant contribution to the literature. In addition, focusing on the general form of number sequences will offer opportunities to derive additional results. The motivation of our research is to examine Fibonacci-like sequences within a certain classification framework. In this direction, the main purpose of our study is to consider two different sequence classes and to reveal the relationship between these sequences. In addition, it is to show that the general and special cases of matrix representations can be obtained with the help of these relations.

2. Main Results

In this section, two different generalized Fibonacci polynomials are discussed. This difference is obtained as a result of taking the coefficients of the core polynomial differently. It is clear that different relations will be obtained by choosing the coefficients of the core polynomial differently. Here, we will deal with the different selection of the first coefficient. By obtaining the relations with the new polynomial resulting from this difference, it will be possible to obtain new relations between many number sequences.

Theorem 2.1. *Let* $t = (t_1, t_2, \ldots, t_k)$ *and* $s = (t_1 + 1, t_2, \ldots, t_k)$*. Then,*

$$F_{k,n}(s) = F_{k,0}(t).F_{k,n-1}(s) + \cdots + F_{k,n-1}(t).F_{k,0}(s) + F_{k,n}(t). \tag{2.1}$$

Proof. We proceed by induction on n. The result clearly holds for $n = 1$. Let us suppose the result is true for all positive integers less than or equal to n, and prove it for $n+1$. From the definition of generalized Fibonacci polynomial and, the equation $F_{k,n}(s) = (t_1 + 1).F_{k,n-1}(s) + \cdots + t_k.F_{k,n-k}(s)$ is valid. Using this equation, we get

$$\begin{aligned} F_{k,n+1}(s) &= (t_1 + 1).F_{k,n}(s) + \cdots + t_k.F_{k,n-k+1}(s) \\ &= (t_1 + 1).[F_{k,0}(t).F_{k,n-1}(s) + F_{k,1}(t).F_{k,n-2}(s) + \cdots \\ &+ F_{k,n-1}(t).F_{k,0}(s) + F_{k,n}(t)] + t_2.[F_{k,0}(t).F_{k,n-2}(s) \\ &+ F_{k,1}(t).F_{k,n-3}(s) + \cdots + F_{k,n-2}(t).F_{k,0}(s) + F_{k,n-1}(t)] \\ &\vdots \\ &+ t_k.[F_{k,0}(t).F_{k,n-k}(s) + F_{k,1}(t).F_{k,n-k-1}(s) + \cdots \\ &+ F_{k,n-k}(t).F_{k,0}(s) + F_{k,n-k+1}(t)] \\ &= F_{k,0}(t).[(t_1 + 1).F_{k,n-1}(s) + \cdots + t_k.F_{k,n-k}(s)] \\ &+ F_{k,1}(t).[(t_1 + 1).F_{k,n-2}(s) + \cdots + t_k.F_{k,n-k-1}(s)] \\ &\vdots \\ &+ F_{k,n-1}(t).[(t_1 + 1).F_{k,0}(s) + t_2] + F_{k,n}(t).(t_1 + 1) \\ &= F_{k,0}(t).F_{k,n}(s) + F_{k,1}(t).F_{k,n-1}(s) + \cdots + F_{k,n-1}(t).(t_1 + 1) \\ &+ F_{k,n}(t).1 + t_1.F_{k,n}(t) + \cdots + t_k.F_{k,n-k+1}(t) \\ &= F_{k,0}(t).F_{k,n}(s) + F_{k,1}(t).F_{k,n-1}(s) + \cdots + F_{k,n-1}(t).F_{k,1}(s) \\ &+ F_{k,n}(t).F_{k,0}(s) + F_{k,n+1}(t). \end{aligned}$$

Therefore, the equation holds for all positive integers n. □

Corollary 2.2. *Let T_n be the nth term of the Tribonacci numbers and tp_n be the nth term of the Tripell numbers. Then,*

$$tp_n = T_0.tp_{n-1} + T_1.tp_{n-2} + \cdots + T_{n-1}.tp_0 + T_n.$$

Proof. If we take $t = (1, 1, 1)$ and $s = (2, 1, 1)$ in Theorem 2.1, we obtain the desired result. □

Corollary 2.3. *Let a_n be the nth term of the sequence A002605 and b_n be the nth term of the sequence A007482 in [16]. Then,*

$$b_n = a_0.b_{n-1} + a_1.b_{n-2} + \cdots + a_{n-1}.b_0 + a_n.$$

Proof. If we take $t = (2, 2)$ and $s = (3, 2)$ in Theorem 2.1, we obtain the desired result. □

Corollary 2.4. *We present the following sequences, the special case of the generalized Fibonacci polynomial, by specifying their relationship.*

	$F_{k,n}(t)$	$F_{k,n}(s)$
$k = 2$	*Fibonacci sequence*	*Pell sequence*
$k = 2$	*Pell sequence*	*Sequence A006190 in [16]*
$k = 2$	*Sequence A006190 in [16]*	*Sequence A001076 in [16]*
$k = 2$	*Sequence A002605 in [16]*	*Sequence A007482 in [16]*
$k = 3$	*Tribonacci Sequence*	*Tripell Sequence*
$k = 3$	*Sequence A077835 in [16]*	*Sequence A077831 in [16]*
$k = 4$	*Tetrabacci Sequence*	*Sequence A103142 in [16]*
$k = 5$	*Pentanacci*	*Generalized Pell Numbers($P(n, 5, 5)$)*

3. Applications

Number sequences and polynomials have many applications from physics to chemistry, economics to social sciences. Matrix and determinant applications are a few of them. The determinant representations of Fibonacci type sequences and polynomials have an important place in the literature and results are obtained with different approaches. When the literature is examined, although there are dozens of studies on this subject, some studies want to obtain a more general situation. In [12] and [7], the authors gave several determinantal representations of the generalized Fibonacci polynomials. The following theorem will allow us to approach this fact in a different way and obtain it possible to make determinant representations of many sequences and polynomials. In continuation, we show that new results can be obtained by considering the method in [6].

Theorem 3.1. *Let $n \geq 1$ be an integer, $t = (t_1, t_2, \ldots, t_k)$, $s = (t_1 + 1, t_2, \ldots, t_k)$ and the $n \times n$ lower Hessenberg matrix sequence $A_n = [a_{i,j}]_{i,j=1,2,\ldots,n}$ be defined by*

$$A_n = \begin{bmatrix} F_{k,0}(t) & -1 & 0 & 0 & 0 \\ F_{k,1}(t) & F_{k,0}(t) & -1 & 0 & 0 \\ F_{k,2}(t) & F_{k,1}(t) & F_{k,0}(t) & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ F_{k,n-1}(t) & F_{k,n-2}(t) & \cdots & F_{k,1}(t) & F_{k,0}(t) \end{bmatrix}.$$

Then,

$$\det A_n = F_{k,n-1}(s).$$

Proof. We proceed by induction on n. The result clearly holds for $n = 1$. Let us suppose the result is true for all positive integers less than or equal to $(n - 1)$, and prove it for n. From Equation (2.1), we get $A_n \cdot [1, F_{k,0}(s), \ldots, F_{k,n-2}(s)]^T = [0, 0, \ldots, 0, F_{k,n-1}(s)]^T$. By Cramer's rule we get

$$F_{k,n-2}(s) = \frac{\det A_{n-1}.F_{k,n-1}(s)}{\det A_n}.$$

Then,

$$F_{k,n-1}(s) = \frac{\det A_n . F_{k,n-2}(s)}{\det A_{n-1}}.$$

From the induction hypothesis we obtain $\det A_n = F_{k,n-1}(s)$. □

Theorem 3.2. *Let n be any integer such that $n \geq 1$,*

$$\widehat{A_n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ & & & \vdots \\ & -(A_{n-1}) & & 0 \\ & & & 1 \end{bmatrix}.$$

Then,

$$(\widehat{A_n})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ F_{k,0}(s) & 1 & 0 & 0 & 0 & 0 \\ F_{k,1}(s) & F_{k,0}(s) & 1 & 0 & 0 & 0 \\ F_{k,2}(s) & F_{k,1}(s) & F_{k,0}(s) & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 & 0 \\ F_{k,n-2}(s) & F_{k,n-2}(s) & \cdots & F_{k,1}(s) & F_{k,0}(s) & 1 \end{bmatrix}.$$

Proof. The proof follows directly from the previous theorem and the equation (2.1). □

Using Theorems 3.1 and 3.2, we can obtain different results for many number sequences and polynomials. Some of these are the relationships between two sequences, obtaining one as a determinant of the other, and obtaining the other using terms of one. We present some of them below.

Corollary 3.3. *Let T_n be the nth term of the Tribonacci sequence, tp_n be the nth term of the Tripell sequence and the $n \times n$ lower Hessenberg matrix sequence $C_n = [c_{i,j}]_{i,j=1,2,\ldots,n}$ be defined by*

$$C_n = \begin{bmatrix} T_0 & -1 & 0 & 0 & 0 \\ T_1 & T_0 & -1 & 0 & 0 \\ T_2 & T_1 & T_0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ T_{n-1} & T_{n-2} & \cdots & T_1 & T_0 \end{bmatrix}.$$

Then, $\det C_n = tp_{n-1}$.

Proof. If we take $t = (1, 1, 1)$ and $s = (2, 1, 1)$ in Theorem 3.1, we obtain the desired result. □

Corollary 3.4. *Let T_n be the nth term of the Tribonacci sequence, tp_n be the nth term of the Tripell sequence and the $n \times n$ lower triangular matrix sequence $\widehat{C_n} = [\widehat{c}_{ij}]_{i,j=1,2,\ldots,n}$ be defined by*

$$\widehat{c}_{ij} = \begin{cases} 1, & i = j, \\ -T_{m-1}, & i - j = m > 0, \\ 0, & i - j < 0. \end{cases}$$

Then, $(\widehat{C_n})^{-1} = [d_{ij}]_{i,j=1,2,\ldots,n}$ are obtaied by

$$d_{ij} = \begin{cases} 1, & i = j, \\ tp_{m-1}, & i - j = m > 0, \\ 0, & i - j < 0. \end{cases}$$

Proof. If we take $t = (1, 1, 1)$ and $s = (2, 1, 1)$ in Theorem 3.2, we obtain the desired result. □

Corollary 3.5. *Let a_n be the nth term of the sequence A002605, b_n be the nth term of the sequence A007482 in [16] and the $n \times n$ lower Hessenberg matrix sequence $B_n = [b_{i,j}]_{i,j=1,2,\ldots,n}$ be defined by*

$$B_n = \begin{bmatrix} a_0 & -1 & 0 & 0 & 0 \\ a_1 & a_0 & -1 & 0 & 0 \\ a_2 & a_1 & a_0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{bmatrix}.$$

Then, $\det B_n = b_{n-1}$.

Proof. If we take $t = (2, 2)$ and $s = (3, 2)$ in Theorem 3.1, we obtain the desired result. □

Corollary 3.6. *Let a_n be the nth term of the sequence A002605, b_n be the nth term of the sequence A007482 in [16] and the $n \times n$ lower triangular matrix sequence $\widehat{B_n} = [\widehat{b}_{ij}]_{i,j=1,2,\ldots,n}$ be defined by*

$$\widehat{b}_{ij} = \begin{cases} 1, & i = j, \\ -a_{m-1}, & i - j = m > 0, \\ 0, & i - j < 0. \end{cases}$$

Then, $(\widehat{B_n})^{-1} = [u_{ij}]_{i,j=1,2,\ldots,n}$ are obtaied by

$$u_{ij} = \begin{cases} 1, & i = j, \\ b_{m-1}, & i - j = m > 0, \\ 0, & i - j < 0. \end{cases}$$

Proof. If we take $t = (2, 2)$ and $s = (3, 2)$ in Theorem 3.2, we obtain the desired result. □

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the manuscript.

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