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A TUTORIAL ON THE SINGULAR VALUE DECOMPOSITION

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ABSTRACT: *An $m \times n$ real matrix* **A** *can be factored as* $\mathbf{UWV}^T$, *where* **U** *and* **V** *are orthonormal, and* **W** *is upper left diagonal. This factorization is called Singular Value Decomposition (SVD). The matrices* **U**, **W**, *and* **V** *are useful in characterizing the matrix* **A**. *In this manuscript geometric characterizations are emphasized. Geometric characterizations are analyzed in terms of subspaces, matrix scaling, and norms. We also present a numerical viewpoint for SVD in order to keep the material self-contained. In the last section we treat a special problem where action of the matrix* **A** *is restricted to a given subspace.*

KEYWORDS: *Singular values, matrix decomposition, matrix*

TEKİL DEĞER AYRIŞTIRMAYA GENEL BAKIŞ

ÖZET: *Gerçel bir* **A** *matrisi* $\mathbf{UWV}^T$ *şeklinde ifade edilebilir; burada* **U** *ve* **V** *ortonormal,* **W** *ise sol üst köşegen matrislerdir. Bu işlem Tekil Değer Ayrıştırma olarak adlandırılır. Bu ayrıştırma* **A** *matrisinin çeşitli özelliklerini belirlemede kullanışlıdır. Bu çalışmada bir matrisin geometrik özelliklerinin belirlenmesi vurgulanmaktadır. Geometrik özellikler alt uzaylar, ölçekleme yeteneği ve normlar türünden incelenmektedir. SVD ye sayısal bir bakış açısının konu bütünlüğü açısından sunulmasının ardından makalenin son kısmında bir matrisi ölçekleme etkisinin verilen bir alt uzayda sınırlandırıldığı bir özel problem incelenmektedir.*

ANAHTAR KELİMELER: *Tekil değerler, matris ayrıştırma, matris*

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I. INTRODUCTION

In this section the *singular value decomposition* (SVD) is introduced through explicit construction. Several properties of symmetric matrices are discussed in the construction process. We also include an example to illustrate the steps in the construction.

In Section II the range and null spaces of a given matrix is obtained from its SVD. Discussing also the complementary subspaces of matrices the results are presented in a table form. In Section III scaling abilities of matrices in various directions are investigated. The results are related to the matrix norms. In section IV and V we discuss several side issues and a numerical viewpoint for SVD as a matter of self-containedness. We treat a special problem in Section VI: We derive procedures to find the maximum amplification abilities of matrices in a given subspace.

For a given matrix $\mathbf{A} \in \mathbf{R}^{m \times n}$, the SVD provides a set of real numbers (will be called *singular values*), and orthonormal bases for very *useful* subspaces in $\mathbf{R}^m$ and $\mathbf{R}^n$. Using these, the matrix $\mathbf{A}$ is characterized geometrically and computationally.

Symmetric matrices play particularly important role in the decomposition process. The following lemma characterizes the eigenvectors of symmetric matrices.

Lemma I.1 *(for instance [4]) Eigenvectors of any symmetric matrix* $\mathbf{A} \in \mathbf{R}^{n \times n}$ *form an orthonormal basis for* $\mathbf{R}^n$. □

Next we present the Singular Value Decomposition theorem. We also present a proof for this theorem in order to establish notation and utilize certain parts in the succeeding sections. For the proofs using alternative combinations of mathematical tools, reader may refer to [2], [3], [4] or [5].

Theorem I.2 *Any* $\mathbf{A} \in \mathbf{R}^{m \times n}$ can be factored as

$$\mathbf{A} = \mathbf{U}\mathbf{W}\mathbf{V}^T \tag{1}$$

such that $\mathbf{U} \in \mathbf{R}^{m \times m}$ and $\mathbf{V} \in \mathbf{R}^{n \times n}$ *are orthonormal, and* $\mathbf{W} \in \mathbf{R}^{m \times n}$ *has the form*

$$\begin{bmatrix} \hat{\mathbf{W}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{2}$$

with $\hat{\mathbf{W}}$ *diagonal and full-rank.*

Proof

We will first construct the matrix $\mathbf{U}^T\mathbf{A}\mathbf{V}$ using orthonormal $\mathbf{U}$ and $\mathbf{V}$, and show that this matrix equals upper-left diagonal matrix $\mathbf{W}$ as in Equation 2. Since $\mathbf{Q}^T = \mathbf{Q}^{-1}$ is valid for any orthonormal $\mathbf{Q}$, we will conclude that $\mathbf{A} = \mathbf{U}\mathbf{W}\mathbf{V}^T$.

Let λ_i, $i = 1,\dots,n$ be the eigenvalues of $\mathbf{A}^T\mathbf{A}$. The matrix $\mathbf{A}^T\mathbf{A}$ is obviously real symmetric. Since real matrices of the form $\mathbf{A}^T\mathbf{A}$ have nonnegative eigenvalues, we are allowed to group them as positive and zero ones. Let $\lambda_i > 0$ for $i = 1,\dots,s$ and $\lambda_i = 0$ for $i = s+1,\dots,n$. Select an orthonormal matrix $\mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2 \cdots \mathbf{v}_n]$ such that its each column $\mathbf{v}_i$ is an eigenvector of $\mathbf{A}^T\mathbf{A}$ corresponding to λ_i, $i = 1,\dots,n$. Note that existence of $\mathbf{V}$ is a consequence of Lemma I.1.

Set $w_i = \left|\sqrt{\lambda_i}\right|$ and $\mathbf{u}_i = \mathbf{A}\mathbf{v}_i / w_i$ for $i = 1,\dots,s$. Together with appropriate $\mathbf{u}_{s+1},\dots,\mathbf{u}_m$, an orthonormal matrix $\mathbf{U} = [\mathbf{u}_1\ \mathbf{u}_2 \cdots \mathbf{u}_m]$ can be formed. The appropriate column vectors can easily be found using Gram-Schmidt process.

Now observe that:

(OBS1) $s \le n$ and $s \le m$

(OBS2) $\mathbf{Av}_i = \mathbf{0}$ for $i = s+1,\dots,m$, since $\|\mathbf{Av}_i\|_2^2 = \lambda_i \cdot \|\mathbf{v}_i\|_2^2 = \lambda_i$, and λ_i is nonzero only if $i = 1,\dots,s$. Hence $\|\mathbf{Av}_i\|_2^2 = 0$ for $i = s+1,\dots,n$, and $\|\mathbf{Av}_i\|_2^2 = 0$ implies $\mathbf{Av}_i = \mathbf{0}$.

(OBS3) $\mathbf{u}_i^T \mathbf{u}_j = 0$ for $i \ne j$ and $i, j = 1,\dots,m$

Let us construct $\mathbf{W} := \mathbf{U}^T\mathbf{AV}$. Its (i,j)-th element $w_{i,j}$ equals $\mathbf{u}_i^T\mathbf{Av}_j$. Partition this matrix into two submatrices $\mathbf{W}_1$ and $\mathbf{W}_2$ such that they contain first s and last n-s columns of $\mathbf{W}$ respectively. The observation (OBS2) implies $\mathbf{W}_2$=0. For $\mathbf{W}_1$, the i-th diagonal element is

$$\mathbf{u}_i^T\mathbf{Av}_i = (\mathbf{Av}_i / w_i)^T \mathbf{Av}_i = \left(\mathbf{v}_i^T\mathbf{A}^T / w_i\right)\mathbf{Av}_i = \mathbf{v}_i^T \lambda_i \mathbf{v}_i / w_i = w_i$$

and nondiagonal elements are zero due to (OBS1) and (OBS3).

Thus $\mathbf{W}$, with the form given by Equation 2, satisfies $\mathbf{W} = \mathbf{U}^T\mathbf{AV}$. This implies $\mathbf{A} = \mathbf{UWV}^T$, which completes the proof. □

Note that the proof shows existence of the decomposition by constructing the factors explicitly. Before proceeding with geometric properties of this decomposition few remarks are in order.

In the proof, without noticing, we obtained eigenvectors of $\mathbf{AA}^T$ from that of $\mathbf{A}^T\mathbf{A}$, and used in the construction of $\mathbf{U}$. Indeed, each column vector $\mathbf{u}_i$, $i = 1,\dots,s$, is an eigenvector of $\mathbf{AA}^T$ associated with the eigenvalue λ_i. Clearly,

$$\mathbf{AA}^T\mathbf{u}_i = \mathbf{AA}^T(\mathbf{Av}_i / w_i) = \mathbf{A}\left(\mathbf{A}^T\mathbf{Av}_i\right)/w_i = \mathbf{A}\lambda_i\mathbf{v}_i / w_i = \lambda_i(\mathbf{Av}_i / w_i) = \lambda_i\mathbf{u}_i$$

proves this assertion. It should be observed that nonzero eigenvalues of $\mathbf{A}^T\mathbf{A}$ and $\mathbf{AA}^T$ are the same.

The diagonal elements of $\hat{\mathbf{W}}$, namely w_i, $i = 1,2,\dots,s$, are called *singular values* of $\mathbf{A}$. A natural consequence of this definition is that the matrices $\mathbf{A}$ and $\mathbf{A}^T$ have the same singular values. Since

$$\mathbf{A}^T = \left(\mathbf{UWV}^T\right)^T = \mathbf{VW}^T\mathbf{U}^T = \mathbf{VWU}^T,$$

singular values are invariant under transposition.

II. GEOMETRIC NOTIONS

The SVD provides two insight-giving subspaces of a decomposed matrix: its range and null spaces. These are denoted by $\mathcal{R}(.)$ and $\mathcal{N}(.)$ respectively. The singular value decomposition provides orthogonal bases for these subspaces (see for instance [1], [2], and [5]). Their orthogonal complements are inherently available in this decomposition.

We justify in simple terms that SVD of $\mathbf{A}$ contains its range and null spaces as follows. From Expression 1 we obtain

$$\mathbf{AV} = \mathbf{UW},$$

that is,

$$\mathbf{A}\left[\mathbf{v}_1\ \mathbf{v}_2\ \cdots \mathbf{v}_n\right] = \left[\mathbf{u}_1\ \mathbf{u}_2\ \cdots \mathbf{u}_m\right]\begin{bmatrix}\hat{\mathbf{W}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix},$$

or

$$\left[\mathbf{Av}_1 \cdots \mathbf{Av}_s \ \mathbf{Av}_{s+1} \cdots \mathbf{Av}_n\right] = \left[w_1\mathbf{u}_1 \cdots w_s\mathbf{u}_s \cdots \mathbf{0} \cdots \mathbf{0}\right]. \qquad (3)$$

To display Equation 3 more intuitively, we equate columns of the left- and right-hand side matrices in two parts:

$$\mathbf{Av}_i = w_i\mathbf{u}_i, \quad i = 1,\ldots,s \qquad (4)$$

and

$$\mathbf{Av}_i = \mathbf{0}, \quad i = s+1,\ldots,n \qquad (5)$$

Nonzero vectors resulted by action of **A** on the columns of **V** form a basis for $\mathcal{R}(\mathbf{A})$. These basis vectors $w_1\mathbf{u}_1,\ldots,w_s\mathbf{u}_s$ (equivalently, $\mathbf{u}_1,\ldots,\mathbf{u}_s$), in fact, form an orthogonal basis for this subspace. For a verification of orthogonality recall how these vectors have been constructed. The null space of **A** is apparently the columns of **V** that are mapped to zero vectors by **A**. These are the last n-s columns of V, that is, $\mathbf{v}_{s+1}, \mathbf{v}_{s+2}, \ldots, \mathbf{v}_n$.

It has been shown that first s columns of **U** and last n-s columns of **V** serve as range and null spaces of **A** respectively. Since each of **U** and **V** is an orthonormal matrix, bases for orthonormal complements of $\mathcal{R}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A})$ are also parts of these matrices. In more precise terms, last n-s columns of **U** form a basis for $\mathcal{R}(\mathbf{A})^{\perp}$. Similarly, first s columns of **V** form a basis for $\mathcal{N}(\mathbf{A})^{\perp}$. The perp sign, $\perp$, denotes orthogonal complementation.

The four subspaces extracted from **U** and **V** are related to each other by $\mathcal{R}(\mathbf{A})^{\perp} = \mathcal{N}(\mathbf{A}^T)$ and $\mathcal{N}(\mathbf{A})^{\perp} = \mathcal{R}(\mathbf{A}^T)$. Table 2.1 summarizes subspaces of **A** and relevant basis vectors.

Subspace	Basis Vectors
$\mathcal{R}(A)$	$\mathbf{u}_1,\ldots,\mathbf{u}_s$
$\mathcal{N}(A)$	$\mathbf{v}_{s+1},\ldots,\mathbf{v}_n$
$\mathcal{R}(\mathbf{A})^{\perp}$	$\mathbf{u}_{s+1},\ldots,\mathbf{u}_m$
$\mathcal{N}(\mathbf{A})^{\perp}$	$\mathbf{v}_1,\ldots,\mathbf{v}_s$

Table 2.1 The subspaces of matrix **A**

In this section columns of **U** and **V** have been related to range and null spaces of **A**. In the next section, the singular values are related to the size of action of **A** in various directions.

III. SVD CHARACTERIZATION OF MATRIX MAGNIFICATION

Operating on a vector a matrix may rotate and scale it. In several engineering applications matrices characterize systems with their extreme abilities in rotating and scaling. For instance, in control engineering a generalization of gain and phase margin to multivariable systems is related to smallest singular value of the return difference matrix ([6], [7]). Also in signal processing the direction where the oriented energy is maximum is related to the largest singular value of the matrix formed by sample vectors [8]. In this section maximum (minimum) scaling ability of matrices is related to their largest (smallest) singular values in simple and intuitive terms.

Consider Eq. (4)

$$\mathbf{Av}_i = w_i\mathbf{u}_i, \quad i = 1,\ldots,s$$

and note that the norms of both $\mathbf{v}_i$ and $\mathbf{u}_i$ are unity. Matrix **A** maps unity-norm $\mathbf{v}_i$ to a vector $w_i\mathbf{u}_i$, whose norm is w_i. Thus

$$\|\mathbf{Av}_i\|_2 = \|w_i\mathbf{u}_i\|_2 = |w_i| \cdot \|\mathbf{u}_i\|_2 = w_i$$

for $i = 1,\ldots,s$, that is, any vector in the direction $\mathbf{v}_i$ is w_i times magnified by $\mathbf{A}$.

We next show that $\max_{\|\mathbf{x}\|=1}\|\mathbf{Ax}\|_2$ is achieved for some $\mathbf{x}\in\{\mathbf{v}_1,\ldots,\mathbf{v}_s\}$, and equals $\max_{i=1,\ldots,s}\{w_i\}$, that is,

$$\|\mathbf{A}\|_2 = \max_{i=1,\ldots,s}\{w_i\}.$$

Since $\mathbf{V}$ is an orthonormal matrix, its columns form an orthonormal basis for $\mathbf{R}^n$. We can write any unity-norm vector x in $\mathbf{R}^n$ as

$$x = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$$

for some α_i, $i = 1,\ldots,n$, such that $\sum_{i=1}^{n}\alpha_i^2 = 1$. Using this

$$\begin{aligned}\|\mathbf{Ax}\|_2^2 &= \|\mathbf{A}(\alpha_1\mathbf{v}_1+\cdots+\alpha_n\mathbf{v}_n)\|_2^2\\ &= \|\alpha_1\mathbf{Av}_1+\cdots+\alpha_n\mathbf{Av}_n\|_2^2\\ &= \|\alpha_1 w_1\mathbf{v}_1+\cdots+\alpha_n w_n\mathbf{v}_n\|_2^2\\ &= \left\|\mathbf{U}\begin{bmatrix}\alpha_1 w_1\\ \vdots\\ \alpha_n w_n\end{bmatrix}\right\|_2^2\end{aligned}$$

Since $\mathbf{U}$ is orthonormal and since orthonormal operators do not change the norm

$$\|\mathbf{Ax}\|_2^2 = \left\|\begin{bmatrix}\alpha_1 w_1\\ \vdots\\ \alpha_n w_n\end{bmatrix}\right\|_2^2$$

$$= \sum_{i=1}^{n}(\alpha_i w_i)^2 \le \max_{i\in\{1,\ldots,n\}}\{w_i^2\}\cdot\sum_{i=1}^{n}\alpha_i^2 = \max_{i\in\{1,\ldots,n\}}\{w_i^2\} = \max_{i\in\{1,\ldots,s\}}\{w_i^2\}$$

This shows that the norm of interest is upper bounded:

$$\|\mathbf{Ax}\|_2 \le \max_{i\in\{1,\ldots s\}}\{w_i\} \text{ for all } \|\mathbf{x}\|_2 = 1 \tag{6}$$

To show that this upper bound is actually achieved let $w_k := \max_{i\in\{1,\ldots,s\}}\{w_i\}$ and select $x=v_k$

$$\|\mathbf{Ax}\|_2 = \|\mathbf{Av_k}\|_2 = \|w_k\mathbf{u_k}\|_2 = |w_k|\|\mathbf{u_k}\|_2 = |w_k| = \max_{i\in\{1,\ldots,s\}}\{w_i\} \tag{7}$$

Equations 6 and 7 show that

$$\|\mathbf{A}\|_2 = \max_{i\in\{1,\ldots,s\}}\{w_i\} \tag{8}$$

We have shown that the greatest singular value is the maximum amplification obtainable by the operator. Likewise, the least singular value can be associated by the least amplification obtainable by full column rank operators:

Theorem III.1 *Let the null space of* $\mathbf{A}\in\mathbf{R}^{m\times n}$ *contains only the zero vector. Then*

$$\min_{\|\mathbf{x}\|_2=1}\|\mathbf{Ax}\|_2 = \min_{i\in\{1,\ldots,s\}}\{w_i\}$$

□

Let us investigate a relation between the least and greatest singular values of nonsingular operators. Let $\mathbf{A}\in\mathbf{R}^{n\times n}$ be nonsingular. Then the SVD of $\mathbf{A}^{-1}$ is

$$\mathbf{A}^{-1} = \mathbf{V}\mathbf{W}^{-1}\mathbf{U}^T$$

with

$$\mathbf{W}^{-1} = \begin{bmatrix} w_1^{-1} & 0 & 0 & \cdots & 0 \\ 0 & w_2^{-1} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & w_n^{-1} \end{bmatrix}$$

Hence the singular values of $\mathbf{A}^{-1}$ are $w_1^{-1}, w_2^{-1}, \ldots, w_n^{-1}$. Together with Theorem III.1 and Eqn. 8 this gives rise to the following corollary.

Corollary III.2 *Let* $\mathbf{A} \in \mathbf{R}^{n\times n}$ *be nonsingular then*

$$\left\|\mathbf{A}^{-1}\right\|_2 = \frac{1}{\min_{\|\mathbf{x}\|_2=1}\|\mathbf{A}\mathbf{x}\|_2}$$

□

IV. A COMPACT REPRESENTATION OF SVD AND EXTENSION TO THE COMPLEX FIELD

In this section existence of a compact representation of SVD will be shown as a corollary of Theorem I.2. Following this, an extension of the SVD to the complex field will be presented.

The following corollary may be proven by a simple inspection. We present the proof in order to form a background for the succeeding comments.

Corollary IV.1 *Any* $\mathbf{A} \in \mathbf{R}^{m\times n}$ *can be factored as*

$$\mathbf{A} = \tilde{\mathbf{U}}\hat{\mathbf{W}}\tilde{\mathbf{V}}^{\mathrm{T}}$$

such that $\tilde{\mathbf{U}} \in \mathbf{R}^{m\times s}$ *and* $\tilde{\mathbf{V}} \in \mathbf{R}^{n\times s}$ *satisfy* $\tilde{\mathbf{U}}^{\mathrm{T}}\tilde{\mathbf{U}} = \tilde{\mathbf{V}}^{\mathrm{T}}\tilde{\mathbf{V}} = \mathbf{I}_s$, *where* $\mathbf{I}_s$ *is the* $s\times s$ *unit matrix with* $s := \mathrm{rank}(\mathbf{A})$. $\hat{\mathbf{W}} \in \mathbf{R}^{s\times s}$ *is diagonal with* $\mathrm{rank}(\mathbf{A}) = \mathrm{rank}(\hat{\mathbf{W}})$.

Proof

From Theorem I.2 it is possible to factorize $\mathbf{A}$ as $\mathbf{A} = \mathbf{U}\mathbf{W}\mathbf{V}^{\mathrm{T}}$. In an explicit form:

$$\mathbf{A} = \begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_s & \mathbf{u}_{s+1} & \cdots \mathbf{u}_m\end{bmatrix}\begin{bmatrix}\hat{\mathbf{W}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{v}_1^{\mathrm{T}} \\ \vdots \\ \mathbf{v}_s^{\mathrm{T}} \\ \mathbf{v}_{s+1}^{\mathrm{T}} \\ \vdots \\ \mathbf{v}_n^{\mathrm{T}}\end{bmatrix}$$

$$= \begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_s & \mathbf{u}_{s+1} & \cdots \mathbf{u}_m\end{bmatrix}\begin{bmatrix}w_1\mathbf{v}_1^{\mathrm{T}} \\ \vdots \\ w_s\mathbf{v}_s^{\mathrm{T}} \\ \mathbf{0} \\ \vdots \\ \mathbf{0}\end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_s & 0 & \cdots 0\end{bmatrix}\begin{bmatrix} w_1\mathbf{v}_1^{\mathrm{T}} \\ \vdots \\ w_s\mathbf{v}_s^{\mathrm{T}} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= \begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_s\end{bmatrix}\begin{bmatrix} w_1\mathbf{v}_1^{\mathrm{T}} \\ \vdots \\ w_s\mathbf{v}_s^{\mathrm{T}} \end{bmatrix} \\
&= \begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_s\end{bmatrix}\hat{\mathbf{W}}\begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_s\end{bmatrix}^{\mathrm{T}} \\
&:= \tilde{\mathbf{U}}\hat{\mathbf{W}}\tilde{\mathbf{V}}^{\mathrm{T}}
\end{aligned}
$$

Since $\tilde{\mathbf{U}}$ and $\tilde{\mathbf{V}}$ are partitions of **U** and **V** in Expression 1, the assertions $\tilde{\mathbf{U}}^{\mathrm{T}}\tilde{\mathbf{U}} = \tilde{\mathbf{V}}^{\mathrm{T}}\tilde{\mathbf{V}} = \mathbf{I}_s$ and rank(A) = rank($\hat{\mathbf{W}}$) follow Theorem I.2. □

This version of SVD also preserves the singular values, however, does not contain columns of **U** and **V** associated with $\mathcal{R}(\mathbf{A}^{\mathrm{T}})$ and $\mathcal{N}(\mathbf{A}^{\mathrm{T}})$. Depending on the application of SVD Theorem IV.2 may be found more practical, since its construction requires less computation and storage. Several numerical analysis softwares use the compact representation of SVD [9].

Thus far we have considered only the real matrices to decompose. The motivations for this are the physical motivations and simplicity in presentation. Generalization of the SVD to complex matrices is straightforward:

Theorem IV.2 *Any* $\mathbf{A} \in \mathbf{C}^{m\times n}$ *can be factored* as

$$\mathbf{A} = \mathbf{U}\mathbf{W}\mathbf{V}^{\mathrm{T}}$$

such that $\mathbf{U} \in \mathbf{C}^{\mathrm{m}\times\mathrm{m}}$ *and* $\mathbf{V} \in \mathbf{C}^{n\times n}$ *are orthonormal, and* $\mathbf{W} \in \mathbf{R}^{\mathrm{m}\times\mathrm{n}}$ *has the form*

$$\begin{bmatrix}\hat{\mathbf{W}} & 0 \\ 0 & 0\end{bmatrix}$$

with $\hat{\mathbf{W}}$ *diagonal and full-rank.* □

Changing the field of **A** to complex numbers has changed the fields of **U** and **V**. However, field of the singular values remained unchanged. In the proof a complex version of Lemma I.1 can be used. For this refer to any textbook showing that symmetric matrices generalizes to Hermitean matrices in complex setting. Corollary IV.1-like version of Theorem IV.2 is straightforward, and omitted.

V. A COMPUTATIONAL VIEWPOINT

In this section we briefly present construction of SVD by computers. Then we discuss the meaning of relatively small singular values in computing the solutions of linear algebraic equations.

In the first section we have presented a systematic method of constructing the SVD. This method uses the eigenvalues of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ in the construction. Lets recall that these eigenvalues are the squares of the singular values of **A**. However, direct computation of the eigenvalues of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ is not used in practice, since squaring **A** may cause loss of information in finite precision machines. Francis is the first addressing this problem and proposes the best solution ever ([10] and [11]): Transform **A** into a form which preserves the singular values, and which allows extraction of the orthogonal factors of SVD using numerically harmless operations. A slightly improved and recapitulated form of this algorithm takes place in the literature few years later [12].

Algorithm of Francis requires bringing the matrix **A** into an upper bidiagonal form via unitary transformations. Indeed there exists orthogonal matrices $\mathbf{P}^T$ and Q such that $\mathbf{P}^T\mathbf{AQ} = \mathbf{A}_{ub}$ is upper bidiagonal. **P** and **Q** can be obtained as products of sequences of appropriate Householder matrices ([12] or any advanced textbook). Then iterative multiplication of $\mathbf{A}_{ub}$ by appropriately selected unitary matrices converges to an upper diagonal matrix $\hat{\mathbf{A}}$ whose upper diagonal entries are the singular values of A. This iterative part of the algorithm is called *QR algorithm*. Denoting the resultant iterative matrices by $\mathbf{R}^T$ and $\mathbf{S}$ we can write $\mathbf{R}^T\mathbf{P}^T\mathbf{AQS} = \hat{\mathbf{A}}$. This yield the SVD of **A**: $(\mathbf{PR})\hat{\mathbf{A}}(\mathbf{QS})^T$.

For parallel machines Kogbetliantz's algorithm is more efficient compared to that of Francis [13]. This algorithm also extends to some other orthogonal decompositions such as the QR decomposition.

The ratio w_{max} / w_{min} is called the *condition number*. Large condition numbers cause error in numerical solutions of linear algebraic equations [12]. This phenomenon may be avoided with a little penalty which is illustrated as follows: Let **A** have the singular value decomposition $\mathbf{UWV}^T$, and let **x** be the unknown to be solved in the linear algebraic equation

$$\mathbf{Ax} = \mathbf{b}. \tag{10}$$

Assume that the condition number of **A** is "large", that is, larger than a certain threshold number which is determined by numerical representation range of the computing machine. Without loss of generality we assume that some singular values are as small as could be affected by the round-off error. Equations corresponding to these singular values are noise-corrupted equations, and they contribute to the error in the solutions. Therefore, the noise-corrupted equations should be disregarded. Substituting the SVD of **A** in Eqn. 10 we obtain

$$\mathbf{UWV}^T\mathbf{x} = \mathbf{b}. \tag{11}$$

Substituting 0 for the small singular values eliminates the corresponding equations in Eqn. 10. Writing Eqn. 11 in scalar notation shows this. Many commercial softwares recommends zero the noise-affected singular values and use least squares method to solve the modified equations [9].

VI. AN APPLICATION: MAXIMUM AMPLIFICATION IN A GIVEN SUBSPACE

Let the singular value decomposition $\mathbf{UWV}^T$ of $\mathbf{A} \in \mathbf{R}^{m \times n}$ be known. For a given $\{\mathbf{x}, \mathbf{y}\} \subset \mathbf{R}^n$ we want to find $\max_{\|\mathbf{m}\|=1} \|\mathbf{Am}\|$ subject to $\mathbf{m} \in \text{span}\{\mathbf{x}, \mathbf{y}\}$.

Using Gram-Schmidt algorithm it is possible to find orthonormal set $\{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\}$ such that $\text{span}\{\mathbf{x}, \mathbf{y}\} = \text{span}\{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\}$. Let **m** be expressed as $\mathbf{m} = \alpha_1\tilde{\mathbf{x}} + \alpha_2\tilde{\mathbf{y}}$ for some $\alpha_1, \alpha_2 \in \mathbf{R}$. Orthogonality of $\{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}\}$ and $\|\mathbf{m}\| = 1$ imply $\alpha_1^2 + \alpha_2^2 = 1$. Let us transform **m** into another basis by $\mathbf{p} = \mathbf{V}^T\mathbf{m}$. Now maximizing $\|\mathbf{Am}\|$ is equivalent to maximizing $\|\mathbf{AVp}\|$. The equality **AV=UW** implies **AVp=UWp**, or explicitly

$$\mathbf{AVp} = p_1(\alpha_1, \alpha_2)w_1\mathbf{u}_1 + \cdots + p_s(\alpha_1, \alpha_2)w_s\mathbf{u}_s \tag{12}$$

where p_i, and $\mathbf{u}_i$, $i = 1,\ldots,s$ (as defined in Section I, s is the number of nonzero singular values) are the first s components of $\mathbf{p}$ and first s columns of $\mathbf{U}$ respectively, and w_i, $i = 1,\ldots,s$ are the (i,i)-th components of $\mathbf{W}$. The problem now is to solve

$$\max_{\alpha_1^2+\alpha_2^2=1} \left\| p_1(\alpha_1,\alpha_2)w_1\mathbf{u}_1 + \cdots + p_s(\alpha_1,\alpha_2)w_s\mathbf{u}_s \right\| \tag{13}$$

for α_1 and α_2. Exploiting that $\mathbf{U}$ is orthonormal Equation 13 is equivalent to

$$\max_{\alpha_1^2+\alpha_2^2=1} \{ |p_1(\alpha_1,\alpha_2)w_1|^2 + \cdots + |p_s(\alpha_1,\alpha_2)w_s|^2 \}. \tag{14}$$

Equation 14 has two variables: α_1 and α_2. In order to reduce it to a single variable, we can utilize the following change of variables

$$\alpha_1 = \cos\theta \tag{15}$$

and

$$\alpha_2 = \sin\theta \tag{16}$$

Using Equations 15 and 16 Equation 14 becomes,

$$\max\{ |r_1(\theta)w_1|^2 + \cdots + |r_s(\theta)w_s|^2 \} \tag{17}$$

for some $r_i(\theta)$, $i = 1,2,\ldots,s$. Note that Equation 17 is a maximization without any constraint.

Example VI.1 *Let the matrix* $\mathbf{A}$ *be given as*

$$\mathbf{A} = \begin{bmatrix} 1/4 & -\sqrt{3}/2 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 1/2 & -3/4 \\ 0 & \sqrt{3} & 0 \\ 3\sqrt{3}/2 & 0 & 3/2 \end{bmatrix}.$$

whose SVD can be computed as

$$\begin{bmatrix} 1/2 & -\sqrt{3}/4 & 0 & 3/4 \\ \sqrt{3}/2 & 1/4 & 0 & -\sqrt{3}/4 \\ 0 & \sqrt{3}/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & -\sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & 1/2 \end{bmatrix}.$$

Let us obtain the unity norm vector $\mathbf{m} \in \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \right\}$ such that $\|\mathbf{Am}\|$ *is maximum.*

The orthonormal set $\left\{ \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} -2\sqrt{5}/45 \\ \sqrt{5}/45 \\ 4\sqrt{5}/9 \end{bmatrix} \right\}$ *span the same subspace spanned by* $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \right\}$.

Now let

$$\mathbf{m} = \alpha_1 \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2\sqrt{5}/45 \\ \sqrt{5}/45 \\ 4\sqrt{5}/9 \end{bmatrix}, \tag{18}$$

and

$$\mathbf{p} = \mathbf{V}^{T}\mathbf{m} = \begin{bmatrix} 1/2 & 0 & -\sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}}\alpha_1 - \frac{2\sqrt{5}}{45}\alpha_2 \\ \frac{2}{\sqrt{5}}\alpha_1 + \frac{\sqrt{5}}{45}\alpha_2 \\ \frac{4\sqrt{5}}{9}\alpha_2 \end{bmatrix} \tag{19}$$

$$= \begin{bmatrix} \frac{1}{2\sqrt{5}}\alpha_1 - \frac{\sqrt{5}}{45}(1+10\sqrt{3})\alpha_2 \\ \frac{2}{\sqrt{5}}\alpha_1 + \frac{\sqrt{5}}{45}\alpha_2 \\ \frac{\sqrt{3}}{2\sqrt{5}}\alpha_1 + \frac{\sqrt{5}}{45}(10-\sqrt{3})\alpha_2 \end{bmatrix}. \tag{20}$$

Using transformations 15 and 16 we obtain

$$\mathbf{r} = \begin{bmatrix} \frac{1}{2\sqrt{5}}\cos\theta - \frac{\sqrt{5}}{45}(1+10\sqrt{3})\sin\theta \\ \frac{2}{\sqrt{5}}\cos\theta + \frac{\sqrt{5}}{45}\sin\theta \\ \frac{\sqrt{3}}{2\sqrt{5}}\cos\theta + \frac{\sqrt{5}}{45}(10-\sqrt{3})\sin\theta \end{bmatrix} \tag{21}$$

whose squared norm is

$$\left(\tfrac{\cos\theta}{2\sqrt{5}} - \tfrac{\sqrt{5}}{45}(1+10\sqrt{3})\sin\theta\right)^2 + 4\left(\tfrac{2}{\sqrt{5}}\cos\theta + \tfrac{\sqrt{5}}{45}\sin\theta\right)^2 + 9\left(\tfrac{\sqrt{3}}{2\sqrt{5}}\cos\theta + \tfrac{\sqrt{5}}{45}(10-\sqrt{3})\sin\theta\right)^2$$

$$= \frac{23}{5}\cos^2\theta + \frac{1232-160\sqrt{3}}{405}\sin^2\theta - \frac{12-80\sqrt{3}}{45}\sin\theta\cos\theta. \tag{22}$$

Using the identities $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$ and $\sin\theta\cos\beta = \frac{1}{2}\left[\sin(\theta+\beta) + \sin(\theta-\beta)\right]$ *this becomes*

$$\frac{40\sqrt{3}-6}{45}\sin 2\theta + \frac{631+160\sqrt{3}}{810}\cos 2\theta + \frac{3095-160\sqrt{3}}{810} \tag{23}$$

Further simplifications is obtained by using $a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\,\sin\left(\theta + \arctan(\frac{a}{b})\right)$:

$$\sqrt{\frac{2041825+46400\sqrt{3}}{656100}}\,\sin\left(2\theta + \arctan\tfrac{631+160\sqrt{3}}{720\sqrt{3}-108}\right) + \frac{3095-160\sqrt{3}}{810}. \tag{24}$$

This norm is maximized by maximizing the sin(.) *term. Maxima are achieved whenever its argument is* $\frac{n\pi}{2}$, $n = 1,3,\ldots$ *Carrying out the operations,*

$$2\theta + \arctan\left(\tfrac{631+160\sqrt{3}}{720\sqrt{3}-108}\right) = \tfrac{\pi}{2} \tag{25}$$

$$\theta = 0.4488674 \tag{26}$$

and from Equations 15 and 16,

$$\alpha_1 = 0.9 \tag{27}$$

and

$$\alpha_2 = 0.434 \tag{28}$$

are obtained.

Therefore the vector **m** *which maximizes* $\|\mathbf{Am}\|$ *in a given two dimensional space is:*

$$\mathbf{m} = \begin{bmatrix} 0.35936 \\ 0.82655 \\ 0.526 \end{bmatrix}. \qquad \square$$

Note that by using transformations 15 and 16 we have removed the constraint $\alpha_1^2 + \alpha_2^2 = 1$. Also the problem is reduced to one unknown θ from α_1 and α_2.

Alternatively, we could have solved the problem without any transformation. In this case Expression 14 can be written as

$$\max \frac{\frac{23}{5}\alpha_1^2 + \frac{80\sqrt{3}-12}{45}\alpha_1\alpha_2 + \frac{1232-160\sqrt{3}}{405}\alpha_2^2}{\alpha_1^2+\alpha_2^2} \tag{29}$$

The normalization is used to remove the constraint. However, maximizing Eqn. 29 we obtain only the direction of the maximum. This maximum satisfies

$$\frac{\partial}{\partial \alpha_1}\left(\frac{\frac{23}{5}\alpha_1^2 + \frac{80\sqrt{3}-12}{45}\alpha_1\alpha_2 + \frac{1232-160\sqrt{3}}{405}\alpha_2^2}{\alpha_1^2+\alpha_2^2}\right) = 0$$

or more explicitly

$$\frac{80\sqrt{3}-12}{45}\alpha_2^2 + \frac{1262+320\sqrt{3}}{405}\alpha_1\alpha_2 + \frac{12-80\sqrt{3}}{45}\alpha_1^2 = 0. \tag{30}$$

Solving Eqn. 30 for α_2 we obtain

$$\alpha_2 = -2.076157769\alpha_1$$

or

$$\alpha_2 = 0.481658969\alpha_1.$$

By testing the second one gives direction for the maximum amplification. In this direction $(\alpha_1, \alpha_2) = (0.9, 0.434)$ satisfies the norm constraint. This result is the same as the one given in Equations 27 and 28.

VII. CONCLUSION

Various geometric characterizations of matrices have been discussed in Singular Value Decomposition (SVD) framework. Construction of SVD, range and null spaces, their complements, and magnification by matrices have been emphasized particularly. Examples have been taken place in the paper for the sake of clarity in presentation.

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