Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume XX (x) (XXXX), 1 – 19
DOI : 10.15672/hujms.xx

Research Article

General solution to a difference equation and the long-term behavior of some of its solutions

Stevo Stević

Mathematical Institute of the Serbian Academy of Sciences, Knez Mihailova 36/III, 11000 Beograd, Serbia
Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan, Republic of China

Abstract

Closed-from formulas for the general solution to a difference equation are given, generalizing some special cases in the literature. We also analyze and give some comments on the results on the long-term behaviour of some solutions of the special cases.

Mathematics Subject Classification (2020). 39A20

Keywords. difference equation, general solution, closed-form formula

1. Introduction and preliminaries

We use the standard notations $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ for the sets of natural, whole and real numbers, respectively. If $l \in \mathbb{Z}$, then $\mathbb{N}_l := \{n \in \mathbb{Z} : n \geq l\}$. If $s, t \in \mathbb{Z}$, $s \leq t$, then $i = \overline{s,t}$, means that i takes the values of $\mathbb{Z}$ such that $s \leq i \leq t$. We understand that $\prod_{i=m}^{m-1} b_i = 1$, for any $m \in \mathbb{Z}$.

Many formulas for solutions to difference equations and systems in closed form can be found in quite old literature [9, 12, 13, 21, 22]. For some old presentations of the methods for finding them see [19, 20]. One can also consult, e.g., [10, 15, 23–25, 27, 28]. Some recent formulas and tricks for finding solutions to nonlinear difference equations and systems can be found, e.g., in [14, 35, 37, 49–60]. The formulas are usually useful in dealing with the solutions to the equations and systems. However, it is a rare situation that an equation or system is solvable. Even if the equation or system is solvable, there is a possibility that obtained formulas are not so useful for investigation of the long-term behavior of their solutions. Therefore, one can try to find some other type of relations, for instance, their invariants [30, 31, 33, 39, 40].

The bilinear difference equation

$$x_{n+1} = \frac{ax_n + b}{cx_n + d}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $a, b, c, d, x_0 \in \mathbb{R}$, is one of the first nonlinear equations for which was shown its solvability [19, 20]. In [52], among other things, we presented some historical facts about the difference equation. Many other facts and connections with other difference equations and systems can be found, for instance, in [1, 2, 10, 11, 18–20, 23, 26–28, 51, 52, 56, 57, 60].

Email address: sscite1@gmail.com
Received: 12.07.2023; Accepted: 14.01.2024

Eq.(1.1) can be solved by transforming it to a linear difference equation of second order. This fact, among other things, is also employed in our studies presented here. Generally speaking, solvability of many difference equations and systems is shown by transforming them to some known solvable ones by some suitable transformations [14,35,49–55,57–60].

The following result was essentially proved in [9] and [12], and can be found in many books and papers (see, e.g., [18,51,61]).

Lemma 1.1. *Consider the equation*

$$x_{n+2}+a_1x_{n+1}+a_0x_n=0,\quad n\in\mathbb{N}_0,$$

where $a_1,x_0,x_1\in\mathbb{R}$ *and* $a_0\in\mathbb{R}\setminus\{0\}$. *Then, the following statements hold.*

(1) *If* $a_1^2\neq 4a_0$, *then*

$$x_n=\frac{(x_1-\lambda_2x_0)\lambda_1^n-(x_1-\lambda_1x_0)\lambda_2^n}{\lambda_1-\lambda_2},\quad n\in\mathbb{N}_0,\tag{1.2}$$

where

$$\lambda_1=\frac{-a_1+\sqrt{a_1^2-4a_0}}{2}\quad and\quad \lambda_2=\frac{-a_1-\sqrt{a_1^2-4a_0}}{2}.$$

(2) *If* $a_1^2=4a_0$, *then*

$$x_n=((x_1-\lambda x_0)n+\lambda x_0)\lambda^{n-1},\quad n\in\mathbb{N}_0,\tag{1.3}$$

where $\lambda=-a_1/2$.

The equation

$$x_{n+1}=ax_n+\frac{bx_nx_{n-4}}{cx_{n-3}+dx_{n-4}},\quad n\in\mathbb{N}_0,\tag{1.4}$$

where $a,b,c,d\in\mathbb{R}$, $x_{-j}\in\mathbb{R}$, $j=\overline{0,4}$, was studied recently in [38], where some formulas for solutions in the cases:

(1) $a=b=c=d=1$;
(2) $a=b=c=1,\ d=-1$;
(3) $a=c=1,\ b=d=-1$;
(4) $a=c=d=1,\ b=-1$,

were presented. Besides, [38] gives some claims on the long-term behavior of solutions to Eq.(1.4).

The purpose of the paper is to show that Eq.(1.4) is a special case of a solvable difference equation, from which the solvability in the cases (1)-(4) follows. We also analyze the claims on the long-term behavior of solutions to Eq.(1.4) formulated in [38] and show that they are false.

2. Solvability of a generalization of Eq. (1.4)

Here we show that Eq.(1.4) is a special case of a solvable difference equation. Before we state our first result note that Eq.(1.4) can be rewritten in the form

$$x_{n+1}=x_n\frac{acx_{n-3}+(ad+b)x_{n-4}}{cx_{n-3}+dx_{n-4}},\quad n\in\mathbb{N}_0.\tag{2.1}$$

The form suggests studying a natural generalization of Eq.(1.4) (see Eq.(2.2) below).

Theorem 2.1. *Let $\alpha,\beta,\gamma,\delta\in\mathbb{R}$, $\alpha^2+\beta^2\neq 0\neq\gamma^2+\delta^2$, and Ψ be a homeomorphism of $\mathbb{R}$ such that $\Psi(0)=0$. Then, the equation*

$$x_{n+1}=\Psi^{-1}\left(\Psi(x_n)\frac{\alpha\Psi(x_{n-3})+\beta\Psi(x_{n-4})}{\gamma\Psi(x_{n-3})+\delta\Psi(x_{n-4})}\right),\quad n\in\mathbb{N}_0, \tag{2.2}$$

is solvable in closed form.

Proof. If there is $n_0\in\mathbb{N}_0$ such that $x_{n_0}=0$, then if x_{n_0+1} is defined it must be equal to zero. But then x_{n_0+5} is not defined. Hence, from now on for a solution $(x_n)_{n\in\mathbb{N}_{-4}}$ to Eq.(2.2) we suppose $x_n\neq 0$ for $n\in\mathbb{N}_{-4}$, which implies

$$\Psi(x_n)\neq 0,\quad \text{for}\quad n\in\mathbb{N}_{-4}. \tag{2.3}$$

Eq.(2.2) along with the conditions posed on Ψ imply

$$\Psi(x_{n+1})=\Psi(x_n)\frac{\alpha\Psi(x_{n-3})+\beta\Psi(x_{n-4})}{\gamma\Psi(x_{n-3})+\delta\Psi(x_{n-4})},\quad n\in\mathbb{N}_0. \tag{2.4}$$

Using the transformation

$$y_n=\frac{\Psi(x_n)}{\Psi(x_{n-1})},\quad n\in\mathbb{N}_{-3}. \tag{2.5}$$

in (2.4) we obtain the equation

$$y_{n+1}=\frac{\alpha y_{n-3}+\beta}{\gamma y_{n-3}+\delta},\quad n\in\mathbb{N}_0. \tag{2.6}$$

Let

$$z_m^{(j)}=y_{4m-j},\quad m\in\mathbb{N}_0,\ j=\overline{0,3}. \tag{2.7}$$

Then

$$z_{m+1}^{(j)}=\frac{\alpha z_m^{(j)}+\beta}{\gamma z_m^{(j)}+\delta},\quad m\in\mathbb{N}_0,\ j=\overline{0,3}. \tag{2.8}$$

Using the change of variables

$$z_m^{(j)}=\frac{u_{m+1}^{(j)}}{u_m^{(j)}}-\frac{\delta}{\gamma},\quad m\in\mathbb{N}_0,\ j=\overline{0,3}, \tag{2.9}$$

where $\gamma\neq 0$, in (2.8) we obtain

$$\gamma^2u_{m+2}^{(j)}-\gamma(\alpha+\delta)u_{m+1}^{(j)}+(\alpha\delta-\beta\gamma)u_m^{(j)}=0,\quad m\in\mathbb{N}_0,\ j=\overline{0,3}. \tag{2.10}$$

There are several cases to be considered.

Case $\alpha\delta\neq\beta\gamma$, $\gamma\neq 0$. Under the assumptions, there are several subcases to be considered.

Case $(\alpha+\delta)^2\neq 4(\alpha\delta-\beta\gamma)$. Employing (1.2) we obtain

$$u_m^{(j)}=\frac{(u_1^{(j)}-\lambda_2u_0^{(j)})\lambda_1^m-(u_1^{(j)}-\lambda_1u_0^{(j)})\lambda_2^m}{\lambda_1-\lambda_2}, \tag{2.11}$$

for $m\in\mathbb{N}_0$, $j=\overline{0,3}$, where

$$\lambda_1=\frac{\alpha+\delta+\sqrt{(\alpha+\delta)^2-4(\alpha\delta-\beta\gamma)}}{2\gamma} \tag{2.12}$$

and

$$\lambda_2=\frac{\alpha+\delta-\sqrt{(\alpha+\delta)^2-4(\alpha\delta-\beta\gamma)}}{2\gamma}. \tag{2.13}$$

Using (2.11) in (2.9) we have

$$z_m^{(j)} = \frac{(z_0^{(j)} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (z_0^{(j)} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(z_0^{(j)} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m} - (z_0^{(j)} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m}} - \frac{\delta}{\gamma},$$

for $m \in \mathbb{N}_0$, $j = \overline{0,3}$, that is

$$y_{4m-j} = \frac{(y_{-j} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (y_{-j} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(y_{-j} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m} - (y_{-j} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m}} - \frac{\delta}{\gamma}, \tag{2.14}$$

for $m \in \mathbb{N}_0$, $j = \overline{0,3}$.

Combining (2.5) and (2.14) we have

$$\Psi(x_{4m-j}) = \left(\frac{(\frac{\Psi(x_{-j})}{\Psi(x_{-j-1})} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (\frac{\Psi(x_{-j})}{\Psi(x_{-j-1})} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{\Psi(x_{-j})}{\Psi(x_{-j-1})} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m} - (\frac{\Psi(x_{-j})}{\Psi(x_{-j-1})} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m}} - \frac{\delta}{\gamma}\right)\Psi(x_{4m-j-1}),$$

for $m \in \mathbb{N}_0$, $j = \overline{0,3}$, as well as

$$\Psi(x_{4m-j}) = y_{4m-j}y_{4m-j-1}y_{4m-j-2}y_{4m-j-3}\Psi(x_{4m-j-4}), \tag{2.15}$$

for $m \in \mathbb{N}$, $j = \overline{1,4}$.

Thus

$$\begin{aligned}
\Psi(x_{4m}) &= \Psi(x_{-4})\prod_{i=0}^{m} y_{4i}y_{4i-1}y_{4i-2}y_{4i-3},\\
\Psi(x_{4m+1}) &= \Psi(x_{-3})\prod_{i=0}^{m} y_{4i+1}y_{4i}y_{4i-1}y_{4i-2},\\
\Psi(x_{4m+2}) &= \Psi(x_{-2})\prod_{i=0}^{m} y_{4i+2}y_{4i+1}y_{4i}y_{4i-1},\\
\Psi(x_{4m+3}) &= \Psi(x_{-1})\prod_{i=0}^{m} y_{4i+3}y_{4i+2}y_{4i+1}y_{4i},
\end{aligned}$$

for $m \in \mathbb{N}_0$, which implies

$$x_{4m} = \Psi^{-1}\left(\Psi(x_{-4})\prod_{i=0}^{m} y_{4i}y_{4i-1}y_{4i-2}y_{4i-3}\right), \tag{2.16}$$

$$x_{4m+1} = \Psi^{-1}\left(\Psi(x_{-3})\prod_{i=0}^{m} y_{4i+1}y_{4i}y_{4i-1}y_{4i-2}\right), \tag{2.17}$$

$$x_{4m+2} = \Psi^{-1}\left(\Psi(x_{-2})\prod_{i=0}^{m} y_{4i+2}y_{4i+1}y_{4i}y_{4i-1}\right), \tag{2.18}$$

$$x_{4m+3} = \Psi^{-1}\left(\Psi(x_{-1})\prod_{i=0}^{m} y_{4i+3}y_{4i+2}y_{4i+1}y_{4i}\right), \tag{2.19}$$

for $m \in \mathbb{N}_0$, where

$$\begin{aligned}
&y_{4m}y_{4m-1}y_{4m-2}y_{4m-3}\\
&=\left(\frac{\left(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m}-\left(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{\left(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m}-\left(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{\left(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m}-\left(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{\left(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m}-\left(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m}}-\frac{\delta}{\gamma}\right),
\end{aligned}\tag{2.20}$$

$$\begin{aligned}
&y_{4m+1}y_{4m}y_{4m-1}y_{4m-2}\\
&=\left(\frac{\left(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+2}-\left(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+2}}{\left(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{\left(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m}-\left(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{\left(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m}-\left(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{\left(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m}-\left(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m}}-\frac{\delta}{\gamma}\right),
\end{aligned}\tag{2.21}$$

$$\begin{aligned}
&y_{4m+2}y_{4m+1}y_{4m}y_{4m-1}\\
&=\left(\frac{\left(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+2}-\left(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+2}}{\left(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{\left(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+2}-\left(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+2}}{\left(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{\left(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m}-\left(\frac{\Psi(x_0)}{\Psi(x_{-1})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{\left(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m+1}-\left(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m+1}}{\left(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_2+\frac{\delta}{\gamma}\right)\lambda_1^{m}-\left(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_1+\frac{\delta}{\gamma}\right)\lambda_2^{m}}-\frac{\delta}{\gamma}\right),
\end{aligned}\tag{2.22}$$

$$\begin{aligned}&y_{4m+3}y_{4m+2}y_{4m+1}y_{4m}\\&=\left(\frac{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\&\times\left(\frac{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\&\times\left(\frac{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\&\times\left(\frac{(\frac{\Psi(x_{0})}{\Psi(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{\Psi(x_{0})}{\Psi(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{\Psi(x_{0})}{\Psi(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{\Psi(x_{0})}{\Psi(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right),\end{aligned}\tag{2.23}$$

for $m\in\mathbb{N}_0$. Hence, (2.16)-(2.23) present the general solution to Eq.(2.2) in this case.

Case $(\alpha+\delta)^2=4(\alpha\delta-\beta\gamma)$. From (1.3) we have

$$u_m^{(j)}=((u_1^{(j)}-\lambda u_0^{(j)})m+\lambda u_0^{(j)})\lambda^{m-1},\tag{2.24}$$

for $m\in\mathbb{N}_0$, $j=\overline{0,3}$, where

$$\lambda=\frac{\alpha+\delta}{2\gamma}\neq 0.$$

Using (2.24) in (2.9) we obtain

$$z_m^{(j)}=\frac{((z_0^{(j)}-\lambda+\frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(z_0^{(j)}-\lambda+\frac{\delta}{\gamma})m+\lambda}-\frac{\delta}{\gamma},$$

for $m\in\mathbb{N}_0$, $j=\overline{0,3}$, that is,

$$y_{4m-j}=\frac{((y_{-j}-\lambda+\frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(y_{-j}-\lambda+\frac{\delta}{\gamma})m+\lambda}-\frac{\delta}{\gamma},\tag{2.25}$$

for $m\in\mathbb{N}_0$, $j=\overline{0,3}$.

Relations (2.5) and (2.25) yield

$$\Psi(x_{4m-j})=\left(\frac{(\frac{\Psi(x_{-j})}{\Psi(x_{-j-1})}-\lambda+\frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(\frac{\Psi(x_{-j})}{\Psi(x_{-j-1})}-\lambda+\frac{\delta}{\gamma})m+\lambda}-\frac{\delta}{\gamma}\right)\Psi(x_{4m-j-1}),\tag{2.26}$$

for $m\in\mathbb{N}_0$, $j=\overline{0,3}$.

We also have

$$\begin{aligned}y_{4m}y_{4m-1}y_{4m-2}y_{4m-3}&=\left(\frac{(\frac{\Psi(x_{0})}{\Psi(x_{-1})}-\lambda+\frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(\frac{\Psi(x_{0})}{\Psi(x_{-1})}-\lambda+\frac{\delta}{\gamma})m+\lambda}-\frac{\delta}{\gamma}\right)\\&\times\left(\frac{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda+\frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})}-\lambda+\frac{\delta}{\gamma})m+\lambda}-\frac{\delta}{\gamma}\right)\\&\times\left(\frac{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda+\frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})}-\lambda+\frac{\delta}{\gamma})m+\lambda}-\frac{\delta}{\gamma}\right)\\&\times\left(\frac{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda+\frac{\delta}{\gamma})(m+1)+\lambda)\lambda}{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})}-\lambda+\frac{\delta}{\gamma})m+\lambda}-\frac{\delta}{\gamma}\right),\end{aligned}\tag{2.27}$$

$$y_{4m+1}y_{4m}y_{4m-1}y_{4m-2} = \left(\frac{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})} - \lambda + \frac{\delta}{\gamma})(m+2) + \lambda)\lambda}{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})} - \lambda + \frac{\delta}{\gamma})(m+1) + \lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_0)}{\Psi(x_{-1})} - \lambda + \frac{\delta}{\gamma})(m+1) + \lambda)\lambda}{(\frac{\Psi(x_0)}{\Psi(x_{-1})} - \lambda + \frac{\delta}{\gamma})m + \lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})} - \lambda + \frac{\delta}{\gamma})(m+1) + \lambda)\lambda}{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})} - \lambda + \frac{\delta}{\gamma})m + \lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})} - \lambda + \frac{\delta}{\gamma})(m+1) + \lambda)\lambda}{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})} - \lambda + \frac{\delta}{\gamma})m + \lambda} - \frac{\delta}{\gamma}\right), \tag{2.28}$$

$$y_{4m+2}y_{4m+1}y_{4m}y_{4m-1} = \left(\frac{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})} - \lambda + \frac{\delta}{\gamma})(m+2) + \lambda)\lambda}{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})} - \lambda + \frac{\delta}{\gamma})(m+1) + \lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})} - \lambda + \frac{\delta}{\gamma})(m+2) + \lambda)\lambda}{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})} - \lambda + \frac{\delta}{\gamma})(m+1) + \lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_0)}{\Psi(x_{-1})} - \lambda + \frac{\delta}{\gamma})(m+1) + \lambda)\lambda}{(\frac{\Psi(x_0)}{\Psi(x_{-1})} - \lambda + \frac{\delta}{\gamma})m + \lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})} - \lambda + \frac{\delta}{\gamma})(m+1) + \lambda)\lambda}{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})} - \lambda + \frac{\delta}{\gamma})m + \lambda} - \frac{\delta}{\gamma}\right), \tag{2.29}$$

$$y_{4m+3}y_{4m+2}y_{4m+1}y_{4m} = \left(\frac{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})} - \lambda + \frac{\delta}{\gamma})(m+2) + \lambda)\lambda}{(\frac{\Psi(x_{-1})}{\Psi(x_{-2})} - \lambda + \frac{\delta}{\gamma})(m+1) + \lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})} - \lambda + \frac{\delta}{\gamma})(m+2) + \lambda)\lambda}{(\frac{\Psi(x_{-2})}{\Psi(x_{-3})} - \lambda + \frac{\delta}{\gamma})(m+1) + \lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})} - \lambda + \frac{\delta}{\gamma})(m+2) + \lambda)\lambda}{(\frac{\Psi(x_{-3})}{\Psi(x_{-4})} - \lambda + \frac{\delta}{\gamma})(m+1) + \lambda} - \frac{\delta}{\gamma}\right)$$
$$\times \left(\frac{(\frac{\Psi(x_0)}{\Psi(x_{-1})} - \lambda + \frac{\delta}{\gamma})(m+1) + \lambda)\lambda}{(\frac{\Psi(x_0)}{\Psi(x_{-1})} - \lambda + \frac{\delta}{\gamma})m + \lambda} - \frac{\delta}{\gamma}\right), \tag{2.30}$$

for $m \in \mathbb{N}_0$. Hence, (2.16)-(2.19), (2.27)-(2.30) present the general solution to Eq.(2.2) in this case.

Case $\gamma = 0$. Under the condition we have $\delta \neq 0$ and

$$y_{n+1} = \frac{\alpha}{\delta} y_{n-3} + \frac{\beta}{\delta}, \quad n \in \mathbb{N}_0, \tag{2.31}$$

that is,

$$z_{m+1}^{(j)} = \frac{\alpha}{\delta} z_m^{(j)} + \frac{\beta}{\delta}, \quad m \in \mathbb{N}_0,\ j = \overline{0,3}. \tag{2.32}$$

Case $\alpha = \delta$. We have

$$z_m^{(j)} = \frac{\beta}{\delta} m + z_0^{(j)}, \quad m \in \mathbb{N}_0,\ j = \overline{0,3}.$$

that is,

$$y_{4m-j}=\frac{\beta}{\delta}m+y_{-j}=\frac{\beta}{\delta}m+\frac{\Psi(x_{-j})}{\Psi(x_{-j-1})},\quad m\in\mathbb{N}_0,\ j=\overline{0,3}.$$

This relation, (2.5) and (2.15) imply

$$\begin{aligned}
\Psi(x_{4m})=&\left(\frac{\beta}{\delta}m+\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\left(\frac{\beta}{\delta}m+\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\\
&\times\left(\frac{\beta}{\delta}m+\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\left(\frac{\beta}{\delta}m+\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\Psi(x_{4m-4}),\\
\Psi(x_{4m+1})=&\left(\frac{\beta}{\delta}(m+1)+\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\left(\frac{\beta}{\delta}m+\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\\
&\times\left(\frac{\beta}{\delta}m+\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\left(\frac{\beta}{\delta}m+\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\Psi(x_{4m-3}),\\
\Psi(x_{4m+2})=&\left(\frac{\beta}{\delta}(m+1)+\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\left(\frac{\beta}{\delta}(m+1)+\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\\
&\times\left(\frac{\beta}{\delta}m+\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\left(\frac{\beta}{\delta}m+\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\Psi(x_{4m-2}),\\
\Psi(x_{4m+3})=&\left(\frac{\beta}{\delta}(m+1)+\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\left(\frac{\beta}{\delta}(m+1)+\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\\
&\times\left(\frac{\beta}{\delta}(m+1)+\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\left(\frac{\beta}{\delta}m+\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\Psi(x_{4m-1})
\end{aligned}$$

for $m\in\mathbb{N}_0$, so that

$$\begin{aligned}
\Psi(x_{4m})=&\Psi(x_{-4})\prod_{j=0}^{m}\left(\frac{\beta}{\delta}j+\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\left(\frac{\beta}{\delta}j+\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\\
&\times\left(\frac{\beta}{\delta}j+\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\left(\frac{\beta}{\delta}j+\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right),\\
\Psi(x_{4m+1})=&\Psi(x_{-3})\prod_{j=0}^{m}\left(\frac{\beta}{\delta}(j+1)+\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\left(\frac{\beta}{\delta}j+\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\\
&\times\left(\frac{\beta}{\delta}j+\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\left(\frac{\beta}{\delta}j+\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right),\\
\Psi(x_{4m+2})=&\Psi(x_{-2})\prod_{j=0}^{m}\left(\frac{\beta}{\delta}(j+1)+\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\left(\frac{\beta}{\delta}(j+1)+\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\\
&\times\left(\frac{\beta}{\delta}j+\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\left(\frac{\beta}{\delta}j+\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right),\\
\Psi(x_{4m+3})=&\Psi(x_{-1})\prod_{j=0}^{m}\left(\frac{\beta}{\delta}(j+1)+\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\left(\frac{\beta}{\delta}(j+1)+\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\\
&\times\left(\frac{\beta}{\delta}(j+1)+\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\left(\frac{\beta}{\delta}j+\frac{\Psi(x_0)}{\Psi(x_{-1})}\right),
\end{aligned}$$

for $m \in \mathbb{N}_0$, and finally

$$\begin{aligned} x_{4m} =& \Psi^{-1}\Big(\Psi(x_{-4}) \prod_{j=0}^{m} \Big(\frac{\beta}{\delta}j + \frac{\Psi(x_0)}{\Psi(x_{-1})}\Big)\Big(\frac{\beta}{\delta}j + \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\Big) \\ & \times \Big(\frac{\beta}{\delta}j + \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\Big)\Big(\frac{\beta}{\delta}j + \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\Big)\Big), \end{aligned} \tag{2.33}$$

$$\begin{aligned} x_{4m+1} =& \Psi^{-1}\Big(\Psi(x_{-3}) \prod_{j=0}^{m} \Big(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\Big)\Big(\frac{\beta}{\delta}j + \frac{\Psi(x_0)}{\Psi(x_{-1})}\Big) \\ & \Big(\frac{\beta}{\delta}j + \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\Big)\Big(\frac{\beta}{\delta}j + \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\Big)\Big), \end{aligned} \tag{2.34}$$

$$\begin{aligned} x_{4m+2} =& \Psi^{-1}\Big(\Psi(x_{-2}) \prod_{j=0}^{m} \Big(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\Big)\Big(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\Big) \\ & \Big(\frac{\beta}{\delta}j + \frac{\Psi(x_0)}{\Psi(x_{-1})}\Big)\Big(\frac{\beta}{\delta}j + \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\Big)\Big), \end{aligned} \tag{2.35}$$

$$\begin{aligned} x_{4m+3} =& \Psi^{-1}\Big(\Psi(x_{-1}) \prod_{j=0}^{m} \Big(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\Big)\Big(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\Big) \\ & \Big(\frac{\beta}{\delta}(j+1) + \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\Big)\Big(\frac{\beta}{\delta}j + \frac{\Psi(x_0)}{\Psi(x_{-1})}\Big)\Big), \end{aligned} \tag{2.36}$$

for $m \in \mathbb{N}_0$. Hence, (2.33)-(2.36) present the general solution to Eq.(2.2) in this case.

Case $\alpha \neq \delta$. Eq.(2.32) implies

$$z_m^{(j)} = \frac{\beta}{\alpha - \delta}\Big(\Big(\frac{\alpha}{\delta}\Big)^m - 1\Big) + \Big(\frac{\alpha}{\delta}\Big)^m z_0^{(j)},$$

for $m \in \mathbb{N}_0$, $j = \overline{0, 3}$, that is,

$$y_{4m-j} = \beta\frac{(\alpha/\delta)^m - 1}{\alpha - \delta} + \Big(\frac{\alpha}{\delta}\Big)^m y_{-j}. \tag{2.37}$$

Relations (2.5), (2.15) and (2.37) imply

$$\begin{aligned}
\Psi(x_{4m}) &= \left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^m\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^m\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^m\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^m\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\Psi(x_{4m-4}),\\
\Psi(x_{4m+1}) &= \left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{m+1}\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^m\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^m\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^m\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\Psi(x_{4m-3}),\\
\Psi(x_{4m+2}) &= \left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{m+1}\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{m+1}\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^m\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^m\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\Psi(x_{4m-2}),\\
\Psi(x_{4m+3}) &= \left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{m+1}\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{m+1}\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^{m+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{m+1}\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\\
&\quad\times\left(\beta\frac{(\alpha/\delta)^m-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^m\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\Psi(x_{4m-1}),
\end{aligned}$$

for $m\in\mathbb{N}_0$.

Hence

$$\begin{aligned}
\Psi(x_{4m}) = \Psi(x_{-4}) \prod_{j=0}^{m} & \left(\beta \frac{(\alpha/\delta)^{j}-1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^{j} \frac{\Psi(x_{0})}{\Psi(x_{-1})}\right) \\
& \times \left(\beta \frac{(\alpha/\delta)^{j}-1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^{j} \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right) \\
& \times \left(\beta \frac{(\alpha/\delta)^{j}-1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^{j} \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right), \\
& \times \left(\beta \frac{(\alpha/\delta)^{j}-1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^{j} \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right), \\
\Psi(x_{4m+1}) = \Psi(x_{-3}) \prod_{j=0}^{m} & \left(\beta \frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^{j+1} \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right) \\
& \times \left(\beta \frac{(\alpha/\delta)^{j}-1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^{j} \frac{\Psi(x_{0})}{\Psi(x_{-1})}\right) \\
& \times \left(\beta \frac{(\alpha/\delta)^{j}-1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^{j} \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right) \\
& \times \left(\beta \frac{(\alpha/\delta)^{j}-1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^{j} \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right), \\
\Psi(x_{4m+2}) = \Psi(x_{-2}) \prod_{j=0}^{m} & \left(\beta \frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^{j+1} \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right) \\
& \times \left(\beta \frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^{j+1} \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right) \\
& \times \left(\beta \frac{(\alpha/\delta)^{j}-1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^{j} \frac{\Psi(x_{0})}{\Psi(x_{-1})}\right) \\
& \times \left(\beta \frac{(\alpha/\delta)^{j}-1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^{j} \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right), \\
\Psi(x_{4m+3}) = \Psi(x_{-1}) \prod_{j=0}^{m} & \left(\beta \frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^{j+1} \frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right) \\
& \times \left(\beta \frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^{j+1} \frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right) \\
& \times \left(\beta \frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^{j+1} \frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right) \\
& \times \left(\beta \frac{(\alpha/\delta)^{j}-1}{\alpha-\delta} + \left(\frac{\alpha}{\delta}\right)^{j} \frac{\Psi(x_{0})}{\Psi(x_{-1})}\right),
\end{aligned}$$

for $m \in \mathbb{N}_0$, so that

$$\begin{aligned} x_{4m} = \Psi^{-1}\Bigg(\Psi(x_{-4})\prod_{j=0}^{m}&\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right),\\ &\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\Bigg), \end{aligned} \tag{2.38}$$

$$\begin{aligned} x_{4m+1} = \Psi^{-1}\Bigg(\Psi(x_{-3})\prod_{j=0}^{m}&\left(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{j+1}\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\Bigg), \end{aligned} \tag{2.39}$$

$$\begin{aligned} x_{4m+2} = \Psi^{-1}\Bigg(\Psi(x_{-2})\prod_{j=0}^{m}&\left(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{j+1}\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{j+1}\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\Bigg), \end{aligned} \tag{2.40}$$

$$\begin{aligned} x_{4m+3} = \Psi^{-1}\Bigg(\Psi(x_{-1})\prod_{j=0}^{m}&\left(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{j+1}\frac{\Psi(x_{-1})}{\Psi(x_{-2})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{j+1}\frac{\Psi(x_{-2})}{\Psi(x_{-3})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^{j+1}-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^{j+1}\frac{\Psi(x_{-3})}{\Psi(x_{-4})}\right)\\ &\times\left(\beta\frac{(\alpha/\delta)^j-1}{\alpha-\delta}+\Big(\frac{\alpha}{\delta}\Big)^j\frac{\Psi(x_0)}{\Psi(x_{-1})}\right)\Bigg), \end{aligned} \tag{2.41}$$

for $m \in \mathbb{N}_0$. Hence, (2.38)-(2.41) present the general solution to Eq.(2.2) in this case.

Case $\alpha\delta = \beta\gamma$, $\alpha = 0$. We have $\beta \neq 0$, $\gamma = 0$ and $\delta \neq 0$. Thus

$$x_{n+1} = \Psi^{-1}\Big(\frac{\beta}{\delta}\Psi(x_n)\Big), \quad n \in \mathbb{N}_0, \tag{2.42}$$

which yields

$$x_n = \Psi^{-1}\left(\left(\frac{\beta}{\delta}\right)^n \Psi(x_0)\right), \quad n \in \mathbb{N}_0. \tag{2.43}$$

Case $\alpha\delta = \beta\gamma$, $\alpha \neq 0$, $\beta = 0$. Under the conditions we have $\delta = 0$ and $\gamma \neq 0$. Thus

$$x_{n+1} = \Psi^{-1}\Big(\frac{\alpha}{\gamma}\Psi(x_n)\Big), \quad n \in \mathbb{N}_0, \tag{2.44}$$

which yields

$$x_n = \Psi^{-1}\left(\left(\frac{\alpha}{\gamma}\right)^n \Psi(x_0)\right), \quad n \in \mathbb{N}_0. \tag{2.45}$$

Case $\alpha\delta = \beta\gamma$, $\delta = 0$. Under the conditions we have $\gamma \neq 0$, $\beta = 0$ and $\alpha \neq 0$, and consequently obtain Eq.(2.44) whose solution is (2.45).

Case $\alpha\delta = \beta\gamma$, $\gamma = 0$. Under the conditions we have $\delta \neq 0$, $\alpha = 0$ and $\beta \neq 0$. and consequently obtain Eq.(2.42) whose solution is (2.43).

Case $\alpha\beta\gamma\delta \neq 0$. In this case we obtain Eq.(2.42), i.e., Eq.(2.44), whose solution is (2.43), i.e., (2.45). □

Remark 2.2. From Theorem 2.1, some calculation and a representation in [51] are obtained the closed-form formulas in [38] for solutions to the special cases (i)-(iv) of Eq.(1.4). We omit the standard problem.

3. On some results on the long-term behavior of solutions to Eq. (1.4)

One of the main problems in the theory of difference equations is describing the long-term behaviour of their solutions (see, e.g., [2, 3, 5–8, 16–18, 23, 26, 29, 34–36, 40–48, 50, 53–55, 58, 59] and the references therein).

Here we analyze the claims on the long-term behaviour of some solutions to the special cases of Eq.(1.4) considered in [38]. The paper starts with finding equilibria of Eq.(1.4) and, unfortunately, made a typical mistake. Namely, for an equilibrium $\bar{x}$ of Eq.(1.4) it has to hold the relation

$$\bar{x} = a\bar{x} + \frac{b\bar{x}^2}{(c+d)\bar{x}}. \tag{3.1}$$

It is wrongly claimed therein that (3.1) is equivalent to the relation

$$\bar{x}^2(1-a)(c+d) = b\bar{x}^2. \tag{3.2}$$

Then it is assumed $(1-a)(c+d) \neq b$, and from (3.2) concluded that $\bar{x} = 0$ is a unique equilibrium point of Eq.(1.4). But $\bar{x} = 0$ is not a solution to (3.1). As a consequence of the wrong claim, the following claim ([38, Theorem 5]) is wrong.

Claim 1. *Assume that* $b(d+3c) < (1-a)(c+d)^2$. *Then the equilibrium point of Eq.*(1.4) *is locally asymptotically stable.*

The following claim is Theorem 6 in [38].

Claim 2. *The equilibrium point* $\bar{x}$ *of Eq.*(1.4) *is global attractor if* $d(1-a) \neq b$.

From the same reason the claim is not well-formulated. Moreover, if we even ignore the problem with the choice of a wrong equilibrium, the claim is not correct. We show this by an example. Before presenting the example note that Eq.(2.1) is obtained from Eq.(2.2) for $\Psi(x) = x$, $\alpha = ac$, $\beta = ad + b$, $\gamma = c$ and $\delta = d$.

Example 3.1. Consider the equation

$$x_{n+1} = x_n \frac{\alpha x_{n-3} + \beta x_{n-4}}{\gamma x_{n-3} + \delta x_{n-4}}, \quad n \in \mathbb{N}_0. \tag{3.3}$$

where $\alpha, \beta, \gamma, \delta$ are positive numbers satisfying the conditions

$$\alpha + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma} > \delta + 2\gamma, \tag{3.4}$$

$$\delta(\gamma - \alpha) \neq \beta\gamma - \delta\alpha, \tag{3.5}$$

whereas the initial values x_{-j}, $j = \overline{0,4}$, satisfy the conditions

$$\frac{x_{-i}}{x_{-(i+1)}} \neq \lambda_2 - \frac{\delta}{\gamma}, \quad i = \overline{0,3}, \tag{3.6}$$

where λ_2 is given in (2.13).

From (3.5) we easily see that in this case $d(1-a) \neq b$, which is a condition in Claim 1.

From (2.16)-(2.23) it follows that for $m \in \mathbb{N}_0$ we have

$$x_{4m} = x_{-4} \prod_{i=0}^{m} y_{4i} y_{4i-1} y_{4i-2} y_{4i-3} \tag{3.7}$$

$$x_{4m+1} = x_{-3} \prod_{i=0}^{m} y_{4i+1} y_{4i} y_{4i-1} y_{4i-2}, \tag{3.8}$$

$$x_{4m+2} = x_{-2} \prod_{i=0}^{m} y_{4i+2} y_{4i+1} y_{4i} y_{4i-1}, \tag{3.9}$$

$$x_{4m+3} = x_{-1} \prod_{i=0}^{m} y_{4i+3} y_{4i+2} y_{4i+1} y_{4i}, \tag{3.10}$$

where

$$\begin{aligned} & y_{4m} y_{4m-1} y_{4m-2} y_{4m-3} \\ &= \left(\frac{(\frac{x_0}{x_{-1}} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (\frac{x_0}{x_{-1}} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_0}{x_{-1}} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m} - (\frac{x_0}{x_{-1}} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m}} - \frac{\delta}{\gamma} \right) \\ &\quad \times \left(\frac{(\frac{x_{-1}}{x_{-2}} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (\frac{x_{-1}}{x_{-2}} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-1}}{x_{-2}} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m} - (\frac{x_{-1}}{x_{-2}} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m}} - \frac{\delta}{\gamma} \right) \\ &\quad \times \left(\frac{(\frac{x_{-2}}{x_{-3}} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (\frac{x_{-2}}{x_{-3}} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-2}}{x_{-3}} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m} - (\frac{x_{-2}}{x_{-3}} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m}} - \frac{\delta}{\gamma} \right) \\ &\quad \times \left(\frac{(\frac{x_{-3}}{x_{-4}} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m+1} - (\frac{x_{-3}}{x_{-4}} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-3}}{x_{-4}} - \lambda_2 + \frac{\delta}{\gamma})\lambda_1^{m} - (\frac{x_{-3}}{x_{-4}} - \lambda_1 + \frac{\delta}{\gamma})\lambda_2^{m}} - \frac{\delta}{\gamma} \right), \end{aligned} \tag{3.11}$$

$$
\begin{aligned}
&y_{4m+1}y_{4m}y_{4m-1}y_{4m-2}\\
&=\left(\frac{(\frac{x_{-3}}{x_{-4}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{x_{-3}}{x_{-4}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{x_{-3}}{x_{-4}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-3}}{x_{-4}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_0}{x_{-1}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_0}{x_{-1}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_0}{x_{-1}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right),
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
&y_{4m+2}y_{4m+1}y_{4m}y_{4m-1}\\
&=\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{(\frac{x_{-3}}{x_{-4}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{x_{-3}}{x_{-4}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{x_{-3}}{x_{-4}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-3}}{x_{-4}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_0}{x_{-1}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_0}{x_{-1}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_0}{x_{-1}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right),
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
&y_{4m+3}y_{4m+2}y_{4m+1}y_{4m}\\
&=\left(\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{(\frac{x_{-3}}{x_{-4}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+2}-(\frac{x_{-3}}{x_{-4}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+2}}{(\frac{x_{-3}}{x_{-4}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-3}}{x_{-4}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}-\frac{\delta}{\gamma}\right)\\
&\quad\times\left(\frac{(\frac{x_0}{x_{-1}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_0}{x_{-1}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_0}{x_{-1}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_0}{x_{-1}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\right),
\end{aligned}
\tag{3.14}
$$

for $m\in\mathbb{N}_0$, where λ_1 and λ_2 are given in (2.12) and (2.13), respectively.

Further, we have

$$\begin{aligned}
&\lim_{m\to+\infty}\frac{(\frac{x_0}{x_{-1}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_0}{x_{-1}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_0}{x_{-1}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_0}{x_{-1}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\\
=&\lim_{m\to+\infty}\frac{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-1}}{x_{-2}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_{-1}}{x_{-2}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\\
=&\lim_{m\to+\infty}\frac{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-2}}{x_{-3}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_{-2}}{x_{-3}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\\
=&\lim_{m\to+\infty}\frac{(\frac{x_{-3}}{x_{-4}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m+1}-(\frac{x_{-3}}{x_{-4}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m+1}}{(\frac{x_{-3}}{x_{-4}}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{m}-(\frac{x_{-3}}{x_{-4}}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{m}}-\frac{\delta}{\gamma}\\
&=\lambda_1-\frac{\delta}{\gamma}>1,
\end{aligned}$$

where first we have used the assumption (3.6), whereas in the last inequality we have used the condition in (3.4), from which along with (3.7)-(3.14) it follows that for such solutions we have $\lim_{n\to+\infty}x_n=+\infty$, that is, the solutions are even unbounded refuting Claim 2.

Remark 3.2. It is easy to find the parameters $\alpha,\beta,\gamma,\delta$ satisfying the conditions (3.4) and (3.5), as well as the positive initial values so that the relations in (3.6) hold. For example, we can chose $\alpha=\gamma=\delta=1$ and $\beta=2$, in which case we have that $a=b=c=d=1$ in Eq.(1.4).

Remark 3.3. The only correct result on the behaviour of solutions to Eq.(1.4) in [38] is Theorem 7 therein, which states that every positive solution to the equation is bounded if $\min\{a,b,c,d\}>0$ and $a+\frac{b}{d}<1$. However, the result is trivial, since in this case from Eq.(1.4) it immediately follows that for each positive solution to the equation we have $0<x_{n+1}\le(a+\frac{b}{d})x_n<x_n$, $n\in\mathbb{N}_0$. Moreover, from the estimate and by a well-known theorem it follows that each positive solution to Eq.(1.4) converges [62], and that the limit is equal to zero, which was not noticed in [38]. For some results on the boundedness of solutions to various difference equations and systems see, for instance, [4–8, 29, 31, 32, 34, 36, 40–43, 45, 47] and the related references therein.

References

[1] D. Adamović, *Problem 194*, Mat. Vesnik, **22** (2), 270, 1970.

[2] D. Adamović, *Solution to problem 194*, Mat. Vesnik, **23**, 236-242, 1971.

[3] A. Andruch-Sobilo and M. Migda, *On the rational recursive sequence* $x_{n+1}=ax_{n-1}/(b+cx_nx_{n-1})$, Tatra Mt. Math. Publ. **43**, 1-9, 2009.

[4] K. Berenhaut, J. Foley and S. Stević, *Boundedness character of positive solutions of a max difference equation*, J. Difference Equ. Appl. **12** (12), 1193-1199, 2006.

[5] K. Berenhaut, J. Foley and S. Stević, *The global attractivity of the rational difference equation* $y_n=1+(y_{n-k}/y_{n-m})$, Proc. Amer. Math. Soc. **135** (1), 1133-1140, 2007.

[6] K. Berenhaut and S. Stević, *The behaviour of the positive solutions of the difference equation* $x_n=A+(x_{n-2}/x_{n-1})^p$, J. Difference Equ. Appl. **12** (9), 909-918, 2006.

[7] L. Berg, *On the asymptotics of nonlinear difference equations*, Z. Anal. Anwendungen, **21** (4), 1061-1074,2002.

[8] L. Berg and S. Stević, *On the asymptotics of the difference equation* $y_n(1+y_{n-1}\cdots y_{n-k+1})=y_{n-k}$, J. Difference Equ. Appl. **17** (4), 577-586, 2011.

[9] D. Bernoulli, *Observationes de seriebus quae formantur ex additione vel substractione quacunque terminorum se mutuo consequentium, ubi praesertim earundem insignis usus pro inveniendis radicum omnium aequationum algebraicarum ostenditur, Commentarii Acad. Petropol. III,* 1728, 85-100, 1732.

[10] G. Boole, *A Treatsie on the Calculus of Finite Differences*, Third Edition, Macmillan and Co., London, 1880.

[11] L. Brand, *A sequence defined by a difference equation*, Amer. Math. Monthly, **62** (7), 489-492, 1955.

[12] A. de Moivre, Miscellanea Analytica de Seriebus et Quadraturis, J. Tonson & J.Watts, Londini, 1730.

[13] L. Euler, *Introductio in Analysin Infinitorum, Tomus Primus*, Lausannae, 1748.

[14] B. Iričanin and S. Stević, *On some rational difference equations*, Ars Combin. **92**, 67-72, 2009.

[15] C. Jordan, *Calculus of Finite Differences*, Chelsea Publishing Company, New York, 1965.

[16] G. L. Karakostas, *Convergence of a difference equation via the full limiting sequences method*, Differ. Eqs. Dyn. Syst. **1** (4), 289-294, 1993.

[17] G. L. Karakostas, *Asymptotic behavior of the solutions of the difference equation* $x_{n+1} = x_n^2 f(x_{n-1})$, J. Differ. Eqs. Appl. **9** (6), 599-602, 2003.

[18] V. A. Krechmar, *A Problem Book in Algebra*, Mir Publishers, Moscow, 1974.

[19] S. F. Lacroix, *Traité des Differénces et des Séries,* J. B. M. Duprat, Paris, 1800. (in French)

[20] S. F. Lacroix, *An Elementary Treatise on the Differential and Integral Calculus, with an Appendix and Notes by J. Herschel*, J. Smith, Cambridge, 1816.

[21] J.-L. Lagrange, Sur l'intégration d'une équation différentielle à différences finies, qui contient la théorie des suites récurrentes, *Miscellanea Taurinensia,* t. I, (1759), 33-42 (Lagrange OEuvres, I, 23-36, 1867).

[22] P. S. Laplace, Recherches sur l'intégration des équations différentielles aux différences finies et sur leur usage dans la théorie des hasards, *Mémoires de l' Académie Royale des Sciences de Paris* 1773, t. VII, (1776) (Laplace OEuvres, VIII, 69-197, 1891). (in French)

[23] H. Levy and F. Lessman, *Finite Difference Equations*, The Macmillan Company, New York, NY, USA, 1961.

[24] A. A. Markoff, *Differenzenrechnung*, Teubner, Leipzig, 1896.

[25] L. M. Milne-Thomson, *The Calculus of Finite Differences*, MacMillan and Co., London, 1933.

[26] D. S. Mitrinović and D. D. Adamović, *Nizovi i Redovi/Sequences and Series*, Naučna Knjiga, Beograd, Serbia, 1980.

[27] D. S. Mitrinović, J. D. Kečkić, *Metodi Izračunavanja Konačnih Zbirova/Methods for Calculating Finite Sums*, Naučna Knjiga, Beograd, 1984.

[28] N. E. Nörlund, *Vorlesungen über Differenzenrechnung*, Berlin, Springer, 1924.

[29] G. Papaschinopoulos and C. J. Schinas, *On a system of two nonlinear difference equations*, J. Math. Anal. Appl. **219** (2), 415-426, 1998.

[30] G. Papaschinopoulos and C. J. Schinas, *Invariants for systems of two nonlinear difference equations*, Differ. Equ. Dyn. Syst. **7**, 181–196, 1999.

[31] G. Papaschinopoulos and C. J. Schinas, *Invariants and oscillation for systems of two nonlinear difference equations*, Nonlinear Anal. Theory Methods Appl. **46**, 967–978, 2001.

[32] G. Papaschinopoulos, C. J. Schinas and G. Stefanidou, *On a difference equation with 3-periodic coefficient*, J. Difference Equ. Appl. **11** (15), 1281-1287, 2005.

[33] G. Papaschinopoulos, C. J. Schinas and G. Stefanidou, *On a k-order system of Lyness-type difference equations*, Adv. Difference Equ. **2007**, 31272, 13 pages, 2007.

[34] G. Papaschinopoulos and G. Stefanidou, *Trichotomy of a system of two difference equations*, J. Math. Anal. Appl. **289**, 216-230, 2004.

[35] G. Papaschinopoulos and G. Stefanidou, *Asymptotic behavior of the solutions of a class of rational difference equations*, Inter. J. Difference Equations, **5** (2), 233-249, 2010.

[36] G. Papaschinopoulos, G. Stefanidou and C. J. Schinas, *Boundedness, attractivity, and stability of a rational difference equation with two periodic coefficients*, Discrete Dyn. Nat. Soc. **2009**, 973714, 23 pages, 2009.

[37] M. H. Rhouma, *The Fibonacci sequence modulo π, chaos and some rational recursive equations*, J. Math. Anal. Appl. **310**, 506-517, 2005.

[38] A. Sanbo and E. M. Elsayed, *Some properties of the solutions of the difference equation $x_{n+1} = ax_n + \frac{bx_n x_{n-4}}{cx_{n-3}+dx_{n-4}}$*, Open J. Discret. Appl. Math. **2** (2), 31-47, 2019.

[39] C. Schinas, *Invariants for difference equations and systems of difference equations of rational form*, J. Math. Anal. Appl. **216**, 164-179, 1997.

[40] C. Schinas, *Invariants for some difference equations*, J. Math. Anal. Appl. **212**, 281-291, 1997.

[41] S. Stević, *A global convergence results with applications to periodic solutions*, Indian J. Pure Appl. Math. **33** (1), 45-53, 2002.

[42] S. Stević, *On the recursive sequence $x_{n+1} = A/\prod_{i=0}^{k} x_{n-i} + 1/\prod_{j=k+2}^{2(k+1)} x_{n-j}$*, Taiwanese J. Math. **7** (2), 249-259, 2003.

[43] S. Stević, *On the recursive sequence $x_{n+1} = \alpha_n + (x_{n-1}/x_n)$ II*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **10** (6), 911-916, 2003.

[44] S. Stević, *Asymptotic periodicity of a higher order difference equation*, Discrete Dyn. Nat. Soc. **2007**, 13737, 9 pages, 2007.

[45] S. Stević, *Boundedness character of a class of difference equations*, Nonlinear Anal. TMA, **70**, 839-848, 2009.

[46] S. Stević, *Global stability of a difference equation with maximum*, Appl. Math. Comput. **210**, 525-529, 2009.

[47] S. Stević, *Global stability of a max-type difference equation*, Appl. Math. Comput. **216**, 354-356, 2010.

[48] S. Stević, *On some periodic systems of max-type difference equations*, Appl. Math. Comput. **218**, 11483-11487, 2012.

[49] S. Stević, *Solutions of a max-type system of difference equations*, Appl. Math. Comput. **218**, 9825-9830, 2012.

[50] S. Stević, *On the system of difference equations $x_n = c_n y_{n-3}/(a_n + b_n y_{n-1} x_{n-2} y_{n-3})$, $y_n = \gamma_n x_{n-3}/(\alpha_n + \beta_n x_{n-1} y_{n-2} x_{n-3})$*, Appl. Math. Comput. **219**, 4755-4764, 2013.

[51] S. Stević, *Representation of solutions of bilinear difference equations in terms of generalized Fibonacci sequences*, Electron. J. Qual. Theory Differ. Equ. **2014** (67), 15 pages, 2014.

[52] S. Stević, *Representations of solutions to linear and bilinear difference equations and systems of bilinear difference equations*, Adv. Difference Equ. **2018**, 474, 21 pages, 2018.

[53] S. Stević, J. Diblik, B. Iričanin and Z. Šmarda, *On a third-order system of difference equations with variable coefficients*, Abstr. Appl. Anal. **2012**, 508523, 22 pages, 2012.

[54] S. Stević, J. Diblik, B. Iričanin and Z. Šmarda, *On a solvable system of rational difference equations*, J. Difference Equ. Appl. **20** (5-6), 811-825, 2014.

[55] S. Stević, J. Diblik, B. Iričanin and Z. Šmarda, *Solvability of nonlinear difference equations of fourth order*, Electron. J. Differential Equations, **2014**, 264, 14 pages, 2014.

[56] S. Stević, B. Iričanin, W. Kosmala and Z. Šmarda, *Note on the bilinear difference equation with a delay*, Math. Methods Appl. Sci. **41**, 9349-9360, 2018.

[57] S. Stević, B. Iričanin, W. Kosmala and Z. Šmarda, *On a solvable class of nonlinear difference equations of fourth order*, Electron. J. Qual. Theory Differ. Equ. **2022**, 37, 17 pages, 2022.

[58] S. Stević, B. Iričanin and Z. Šmarda, *On a product-type system of difference equations of second order solvable in closed form*, J. Inequal. Appl. **2015**, 327, 15 pages, 2015.

[59] S. Stević, B. Iričanin and Z. Šmarda, *Solvability of a close to symmetric system of difference equations*, Electron. J. Differential Equations, **2016**, 159, 13 pages, 2016.

[60] S. Stević, B. Iričanin and Z. Šmarda, *On a symmetric bilinear system of difference equations*, Appl. Math. Lett. **89**, 15-21, 2019.

[61] N. N. Vorobiev, *Fibonacci Numbers*, Birkhäuser, Basel, 2002.

[62] V. A. Zorich, *Mathematical Analysis I*, Springer, Berlin, Heidelberg, 2004.