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Research Article

Generalizations of the drift Laplace equation over the quaternions in a class of Grushin-type spaces

THOMAS BIESKE* AND KELLER L. BLACKWELL

ABSTRACT. Beals, Gaveau, and Greiner established a formula for the fundamental solution to the Laplace equation with drift term in Grushin-type planes. The first author and Childers expanded these results by invoking a p-Laplace-type generalization that encompasses these formulas while the authors explored a different natural generalization of the p-Laplace equation with drift term that also encompasses these formulas. In both, the drift term lies in the complex domain. We extend these results by considering a drift term in the quaternion realm and show our solutions are stable under limits as p tends to infinity.

Keywords: p-Laplace equation, Grushin-type plane.

2020 Mathematics Subject Classification: 53C17, 35H20, 35A09, 35R03, 17B70.

1. MOTIVATION AND BACKGROUND

1.1. **Motivation.** In [2], Beals, Gaveau, and Greiner established a formula for the fundamental solution to the Laplace equation with drift term in a large class of sub-Riemannian spaces, which includes the so-called Grushin-type planes. In [4], the first author and Childers expanded these results by invoking a p-Laplace-type generalization that encompasses the formulas of [2] while in [3], the authors explored a different natural generalization of the p-Laplace equation with drift term that also encompasses the formulas of [2]. In both cases, the drift term lies in the complex domain. In this paper, we will consider both approaches, but with a drift term in the quaternion realm and create an extension of both cases. We will then show our solutions are stable under limits when $\mathrm{p} \to \infty$.

1.2. **Grushin-type planes.** We begin with a brief discussion of our environment. The Grushin-type planes are a class of sub-Riemannian spaces lacking an algebraic group law. We begin with $\mathbb{R}^2$ possessing coordinates (y_1, y_2), $a \in \mathbb{R}$, $c \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$. We use them to construct the vector fields

$$Y_1 = \frac{\partial}{\partial y_1} \text{ and } Y_2 = c(y_1 - a)^n \frac{\partial}{\partial y_2}.$$

For these vector fields, the only (possibly) nonzero Lie bracket is

$$[Y_1, Y_2] = cn(y_1 - a)^{n-1} \frac{\partial}{\partial y_2}.$$

Because $n \in \mathbb{N}$, it follows that Hörmander's condition (see, for example, [1]) is satisfied by these vector fields.

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*Corresponding author: Thomas Bieske; tbieske@mail.usf.edu
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We endow $\mathbb{R}^2$ with a (singular) inner product, denoted $\langle \cdot, \cdot \rangle$, with related norm $\| \cdot \|$, so that the collection $\{Y_1, Y_2\}$ forms an orthonormal basis. We then have a sub-Riemannian space that we will call g_n, which is also the tangent space to a generalized Grushin-type plane $\mathbb{G}_n$. Points in $\mathbb{G}_n$ will also be denoted by $p = (y_1, y_2)$. The Carnot-Carathéodory distance on $\mathbb{G}_n$ is defined for points p and q as follows:

$$d_{\mathbb{G}}(p,q) = \inf_{\Gamma} \int \|\gamma'(t)\| \, dt$$

with Γ the set of curves γ such that $\gamma(0) = p$, $\gamma(1) = q$ and $\gamma'(t) \in \operatorname{span}\{Y_1(\gamma(t)), Y_2(\gamma(t))\}$. By Chow's theorem, this is an honest metric.

We shall now discuss calculus on the Grushin-type planes. Given a smooth function f on $\mathbb{G}_n$, we define the horizontal gradient of f as

$$\nabla_0 f(p) = \big(Y_1 f(p), Y_2 f(p)\big).$$

Using these derivatives, we consider a key operator on $C^2_{\mathbb{G}}$ functions, namely the p-Laplacian for $1 < \mathrm{p} < \infty$, given by

$$\begin{aligned} \Delta_{\mathrm{p}} f &= \operatorname{div}(\|\nabla_0 f\|^{\mathrm{p}-2} \nabla_0 f) = Y_1\big(\|\nabla_0 f\|^{\mathrm{p}-2} Y_1 f\big) + Y_2\big(\|\nabla_0 f\|^{\mathrm{p}-2} Y_2 f\big) \\ &= \frac{1}{2}(\mathrm{p}-2)\|\nabla_0 f\|^{\mathrm{p}-4}\big(Y_1\|\nabla_0 f\|^2 Y_1 f + Y_2\|\nabla_0 f\|^2 Y_2 f\big) \\ &+ \|\nabla_0 f\|^{\mathrm{p}-2}\big(Y_1 Y_1 f + Y_2 Y_2 f\big). \end{aligned} \tag{1.1}$$

For more recent results concerning Grushin-type spaces, see [6] and references therein.

2. Motivating Results

2.1. **Grushin-type Planes.** The first author and Gong [5] proved the following in the Grushin-type planes.

Theorem 2.1 ([5]). *Let $1 < p < \infty$ and define*

$$f(y_1, y_2) = c^2 (y_1 - a)^{(2n+2)} + (n+1)^2 (y_2 - b)^2.$$

For $p \neq n+2$, consider

$$\tau_p = \frac{n+2-p}{(2n+2)(1-p)}$$

so that in $\mathbb{G}_n \setminus \{(a,b)\}$ we have the well-defined function

$$\psi_p = \begin{cases} f(y_1,y_2)^{\tau_p}, & p \neq n+2 \\ \log f(y_1,y_2), & p = n+2 \end{cases}.$$

Then, $\Delta_p \psi_p = 0$ in $\mathbb{G}_n \setminus \{(a,b)\}$.

In the Grushin-type planes, Beals, Gaveau and Greiner [2] extended this equation as shown in the following theorem.

Theorem 2.2 ([2]). *Let $L \in \mathbb{R}$. Consider the following quantities*

$$\alpha = \frac{-n}{(2n+2)}(1+L) \ \ \textit{and} \ \ \beta = \frac{-n}{(2n+2)}(1-L).$$

We use these constants with the functions

$$\begin{aligned} g(y_1,y_2) &= c(y_1 - a)^{n+1} + i(n+1)(y_2 - b) \\ h(y_1,y_2) &= c(y_1 - a)^{n+1} - i(n+1)(y_2 - b) \end{aligned}$$

to define our main function $f(y_1, y_2)$*, given by*

$$f(y_1, y_2) = g(y_1, y_2)^{\alpha} h(y_1, y_2)^{\beta}.$$

Then, $\mathcal{D}(f) := \Delta_2 f + iL[Y_1, Y_2]f = 0$ *in* $\mathbb{G}_n \setminus \{(a, b)\}$*.*

Non-linear generalizations of Theorem 2.2 have been explored by the first author and Childers in [4] and by the authors in [3]. The following theorem extends Theorem 2.2 through a p-Laplace type divergence form.

Theorem 2.3 ([4])**.** *For* $L \in \mathbb{R}$ *with* $L \neq \pm 1$*, consider the following parameters for* $p \neq n+2$*:*

$$\alpha = \frac{n+2-p}{(1-p)(2n+2)}(1+L) \quad \text{and} \quad \beta = \frac{n+2-p}{(1-p)(2n+2)}(1-L)$$

with the functions:

$$g(y_1, y_2) = c(y_1 - a)^{n+1} + i(n+1)(y_2 - b)$$
$$h(y_1, y_2) = c(y_1 - a)^{n+1} - i(n+1)(y_2 - b)$$

to define the main function:

$$f_{p,L} = \begin{cases} g(y_1, y_2)^{\alpha} h(y_1, y_2)^{\beta}, & p \neq n+2 \\ \log\left(g(y_1, y_2)^{1+L} h(y_1, y_2)^{1-L}\right), & p = n+2 \end{cases}.$$

Then

$$\overline{\Delta_p} f_{p,L} := \operatorname{div}\left(\left\| \begin{matrix} Y_1 f_{p,L} + iLY_2 f_{p,L} \\ Y_2 f_{p,L} - iLY_1 f_{p,L} \end{matrix} \right\|^{p-2} \begin{pmatrix} Y_1 f_{p,L} + iLY_2 f_{p,L} \\ Y_2 f_{p,L} - iLY_1 f_{p,L} \end{pmatrix} \right) = 0.$$

The following theorem of the authors takes an alternative approach to extending Theorem 2.2 through a generalization of the drift term.

Theorem 2.4 ([3])**.** *For* $L \in \mathbb{R}$ *with:*

$$L \neq \pm \frac{n+2-p}{n(1-p)}$$

consider the parameters:

$$\alpha = \frac{n+2-p-Ln(1-p)}{2(n+1)(1-p)} \text{ and } \beta = \frac{n+2-p+Ln(1-p)}{2(n+1)(1-p)}$$

with the functions

$$g(y_1, y_2) = c(y_1 - a)^{n+1} + i(n+1)(y_2 - b)$$
$$h(y_1, y_2) = c(y_1 - a)^{n+1} - i(n+1)(y_2 - b)$$

to define the main function:

$$f_{p,L}(y_1, y_2) = g(y_1, y_2)^{\alpha} h(y_1, y_2)^{\beta}. \tag{2.2}$$

Then on $\mathbb{G}_n \setminus \{(a, b)\}$*, we have:*

$$\mathcal{G}_{p,L}(f_{p,L}) := \Delta_p f_{p,L} + iL[Y_1, Y_2]\left(\|\nabla_0 f_{p,L}\|^{p-2} f_{p,L} \right) = 0.$$

Main Question. *We wish to extend the preceding generalizations of Theorem 2.2 over the quaternions, denoted* $\mathbb{H}$*. Recall that the solved partial differential equation of Theorem 2.2, namely*

$$\Delta_2 f + iL[Y_1, Y_2]f = 0,$$

features a drift term bearing the purely complex-imaginary coefficient $iL \in \mathbb{C}$. We ask if this coefficient can be generalized to a purely quaternion-imaginary coefficient of the form

$$Q = Li + Mj + Nk \in \mathbb{H} \setminus \mathbb{R},$$

where the case of $Q = 0$ reduces to the result of Theorem 2.1. With respect to Theorem 2.3, we explore smooth solutions to the generalization

$$\overline{\Delta_p} f := \operatorname{div}\left(\left\|\begin{matrix} Y_1 f + QY_2 f \\ Y_2 f - QY_1 f \end{matrix}\right\|^{p-2} \begin{pmatrix} Y_1 f + QY_2 f \\ Y_2 f - QY_1 f \end{pmatrix}\right) = 0.$$

With respect to Theorem 2.4, we explore smooth solutions to the generalization

$$\mathcal{G}_{p,Q}(f) := \Delta_p f + Q\,[Y_1, Y_2]\left(\|\nabla_0 f\|^{p-2} f\right) = 0.$$

3. A p-Laplacian Type Generalization over $\mathbb{H}$

3.1. Case I: $\mathbf{L + M + N \neq 0}$.

Let $Q = Li + Mj + Nk \in \mathbb{H} \setminus \mathbb{R}$ with $L + M + N \neq 0$. We consider the following parameters:

$$\begin{aligned}
\mu &= \frac{\sqrt{|Q^2|}}{|L+M+N|} \\
\omega &= \frac{Q}{L+M+N} \\
\xi &= \sqrt{|Q^2|}(L+M+N) \\
\alpha &= \frac{n+2-\mathrm{p}}{(1-\mathrm{p})(2n+2)}(1+\xi) \\
\text{and } \beta &= \frac{n+2-\mathrm{p}}{(1-\mathrm{p})(2n+2)}(1-\xi),
\end{aligned}$$

where $\xi \neq \pm 1$. We use these constants with the functions:

$$\begin{aligned}
g(y_1, y_2) &= \mu c (y_1 - a)^{n+1} + \omega(n+1)(y_2 - b) \\
h(y_1, y_2) &= \mu c (y_1 - a)^{n+1} - \omega(n+1)(y_2 - b)
\end{aligned}$$

to define our main function:

$$f_{\mathrm{p},Q}(y_1, y_2) = \begin{cases} g(y_1, y_2)^{\alpha} h(y_1, y_2)^{\beta}, & \mathrm{p} \neq n+2 \\ \log\left(g(y_1, y_2)^{1+\xi} h(y_1, y_2)^{1-\xi}\right), & \mathrm{p} = n+2 \end{cases}. \tag{3.3}$$

Using equation 3.3, we have the following theorem.

Theorem 3.5. *Let $Q = Li + Mj + Nk \in \mathbb{H} \setminus \mathbb{R}$ with $L + M + N \neq 0$. On $G_n \setminus \{(a, b)\}$, we have:*

$$\overline{\Delta_p} f_{p,Q} := \operatorname{div}_{\mathbb{G}}\left(\left\|\begin{matrix} Y_1 f_{p,Q} + QY_2 f_{p,Q} \\ Y_2 f_{p,Q} - QY_1 f_{p,Q} \end{matrix}\right\|^{p-2} \begin{pmatrix} Y_1 f_{p,Q} + QY_2 f_{p,Q} \\ Y_2 f_{p,Q} - QY_1 f_{p,Q} \end{pmatrix}\right) = 0.$$

Proof. Suppressing arguments and subscripts, we let:

$$\Upsilon := \begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \end{pmatrix} = \begin{pmatrix} Y_1 f + QY_2 f \\ Y_2 f - QY_1 f \end{pmatrix}.$$

Observing that:

$$\begin{aligned}\overline{\Delta_{\mathrm{p}}}f &= \operatorname{div}\left(\|\Upsilon\|^{\mathrm{p}-2}\Upsilon\right)\\ &= \|\Upsilon\|^{\mathrm{p}-4}\left(\frac{\mathrm{p}-2}{2}\sum_{s=1}^{2}Y_s\|\Upsilon\|^2\Upsilon_s + \|\Upsilon\|^2(Y_1\Upsilon_1+Y_2\Upsilon_2)\right)\end{aligned}$$

it suffices to show:

$$\Lambda := \frac{\mathrm{p}-2}{2}\sum_{s=1}^{2}Y_s\|\Upsilon\|^2\Upsilon_s + \|\Upsilon\|^2(Y_1\Upsilon_1+Y_2\Upsilon_2) = 0.$$

For $\mathrm{p}\neq n+2$, we compute the following:

$$\begin{aligned}Y_1 f &= \mu c(n+1)(y_1-a)^n g^{\alpha-1}h^{\beta-1}(\alpha h+\beta g)\\ Y_2 f &= \omega c(n+1)(y_1-a)^n g^{\alpha-1}h^{\beta-1}(\alpha h-\beta g)\\ Y_1 f + QY_2 f &= \mu c(n+1)(y_1-a)^n g^{\alpha-1}h^{\beta-1}(\alpha h(1-\xi)+\beta g(1+\xi))\\ Y_2 f - QY_1 f &= \omega c(n+1)(y_1-a)^n g^{\alpha-1}h^{\beta-1}(\alpha h(1-\xi)-\beta g(1+\xi))\\ \text{and } \|\Upsilon\|^2 &= 2\mu^2c^2(n+1)^2(y_1-a)^{2n}g^{\alpha+\beta-1}h^{\alpha+\beta-1}\left(\alpha^2(1-\xi)^2+\beta^2(1+\xi)^2\right).\end{aligned}$$

We then calculate:

$$\begin{aligned}Y_1\Upsilon_1+Y_2\Upsilon_2 &= \frac{1}{(-1+\mathrm{p})^2gh}\mu^2c^2(-1+\xi^2)(1+n)(2+n-\mathrm{p})(-2+\mathrm{p})(y_1-a)^{2n}g^{\alpha}h^{\beta}\\ Y_1\|\Upsilon\|^2 &= -\frac{1}{(-1+\mathrm{p})^3gh}\Big(2\mu^2c^2(1-\xi^2)^2(n+1)(n+2-\mathrm{p})^2(y_1-a)^{2n-1}\\ &\quad\times g^{\alpha+\beta-1}h^{\alpha+\beta-1}\left(\mu^2c^2(y_1-a)^{2n+2}-\mu^2 n(n+1)(-1+\mathrm{p})(y_2-b)^2\right)\Big)\\ \text{and } Y_2\|\Upsilon\|^2 &= \frac{1}{(-1+\mathrm{p})^3gh}2\mu^4c^3(1-\xi^2)^2(n+1)(n+2-\mathrm{p})^2(1+n\mathrm{p})\\ &\quad\times (y_1-a)^{3n}(b-y_2)g^{\alpha+\beta-1}h^{\alpha+\beta-1}.\end{aligned}$$

Using the above quantities, we compute:

$$\begin{aligned}\frac{\mathrm{p}-2}{2}\sum_{s=1}^{2}Y_s\|\Upsilon\|^2\Upsilon_s &= -\frac{1}{(-1+\mathrm{p})^4}\mu^4c^4(-1+\xi^2)^3(n+1)(n+2-\mathrm{p})^3\\ &\quad\times(y_1-a)^{4n}g^{2\alpha+\beta-2}h^{\alpha+2\beta-2}(\mathrm{p}-2)\\ \text{and } \|\Upsilon\|^2(Y_1\Upsilon_1+Y_2\Upsilon_2) &= \frac{1}{(-1+\mathrm{p})^4}\mu^4c^4(n+1)(y_1-a)^{4n}g^{2\alpha+\beta-2}h^{\alpha+2\beta-2}\\ &\quad\times(n+2-\mathrm{p})^3(-1+\xi^2)^3(\mathrm{p}-2)\end{aligned} \tag{3.4}$$

whereby it follows that $\Lambda=0$, as desired. The case $\mathrm{p}=n+2$ is similar and omitted. □

3.2. **Case II: $\mathbf{L}+\mathbf{M}+\mathbf{N}=\mathbf{0}$.**

Let $Q = Li + Mj + Nk \in \mathbb{H} \setminus \mathbb{R}$ with $L + M + N = 0$. We consider the following parameters:

$$\xi = \sqrt{2|LM + LN + MN|}$$
$$\alpha = \frac{n+2-\mathrm{p}}{(1-\mathrm{p})(2n+2)}(1+\xi)$$
$$\text{and } \beta = \frac{n+2-\mathrm{p}}{(1-\mathrm{p})(2n+2)}(1-\xi),$$

where $\xi \neq \pm 1$. We use these constants with the functions:

$$g(y_1, y_2) = \xi c(y_1 - a)^{n+1} + Q(n+1)(y_2 - b)$$
$$h(y_1, y_2) = \xi c(y_1 - a)^{n+1} - Q(n+1)(y_2 - b)$$

to define our main function:

$$f_{\mathrm{p},Q}(y_1, y_2) = \begin{cases} g(y_1, y_2)^{\alpha} h(y_1, y_2)^{\beta}, & \mathrm{p} \neq n+2 \\ \log\left(g(y_1, y_2)^{1+\xi} h(y_1, y_2)^{1-\xi}\right), & \mathrm{p} = n+2 \end{cases}. \tag{3.5}$$

Using equation 3.5, we have the following theorem.

Theorem 3.6. *Let $Q = Li + Mj + Nk \in \mathbb{H} \setminus \mathbb{R}$ with $L + M + N = 0$. On $G_n \setminus \{(a, b)\}$, we have:*

$$\overline{\Delta_p} f_{p,Q} := \mathrm{div}_{\mathbb{G}}\left(\left\| \begin{matrix} Y_1 f_{p,Q} + QY_2 f_{p,Q} \\ Y_2 f_{p,Q} - QY_1 f_{p,Q} \end{matrix} \right\|^{p-2} \begin{pmatrix} Y_1 f_{p,Q} + QY_2 f_{p,Q} \\ Y_2 f_{p,Q} - QY_1 f_{p,Q} \end{pmatrix}\right) = 0.$$

Proof. The proof of Theorem 3.6 is similar to that of Theorem 3.5 and left to the reader. □

We then conclude the following corollary.

Corollary 3.1. *Let $p > n + 2$. The function $f_{p,Q}$, as above, is a nontrivial smooth solution to the Dirichlet problem*

$$\begin{cases} \overline{\Delta_p} f_{p,Q}(\mathbf{y}) = 0, & \mathbf{y} \in \mathbb{G}_n \setminus \{(a, b)\} \\ 0, & \mathbf{y} = (a, b) \end{cases}.$$

4. A Generalization of the Drift Term over $\mathbb{H}$

4.1. Case I: $\mathbf{L + M + N \neq 0}$.

Let $Q = Li + Mj + Nk \in \mathbb{H} \setminus \mathbb{R}$ with $L + M + N \neq 0$. We consider the following parameters:

$$\mu = \frac{\sqrt{|Q^2|}}{|L + M + N|}$$
$$\omega = \frac{Q}{L + M + N}$$
$$\xi = \sqrt{|Q^2|}(L + M + N)$$
$$\alpha = \frac{n+2-\mathrm{p}-\xi n(1-\mathrm{p})}{2(n+1)(1-\mathrm{p})}$$
$$\text{and } \beta = \frac{n+2-\mathrm{p}+\xi n(1-\mathrm{p})}{2(n+1)(1-\mathrm{p})},$$

where:

$$\xi \neq \pm\frac{n+2-\mathrm{p}}{n(\mathrm{p}-1)}.$$

We use these constants with the functions:

$$g(y_1, y_2) = \mu c(y_1 - a)^{n+1} + \omega(n+1)(y_2 - b)$$
$$h(y_1, y_2) = \mu c(y_1 - a)^{n+1} - \omega(n+1)(y_2 - b)$$

to define our main function:

$$f_{p,Q}(y_1, y_2) = g(y_1, y_2)^{\alpha} h(y_1, y_2)^{\beta}. \tag{4.6}$$

Using equation 4.6, we have the following theorem.

Theorem 4.7. *Let $Q = Li + Mj + Nk \in \mathbb{H} \setminus \mathbb{R}$ with $L + M + N \neq 0$. On $G_n \setminus \{(a, b)\}$, we have:*

$$\mathcal{G}_{p,Q}\left(f_{p,Q}\right) := \Delta_p f_{p,Q} + Q\left[Y_1, Y_2\right]\left(\|\nabla_0 f_{p,Q}\|^{p-2} f_{p,Q}\right) = 0.$$

Proof. Suppressing arguments and subscripts, we compute the following:

$$Y_1 f = \mu c(n+1)(y_1 - a)^n g^{\alpha-1} h^{\beta-1}(\alpha h + \beta g) \tag{4.7}$$
$$\overline{Y_1 f} = \mu c(n+1)(y_1 - a)^n g^{\beta-1} h^{\alpha-1}(\alpha g + \beta h)$$
$$Y_2 f = \omega c(n+1)(y_1 - a)^n g^{\alpha-1} h^{\beta-1}(\alpha h - \beta g) \tag{4.8}$$
$$\overline{Y_2 f} = -\omega c(n+1)(y_1 - a)^n g^{\beta-1} h^{\alpha-1}(\alpha g - \beta h)$$
and $\|\nabla_0 f\|^2 = 2\mu^2 c^2 (n+1)^2 (y_1 - a)^{2n} g^{\alpha+\beta-1} h^{\alpha+\beta-1}\left(\alpha^2 + \beta^2\right).$

Using the above, we compute:

$$\begin{aligned}
Y_1 Y_1 f &= \mu c(n+1)(y_1 - a)^{n-1} g^{\alpha-2} h^{\beta-2} \\
&\quad \times \Big(ngh(\alpha h + \beta g) + \mu c(n+1)(y_1 - a)^{n+1} \\
&\quad \times \big((\alpha h + \beta g)\big((\alpha - 1)h + (\beta - 1)g\big) + gh(\alpha + \beta)\big)\Big) \\
Y_2 Y_2 f &= -\mu^2 c^2 (n+1)^2 (y_1 - a)^{2n} g^{\alpha-2} h^{\beta-2} \\
&\quad \times \big((\alpha h - \beta g)\big((\alpha - 1)h - (\beta - 1)g\big) - gh(\alpha + \beta)\big)
\end{aligned}$$

$$Y_1 \|\nabla_0 f\|^2 = 4\mu^2 c^2 (n+1)^2 (y_1 - a)^{2n-1} g^{\alpha+\beta-2} h^{\alpha+\beta-2}(\alpha^2 + \beta^2)x \times \big(ngh + \mu^2 c^2 (n+1)(y_1 - a)^{2n+2}(\alpha + \beta - 1)\big) \tag{4.9}$$

$$Y_2 \|\nabla_0 f\|^2 = -4\omega^2 \mu^2 c^3 (n+1)^4 (y_1 - a)^{3n}(y_2 - b) g^{\alpha+\beta-2} h^{\alpha+\beta-2} \times \big(\alpha^2 + \beta^2\big)(\alpha + \beta - 1) \tag{4.10}$$

and

$$\begin{aligned}
\sum_{s=1}^{2} Y_s \|\nabla_0 f\|^2 (Y_s f) &= 4\mu^3 c^3 (n+1)^3 (y_1 - a)^{3n-1} g^{2\alpha+\beta-3} h^{\alpha+2\beta-3}(\alpha^2 + \beta^2) \\
&\quad \times \big((\alpha h + \beta g)\big(ngh + \mu^2 c^2 (n+1)(y_1 - a)^{2n+2}(\alpha + \beta - 1)\big) \\
&\quad + \omega \mu c(n+1)^2 (y_1 - a)^{n+1}(y_2 - b)(\alpha + \beta - 1)(\alpha h - \beta g)\big) \\
\|\nabla_0 f\|^2 (Y_1 Y_1 + Y_2 Y_2 f) &= 2\mu^3 c^3 (n+1)^3 (y_1 - a)^{3n-1} g^{2\alpha+\beta-3} h^{\alpha+2\beta-3} \\
&\quad \times \big(\alpha^2 + \beta^2\big)\big(ngh(\alpha h + \beta g) + 4\mu c(n+1)(y_1 - a)^{n+1} gh\alpha\beta\big)
\end{aligned}$$

so that

$$\begin{aligned}
\Delta_{\mathrm{p}} f &= \|\nabla_0 f\|^{\mathrm{p}-4}\left(\frac{(\mathrm{p}-2)}{2}\sum_{s=1}^{2} Y_s\|\nabla_0 f\|^2 (Y_s f) + \|\nabla_0 f\|^2 (Y_1Y_1 f + Y_2Y_2 f)\right)\\
&= -\xi 2^{\frac{\mathrm{p}-2}{2}}\mu^{\mathrm{p}-1}c^{\mathrm{p}-1}n^2(n+1)^{\mathrm{p}-2}(y_1-a)^{n(\mathrm{p}-1)-1} g^{\frac{\alpha\mathrm{p}+\beta(\mathrm{p}-2)-\mathrm{p}}{2}} h^{\frac{\alpha(\mathrm{p}-2)+\beta\mathrm{p}-\mathrm{p}}{2}} \left(\alpha^2+\beta^2\right)^{\frac{\mathrm{p}-2}{2}}\\
&\times \left(\xi\mu c(y_1-a)^{n+1} + \omega(1-\mathrm{p})(n+1)(y_2-b)\right).
\end{aligned}$$

We then compute:

$$\begin{aligned}
&Q[Y_1,Y_2]\left(\|\nabla_0 f\|^{\mathrm{p}-2} f\right) =\\
&Q 2^{\frac{p-2}{2}}\mu^{p-2}c^{p-1}n(n+1)^{p-2}(y_1-a)^{n(p-1)-1}\left(\alpha^2+\beta^2\right)^{\frac{p-2}{2}}\\
&\times \frac{\partial}{\partial y_2}\left(g^{\frac{\alpha\mathrm{p}+\beta(\mathrm{p}-2)-(\mathrm{p}-2)}{2}} h^{\frac{\alpha(\mathrm{p}-2)+\beta\mathrm{p}-(\mathrm{p}-2)}{2}}\right)\\
&= \xi 2^{\frac{\mathrm{p}-2}{2}}\mu^{\mathrm{p}-1}c^{\mathrm{p}-1}n^2(n+1)^{\mathrm{p}-2}(y_1-a)^{n(\mathrm{p}-1)-1} g^{\frac{\alpha\mathrm{p}+\beta(\mathrm{p}-2)-\mathrm{p}}{2}} h^{\frac{\alpha(\mathrm{p}-2)+\beta\mathrm{p}-\mathrm{p}}{2}}\\
&\times \left(\alpha^2+\beta^2\right)^{\frac{\mathrm{p}-2}{2}}\left(\xi\mu c(y_1-a)^{n+1} + \omega(1-\mathrm{p})(n+1)(y_2-b)\right)\\
&= -\Delta_{\mathrm{p}} f.
\end{aligned}$$

□

4.2. Case II: $\mathbf{L}+\mathbf{M}+\mathbf{N}=\mathbf{0}$.

Let $Q = Li + Mj + Nk \in \mathbb{H}\setminus\mathbb{R}$ with $L+M+N=0$. We consider the following parameters:

$$\begin{aligned}
\xi &= \sqrt{2|LM+LN+MN|}\\
\alpha &= \frac{n+2-\mathrm{p}-\xi n(1-\mathrm{p})}{2(n+1)(1-\mathrm{p})}\\
\text{and } \beta &= \frac{n+2-\mathrm{p}+\xi n(1-\mathrm{p})}{2(n+1)(1-\mathrm{p})},
\end{aligned}$$

where:

$$\xi \neq \pm\frac{n+2-\mathrm{p}}{n(\mathrm{p}-1)}.$$

We use these constants with the functions:

$$\begin{aligned}
g(y_1,y_2) &= \xi c(y_1-a)^{n+1} + Q(n+1)(y_2-b)\\
h(y_1,y_2) &= \xi c(y_1-a)^{n+1} - Q(n+1)(y_2-b)
\end{aligned}$$

to define our main function:

$$f_{\mathrm{p},Q}(y_1,y_2) = g(y_1,y_2)^{\alpha} h(y_1,y_2)^{\beta}. \tag{4.11}$$

Using equation 4.11, we have the following theorem.

Theorem 4.8. *Let $Q = Li + Mj + Nk \in \mathbb{H}\setminus\mathbb{R}$ with $L+M+N=0$. On $G_n\setminus\{(a,b)\}$, we have:*

$$\mathcal{G}_{p,Q}\left(f_{p,Q}\right) := \Delta_p f_{p,Q} + Q\left[Y_1,Y_2\right]\left(\|\nabla_0 f_{p,Q}\|^{p-2} f_{p,Q}\right) = 0.$$

Proof. The computations proving Theorem 4.8 are similar to those of the proof of Theorem 4.7 and are left to the reader. □

Observing that

$$\xi \neq \pm\frac{n(\mathrm{p}-1)}{n+2-\mathrm{p}} \quad \text{implies} \quad \mathrm{p} \neq \left|\frac{\xi(n+2)+n}{n+\xi}\right|, \left|\frac{\xi(n+2)-n}{n-\xi}\right|$$

we have immediately the following corollary.

Corollary 4.2. *Let* $p > \max\left\{\left|\frac{\xi(n+2)+n}{n+\xi}\right|, \left|\frac{\xi(n+2)-n}{n-\xi}\right|\right\}$. *Then the function* $f_{p,Q}$ *of equation 4.6 is a nontrivial smooth solution to the Dirichlet problem*

$$\begin{cases} \mathcal{G}_{p,Q}\left(f_{p,Q}(\mathbf{y})\right) = 0, & \mathbf{y} \in \mathbb{G}_n \setminus \{(a,b)\} \\ 0, & \mathbf{y} = (a,b) \end{cases}.$$

5. The Limit as $\mathrm{p} \to \infty$

5.1. **p-Laplacian Type Generalization over $\mathbb{H}$.** Recall that on $\mathbb{G}_n \setminus \{(a,b)\}$, we have

$$\begin{aligned}\overline{\Delta_{\mathrm{p}}} f &= \operatorname{div}_G(\|\Upsilon\|^{\mathrm{p}-2}\Upsilon) \\ &= \|\Upsilon\|^{\mathrm{p}-4}\left(\frac{1}{2}(\mathrm{p}-2)\big(Y_1\|\Upsilon\|^2\Upsilon_1 + Y_2\|\Upsilon\|^2\Upsilon_2\big) + \|\Upsilon\|^2\big(Y_1\Upsilon_1 + Y_2\Upsilon_2\big)\right),\end{aligned}$$

where Υ defined by

$$\Upsilon := \begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \end{pmatrix} = \begin{pmatrix} Y_1 f + QY_2 f \\ Y_2 f - QY_1 f \end{pmatrix}.$$

Formally letting $\mathrm{p} \to \infty$, we obtain:

$$\overline{\Delta_\infty} f = (Y_1\|\Upsilon\|^2)\Upsilon_1 + (Y_2\|\Upsilon\|^2)\Upsilon_2.$$

5.1.1. *Case I: $L+M+N \neq 0$.*

Formally letting $\mathrm{p} \to \infty$ in equation 3.3, we obtain:

$$f_{\infty,Q}(y_1,y_2) = g(y_1,y_2)^{\frac{1+\xi}{2n+2}} h(y_1,y_2)^{\frac{1-\xi}{2n+2}},$$

where we recall the functions $g(y_1,y_2)$ and $h(y_1,y_2)$ are given by:

$$\begin{aligned} g(y_1,y_2) &= \mu c(y_1-a)^{n+1} + \omega(n+1)(y_2-b) \\ h(y_1,y_2) &= \mu c(y_1-a)^{n+1} - \omega(n+1)(y_2-b). \end{aligned}$$

We then have the following theorem.

Theorem 5.9. *The function* $f_{\infty,Q}$*, as above, is a smooth solution to the Dirichlet problem*

$$\begin{cases} \overline{\Delta_\infty} f_{\infty,Q}(\mathbf{y}) = 0, & \mathbf{y} \in \mathbb{G}_n \setminus \{(a,b)\} \\ 0, & \mathbf{y} = (a,b) \end{cases}.$$

Proof. We may prove this theorem by letting $\mathrm{p} \to \infty$ in a prudent multiple of Equation (3.4) and invoking continuity (cf. Corollary 3.1). For completeness, though, we compute formally. We let:

$$A = \frac{1+\xi}{2n+2} \quad \text{and} \quad B = \frac{1-\xi}{2n+2}$$

and compute:

$$\begin{aligned}
Y_1 f &= \mu c(n+1)(y_1-a)^n g^{A-1}h^{B-1}(Ah+Bg)\\
Y_2 f &= \omega c(n+1)(y_1-a)^n g^{A-1}h^{B-1}\left(Ah-Bg\right)\\
Y_1 f + QY_2 f &= \mu c(n+1)(y_1-a)^n g^{A-1}h^{B-1}\left(Ah(1-\xi)+Bg(1+\xi)\right)\\
Y_2 f - QY_1 f &= \omega c(n+1)(y_1-a)^n g^{A-1}h^{B-1}\left(Ah(1-\xi)-Bg(1+\xi)\right)\\
\|\Upsilon\|^2 &= 2\mu^2c^2(n+1)^2(y_1-a)^{2n} g^{A+B-1}h^{A+B-1}\left(A^2(1-\xi)^2+B^2(1+\xi)^2\right).
\end{aligned}$$

We then have:

$$\begin{aligned}
Y_1\|\Upsilon\|^2 &= 2\mu^2c^2(1-\xi^2)^2 n(n+1)^2(y_1-a)^{2n-1}(y_2-b)^2(gh)^{\frac{-1-2n}{n+1}}\\
Y_2\|\Upsilon\|^2 &= 2\omega\mu c^3(1-\xi^2)^2 n(n+1)(y_1-a)^{3n}(y_2-b)(gh)^{\frac{-1-2n}{n+1}}
\end{aligned}$$

so that:

$$\begin{aligned}
Y_1\|\xi\|^2\xi_1 &= 2\mu^3c^4(1-\xi^2)^3 n(n+1)^2(y_1-a)^{4n}(y_2-b)^2(gh)^{\frac{-1-2n}{n+1}}g^{A-1}h^{B-1}\\
Y_2\|\xi\|^2\xi_2 &= -2\mu^3c^4(1-\xi^2)^3 n(n+1)^2(y_1-a)^{4n}(y_2-b)^2(gh)^{\frac{-1-2n}{n+1}}g^{A-1}h^{B-1}.
\end{aligned}$$

The theorem follows. □

5.1.2. *Case II: $L+M+N=0$.*

Formally letting $\mathrm{p}\to\infty$ in equation 3.5, we obtain:

$$f_{\infty,Q}(y_1,y_2) = g(y_1,y_2)^{\frac{1+\xi}{2n+2}}h(y_1,y_2)^{\frac{1-\xi}{2n+2}},$$

where we recall the functions $g(y_1,y_2)$ and $h(y_1,y_2)$ are given by:

$$\begin{aligned}
g(y_1,y_2) &= \xi c(y_1-a)^{n+1}+Q(n+1)(y_2-b)\\
h(y_1,y_2) &= \xi c(y_1-a)^{n+1}-Q(n+1)(y_2-b).
\end{aligned}$$

We then have the following theorem.

Theorem 5.10. *The function $f_{\infty,Q}$, as above, is a smooth solution to the Dirichlet problem*

$$\begin{cases} \overline{\Delta_\infty} f_{\infty,Q}(\mathbf{y})=0, & \mathbf{y}\in\mathbb{G}_n\setminus\{(a,b)\}\\ 0, & \mathbf{y}=(a,b)\end{cases}.$$

Proof. The proof of Theorem 5.10 is similar to that of Theorem 5.9 and omitted. □

5.2. **Generalization of the Drift Term over $\mathbb{H}$.** Recall that the drift p-Laplace equation in the Grushin-type planes $\mathbb{G}_n$ is given by:

$$\mathcal{G}_{\mathrm{p},Q}(f) := \Delta_{\mathrm{p}} f + Q[Y_1,Y_2]\left(\|\nabla_0 f\|^{\mathrm{p}-2}f\right)=0\,.$$

A routine expansion of the drift term yields the observation

$$\begin{aligned}
\mathcal{G}_{\mathrm{p},Q}(f) &= \Delta_{\mathrm{p}} f + Qcn(y_1-a)^{n-1}\\
&\quad\times\left(\frac{\mathrm{p}-2}{2}\|\nabla_0 f\|^{\mathrm{p}-4}\left(\frac{\partial}{\partial y_2}\|\nabla_0 f\|^2\right)f+\|\nabla_0 f\|^{\mathrm{p}-2}\frac{\partial}{\partial y_2}f\right)\\
&= 0.
\end{aligned}$$

Dividing through by $\frac{\mathrm{p}-2}{2}\|\nabla_0 f\|^{\mathrm{p}-4}$ and formally taking the limit $\mathrm{p}\to\infty$, we obtain:

$$\mathcal{G}_{\infty,Q}(f) = \Delta_\infty f + Q[Y_1,Y_2]\left(\|\nabla_0 f\|^2\right)f.$$

5.2.1. *Case I: $L+M+N\neq 0$.* Considering equation 4.6 and formally letting $\mathrm{p}\to\infty$ yields:

$$f_{\infty,Q}(y_1,y_2)=g(y_1,y_2)^{\frac{1}{2(n+1)}(1-n\xi)}h(y_1,y_2)^{\frac{1}{2(n+1)}(1+n\xi)},$$

where we recall the functions $g(y_1,y_2)$ and $h(y_1,y_2)$ are given by:

$$\begin{aligned} g(y_1,y_2) &= \mu c(y_1-a)^{n+1}+\omega(n+1)(y_2-b)\\ h(y_1,y_2) &= \mu c(y_1-a)^{n+1}-\omega(n+1)(y_2-b). \end{aligned}$$

We have the following theorem.

Theorem 5.11. *The function $f_{\infty,Q}$, as above, is a smooth solution to the Dirichlet problem*

$$\begin{cases} \mathcal{G}_{\infty,Q}f_{\infty,Q}(\mathbf{y})=0, & \mathbf{y}\in\mathbb{G}_n\setminus\{(a,b)\}\\ 0, & \mathbf{y}=(a,b) \end{cases}.$$

Proof. We may prove this theorem by letting $\mathrm{p}\to\infty$ in Equations (4.7), (4.8), (4.9), (4.10) and invoking continuity (cf. Corollary 4.2). However, for completeness we compute formally. We let:

$$A=\frac{1}{2(n+1)}(1-n\xi)\text{and} \qquad B=\frac{1}{2(n+1)}(1+n\xi)$$

and, suppressing arguments and subscripts, compute:

$$\begin{aligned} Y_1f &= \mu c(n+1)(y_1-a)^n g^{A-1}h^{B-1}(Ah+Bg)\\ Y_2f &= \omega c(n+1)(y_1-a)^n g^{A-1}h^{B-1}(Ah-Bg)\\ \|\nabla_0 f\|^2 &= 2\mu^2c^2(n+1)^2(y_1-a)^{2n}g^{A+B-1}h^{A+B-1}\left(A^2+B^2\right)\\ Y_1\|\nabla_0 f\|^2 &= 4\mu^2c^2(n+1)^2(y_1-a)^{2n-1}g^{A+B-2}h^{A+B-2}(A^2+B^2)\\ &\quad\times\left(ngh+\mu^2c^2(n+1)(y_1-a)^{2n+2}(A+B-1)\right)\\ Y_2\|\nabla_0 f\|^2 &= -4\omega^2\mu^2c^3(n+1)^4(y_1-a)^{3n}(y_2-b)\left(A^2+B^2\right)(A+B-1)\\ &\quad\times g^{A+B-2}h^{A+B-2} \end{aligned}$$

so that:

$$\begin{aligned} \Delta_\infty f &= Y_1\|\nabla_0 f\|^2 Y_1 f+Y_2\|\nabla_0 f\|^2 Y_2 f\\ &= 4\mu^3c^3(n+1)^3(A^2+B^2)(y_1-a)^{3n-1}g^{2A+B-3}h^{A+2B-3}\\ &\times\Big((Ah+Bg)\big(ngh+\mu^2c^2(n+1)(A+B-1)(y_1-a)^{2n+2}\big)\\ &+\ \omega\mu c(n+1)^2(y_1-a)^{n+1}(y_2-b)(A+B-1)(Ah-Bg)\Big)\\ &= 4\xi\omega\mu^3c^3n^2(n+1)^3(y_1-a)^{3n-1}(y_2-b)g^{2A+B-2}h^{A+2B-2}(A^2+B^2). \end{aligned}$$

We also compute:

$$\begin{aligned} Q[Y_1,Y_2]\left(\|\nabla_0 f\|^2\right)f &= Qg^Ah^B\left(cn(y_1-a)^{n-1}\frac{\partial}{\partial y_2}\|\nabla_0 f\|^2\right)\\ &= -4\xi\omega\mu^3c^3n^2(n+1)^3(y_1-a)^{3n-1}(y_2-b)\left(A^2+B^2\right)\\ &\times g^{A+B-2}h^{A+B-2} \end{aligned}$$

The theorem follows. □

5.2.2. *Case II: $L + M + N = 0$.* Considering equation 4.11 and formally letting $p \to \infty$ yields:

$$f_{\infty,Q}(y_1, y_2) = g(y_1, y_2)^{\frac{1}{2(n+1)}(1-n\xi)} h(y_1, y_2)^{\frac{1}{2(n+1)}(1+n\xi)},$$

where we recall the functions $g(y_1, y_2)$ and $h(y_1, y_2)$ are given by:

$$g(y_1, y_2) = \xi c(y_1 - a)^{n+1} + Q(n+1)(y_2 - b)$$
$$h(y_1, y_2) = \xi c(y_1 - a)^{n+1} - Q(n+1)(y_2 - b).$$

We have the following theorem.

Theorem 5.12. *The function $f_{\infty,Q}$, as above, is a smooth solution to the Dirichlet problem*

$$\begin{cases} \mathcal{G}_{\infty,Q} f_{\infty,Q}(\mathbf{y}) = 0, & \mathbf{y} \in \mathbb{G}_n \setminus \{(a,b)\} \\ 0, & \mathbf{y} = (a,b) \end{cases}.$$

Proof. The proof of Theorem 5.12 is similar to that of Theorem 5.11 and omitted. □

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REFERENCES

[1] A. Bellāche: *The Tangent Space in Sub-Riemannian Geometry*, in: Sub-Riemannian Geometry; A. Bellāche, J. J. Risler, Eds.; Progress in Mathematics; Birkhäuser: Basel, Switzerland, **144** (1996), 1–78.
[2] R. Beals, B. Gaveau, P. Greiner: *On a Geometric Formula for the Fundamental Solution of Subelliptic Laplacians*, Math. Nachr., **181** (1996), 81–163.
[3] T. Bieske, K. Blackwell: *Generalizations of the Drift p-Laplace Equation in the Heisenberg Group and a Class of Grushin-type Planes*, (2019), submitted for publication, preprint available at https://arxiv.org/abs/1906.01467.
[4] T. Bieske, K. Childers: *Generalizations of a Laplacian-type Equation in the Heisenberg Group and a Class of Grushin-type Spaces*, Proc. Amer. Math. Soc., **142** (3) (2013), 989–1003.
[5] T. Bieske, J. Gong: *The p-Laplacian Equation on a Class of Grushin-Type Spaces*, Amer. Math. Society, **134** (2006), 3585–3594.
[6] T. Bieske, Z. Forrest: *Existence and uniqueness of viscosity solutions to the infinity Laplacian relative to a class of Grushin-type vector fields*, Constr. Math. Anal., **6** (2) (2023), 77–89.

THOMAS BIESKE
UNIVERSITY OF SOUTH FLORIDA
DEPARTMENT OF MATHEMATICS
TAMPA, FL 33620, USA
ORCID: 0000-0003-2029-0562
Email address: tbieske@mail.usf.edu

KELLER L. BLACKWELL
STANFORD UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE
STANFORD, CA 94305, USA
ORCID: 0000-0003-3588-9199
Email address: kellerb@stanford.edu