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Research Article

ON ρ −STATISTICAL CONVERGENCE OF ORDER (α,β) FOR SEQUENCES OF FUZZY NUMBERS

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Abstract: *In this study, we first give the definition of ρ −statistical convergence of order (α,β) for sequences of fuzzy numbers. We also define the strongly $w(\rho,F,q)$ −summable of order (α,β) and the strongly $w(\rho,F,q,f)$ −summable of order (α,β), defined by a modulus function f for sequences of fuzzy numbers. Later we give some inclusion theorems between these sets and the set $S_\alpha^\beta(\rho,F)$.*

Keywords*: Modulus function, Sequence of fuzzy numbers, Statistical convergence.*

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1. Introduction

The concept of statistical convergence was independently defined by Fast [1] and Steinhaus [2] in 1951. Schoenberg [3] redefined the concept of statistical convergence and provided some of its properties. Subsequently, statistical convergence has been used by many researchers in statistical measurement theory, summability theory, Banach spaces, trigonometric series, and fuzzy set theory. Several researchers, including ([4]- [7]) , have conducted studies on this concept.

Zadeh [8] first introduced fuzzy set theory. Matloka [9] provided the definition of fuzzy number sequences and defined the concepts of boundedness and convergence of sequences of fuzzy numbers, along with some properties. He showed that many properties valid for real number sequences also hold for sequences of fuzzy numbers. Since then, numerous studies have been conducted and continue to be conducted on sequences of fuzzy numbers.

The concept of statistical convergence for sequences of fuzzy numbers was introduced by Nuray & Savaş [10]. Nuray & Savaş [10] and Kwon [11] examined the relationship between statistical convergence, convergence, lacunary statistical convergence, and strong Cesàro convergence for sequences of fuzzy numbers.

The degree of statistical convergence of a sequence was provided by Gadjiev & Orhan [12]. Subsequently, the idea of statistical convergence of order α and strong p-Cesàro summability of order α was proposed by Çolak [13]. Statistical convergence of order (α, β) was first defined by Şengül [14]. Altınok & Et [15] introduced the definition of statistical convergence of order (β,γ) for sequences of fuzzy numbers.

Çakallı [16] defined ρ-statistical convergence in real number sequences. Subsequently, several researchers, including ([17]- [22]) , have conducted studies on this topic. The aim of this study is to generalize Aral's work [17] on real number sequences to sequences of fuzzy number.

2. Definitions and Preliminaries

In this section, we have discussed the fundamental concepts that we will use throughout this study.

A fuzzy number is a fuzzy set that maps from the real numbers $\mathbb{R}$ to the closed interval [0,1], satisfying the following properties:

(i) Z is normal, which means there exists $z_0 \in \mathbb{R}$ such that $Z(z_0) = 1$.

(ii) Z is fuzzy convex, which means for $z, t \in \mathbb{R}$ and $0 \leq \beta \leq 1$, we have $Z(\beta z + (1-\beta)t) \geq \min[Z(z), Z(t)]$.

(iii) Z is upper semi-continuous.

(iv) The support of Z, denoted by suppZ, is defined as the closure of the set {z∈R:Z(z)>0}, which is a compact set.

An α-level set of a fuzzy number, denoted as $[Z]^\alpha$, is defined as follows:

$$[Z]^\alpha = \begin{cases} \{z \in \mathbb{R}: Z(z) \geq \alpha\}, & if\ \alpha \in (0,1], \\ \quad\quad\quad \text{supp}Z, & if\ \alpha = 0. \end{cases}$$

For a number Z to be a fuzzy number, the necessary and sufficient condition is that for each $\alpha \in [0,1]$, the set $[Z]^\alpha$ is a closed interval, and $[Z]^1 \neq \emptyset$ is obvious. We will denote the space of all fuzzy numbers with real terms as $L(\mathbb{R})$.

The distance between fuzzy numbers Z and T is calculated using the metric:

$$d(Z,T) = \text{sup}_{0 \leq \alpha \leq 1} d_H([Z]^\alpha, [T]^\alpha)$$

where d_H is the Hausdorff metric and for $Z^\alpha = [\underline{Z}^\alpha, \bar{Z}^\alpha]$ and $T^\alpha = [\underline{T}^\alpha, \bar{T}^\alpha]$, it is defined as:

$$d_H([Z]^\alpha, [T]^\alpha) = \max\{|\underline{Z}^\alpha - \underline{T}^\alpha|, |\bar{Z}^\alpha - \bar{T}^\alpha|\}.$$

The distance d is a metric on $L(\mathbb{R})$ and it is complete.

A sequence $Z = (Z_k)$ of fuzzy number is a function Z from the set $\mathbb{N}$ of all natural numbers into $L(\mathbb{R})$ that is $Z: \mathbb{N} \to L(\mathbb{R})$ [9]. In this case, each term of the sequence (Z_k) corresponds to a fuzzy number.

A sequence $Z = (Z_k)$ of fuzzy numbers is said to be statistically convergent to a fuzzy number Z_0 if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n: d(Z_k, Z_0) \geq \varepsilon\}| = 0.$$

where the vertical bars indicate the number of elements in the enclosed set. We denote the set of all statistically convergent sequences of fuzzy numbers as $S(F)$.

The (α, β) natural density of a subset E, which is a subset of the set of natural numbers $\mathbb{N}$, is defined as follows:

$$\delta_\alpha^\beta(E) = \lim_n \frac{1}{n^\alpha} |\{k \leq n: k \in E\}|^\beta.$$

Here, the expression $|\{k \leq n: k \in E\}|^\beta$ represents the β power of the number of elements in E that are not greater than n.

Let $Z = (Z_k)$ be a sequence of fuzzy numbers and $0 < \alpha \leq \beta \leq 1$. If for every $\varepsilon > 0$, there exists a fuzzy number Z_0 such that

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \leq n: d(Z_k, Z_0) \geq \varepsilon\}|^\beta = 0$$

the sequence $Z = (Z_k)$ is said to be statistically convergent of order (α, β) to the fuzzy number Z_0.

Let $Z = (Z_k)$ be a sequence of points in the fuzzy number set $L(\mathbb{R})$ and $0 < \alpha \leq 1$. If for every $\varepsilon > 0$, there exists a fuzzy number Z_0 such that

$$\lim_{n\to\infty} \frac{1}{\rho_n^\alpha} |\{k \leq n: d(Z_k, Z_0) \geq \varepsilon\}| = 0$$

the sequence $Z=(Z_k)$ is said to be ρ −statistically convergent of order α to the fuzzy number Z_0. Here, $\rho=(\rho_n)$ is an non-decreasing sequence of positive real numbers that approaches to ∞, satisfying $\limsup_{n\to\infty}\frac{\rho_n^\alpha}{n}<\infty$, $\Delta\rho_n^\alpha=O(1)$ and $\Delta Z_n^\alpha=Z_{n+1}^\alpha-Z_n^\alpha$ for every positive integer n. In this case, it is denoted as $S_\rho^\alpha(F)-\lim Z_k=Z_0$.

If $\rho=(\rho_n)=n$ and $\alpha=1$, there is no difference between ρ −statistical convergence of order α and statistical convergence.

Throughout this study, let $\rho=(\rho_n)$ be a sequence as given above.

The concept of modulus function was first introduced by Nakano [23]. If a function $f:[0,\infty)\to[0,\infty)$ satisfies the following properties:

(i) $f(x)=0$ if and only if $x=0$,

(ii) for every $x,y\geq 0$, $f(x+y)\leq f(x)+f(y)$,

(iii) f is right-continuous at , $x=0$,

(iv) f is increasing, then f is called a modulus function.

Let (p_k) be a positive and bounded real number sequence with $\sup_k p_k=N$. Let $K=max(1,2^{N-1})$ and $a_k,b_k\in\mathbb{C}$. The inequality

$$|a_k+b_k|^{p_k}\leq K(|a_k|^{p_k}+|b_k|^{p_k}) \tag{1}$$

given by Maddox [24] will be used throughout this study.

3. Main Results

In this section, we first introduced definition of ρ −statistically convergent of order (α,β) definition for sequences of fuzzy numbers. We also defined the sets of strongly $w_\alpha^\beta(\rho,F,q)$ −summable sequences and strongly $w_\alpha^\beta(\rho,F,q,f)$ −summable sequences with respect to the modulus function f. Furthermore, we provided some inclusion theorems between these sets and the set $S_\alpha^\beta(\rho,F)$.

Definition 3.1. Let $Z=(Z_k)$ be a sequence of fuzzy numbers and $0<\alpha\leq\beta\leq 1$. If there exists a fuzzy number Z_0 such that

$$\lim_{n\to\infty}\frac{1}{\rho_n^\alpha}|\{k\leq n: d(Z_k,Z_0)\geq\varepsilon\}|^\beta=0$$

then the sequence $Z=(Z_k)$ is said to be ρ −statistically convergent of order (α,β) to Z_0 (or $S_\alpha^\beta(\rho,F)$ −convergent to Z_0, where ρ_n^α denotes the αth power of ρ_n , that is, $\rho^n=(\rho_n^\alpha)=(\rho_1^\alpha,\rho_2^\alpha,\dots\rho_n^\alpha,\dots)$. In this case, we write $S_\alpha^\beta(\rho,F)-\lim Z_k=Z_0$. $S_\alpha^\beta(\rho,F)$ will denote the set of all ρ −statistically convergent of order (α,β) for sequences of fuzzy numbers

Definition 3.2. Let $Z=(Z_k)$ be a sequence of fuzzy numbers, $0<\alpha\leq\beta\leq 1$ and q is a positive real number. If there exists a fuzzy number Z_0 such that

$$\lim_{n\to\infty}\frac{1}{\rho_n^\alpha}\left(\sum_{k=1}^{n}[d(Z_k,Z_0)]^q\right)^\beta=0$$

then the sequence $Z=(Z_k)$ is said to be strongly $w(\rho,F,q)$ −summable of order (α,β) to Z_0 (or strongly $w_\alpha^\beta(\rho,F,q)$ −summable). In this case, we write $w_\alpha^\beta(\rho,F,q)-\lim Z_k=Z_0$. $w_\alpha^\beta(\rho,F,q)$ will denote the set of all strongly $w(\rho,F,q)$ −summable of order (α,β) for sequences of fuzzy numbers When $Z_0=\bar{0}$, we use $w^{\beta}_{\alpha,0}(\rho,F,q)$ instead of $w^{\beta}_{\alpha}(\rho,F,q)$.

Definition 3.3. Let f be a modulus function, $q=(q_k)$ be a sequence of positive real numbers, and $0<\alpha\le\beta\le 1$. If there exists a fuzzy number Z_0 such that

$$\lim_{n\to\infty}\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}\right)^{\beta}=0$$

then the sequence $Z=(Z_k)$ is said to be strongly $w(\rho,F,q,f)$ −summable of order (α,β) to Z_0 (or strongly $w^{\beta}_{\alpha}(\rho,F,q,f)$ −summable). In this case, we write $w^{\beta}_{\alpha}(\rho,F,q,f)-\lim Z_k=Z_0$. $w^{\beta}_{\alpha}(\rho,F,q,f)$ will denote the set of all strongly $w(\rho,F,q,f)$ −summable of order (α,β) for sequences of fuzzy numbers. When $Z_0=\bar{0}$, we use $w^{\beta}_{\alpha,0}(\rho,F,q,f)$ instead of $w^{\beta}_{\alpha}(\rho,F,q,f)$.

In the following theorems, we will assume that $q=(q_k)$ is a bounded sequence with $0<r=\inf_k q_k\le q_k\le\sup_k q_k=R<\infty$.

Theorem 3.1. Let $Z=(Z_k)$ be a sequence of fuzzy numbers, f be a modulus function and $\alpha_1,\alpha_2,\beta_1,\beta_2\in(0,1]$ be real numbers such that $0<\alpha_1\le\alpha_2\le\beta_1\le\beta_2\le 1$. In this case, we have $w^{\beta_2}_{\alpha_1}(\rho,F,q,f)\subset S^{\beta_1}_{\alpha_2}(\rho,F)$.

Proof. Let $Z\in w^{\beta_2}_{\alpha_1}(\rho,F,q,f)$ and $\varepsilon>0$ be given. Let the sums $\sum_1$ ve $\sum_2$ represent the sums over $k\le n$, where $d(Z_k,Z_0)\ge\varepsilon$ and $d(Z_k,Z_0)<\varepsilon$ respectively. Since for every n, we have $\rho_n^{\alpha_1}\le\rho_n^{\alpha_2}$,

$$\begin{aligned}\frac{1}{\rho_n^{\alpha_1}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}\right)^{\beta_2}&=\frac{1}{\rho_n^{\alpha_1}}\left(\sum_{1}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}+\sum_{2}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}\right)^{\beta_2}\\&\ge\frac{1}{\rho_n^{\alpha_2}}\left(\sum_{1}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}+\sum_{2}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}\right)^{\beta_2}\\&\ge\frac{1}{\rho_n^{\alpha_2}}\left(\sum_{1}[f(\varepsilon)]^{q_k}\right)^{\beta_2}\\&\ge\frac{1}{\rho_n^{\alpha_2}}\left(\sum_{1}\min\left([f(\varepsilon)]^{r},[f(\varepsilon)]^{R}\right)\right)^{\beta_2}\\&\ge\frac{1}{\rho_n^{\alpha_2}}\left|\{k\le n: d(Z_k,Z_0)\ge\varepsilon\}\right|^{\beta_1}[\min([f(\varepsilon)]^{r},[f(\varepsilon)]^{R}])^{\beta_1}.\end{aligned}$$

Therefore $Z\in S^{\beta_1}_{\alpha_2}(\rho,F)$.

Theorem 3.2. Let $Z=(Z_k)$ be a sequence of fuzzy numbers, f be a bounded modulus function and $\lim_{n\to\infty}\frac{\rho_n^{\beta_2}}{\rho_n^{\alpha_1}}=1$. Then, $S^{\beta_2}_{\alpha_1}(\rho,F)\subset w^{\beta_1}_{\alpha_2}(\rho,F,q,f)$.

Proof. Let $Z\in S^{\beta_2}_{\alpha_1}(\rho,F)$, f is a bounded modulus function. Since f is bounded, there exists an integer M such that $f(x)\le M$ for all x. For any $\varepsilon>0$, we can write:

$$
\begin{aligned}
\frac{1}{\rho_n^{\alpha_2}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}\right)^{\beta_1} &\le \frac{1}{\rho_n^{\alpha_1}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}\right)^{\beta_1} \\
&= \frac{1}{\rho_n^{\alpha_1}}\left(\sum_{1}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}+\sum_{2}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}\right)^{\beta_1} \\
&\le \frac{1}{\rho_n^{\alpha_2}}\left(\sum_{1}\max\left(M^r,M^R\right)+\sum_{2}[f(\varepsilon)]^{q_k}\right)^{\beta_1} \\
&\le \left(\max\left(M^r,M^R\right)\right)^{\beta_2}\frac{1}{\rho_n^{\alpha_1}}\left|\{k\le n: d(Z_k,Z_0)\ge\varepsilon\}\right|^{\beta_2} \\
&\quad+\frac{\rho_n^{\beta_2}}{\rho_n^{\alpha_1}}\left(\max(f(\varepsilon)^R,(f(\varepsilon)^R)\right)^{\beta_2}.
\end{aligned}
$$

Thus, we can conclude that $Z\in w_{\alpha_2}^{\beta_1}(\rho,F,q,f)$.

Theorem 3.3. Let $Z=(Z_k)$ be a sequence of fuzzy numbers, f be a modulus function, $0<\alpha\le\beta\le 1$ and $\lim\inf q_k>0$. If the sequence $Z=(Z_k)$ is convergent to a fuzzy number Z_0, then the sequence $Z=(Z_k)$ is strongly $w(\rho,F,q,f)-$summable of order (α,β).

Proof. Let $Z_k\to Z_0$. Since f is a modulus function, $f\left(d(Z_k,Z_0)\right)\to\bar{0}$. As $\lim\inf q_k>0$, $\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}\to\bar{0}$. Thus, we have $w_\alpha^\beta(\rho,F,q,f)-\lim Z_k=Z_0$. This is what we wanted to prove.

Theorem 3.4. Let $Z=(Z_k)$ be a sequence of fuzzy numbers, f be a modulus function, $0<\alpha=\beta\le 1$, $q>1$ and $\lim\inf_{u\to\infty}\frac{f(u)}{u}>0$. Then $w_\alpha^\beta(\rho,F,q,f)\subset w_\alpha^\beta(\rho,F,q)$.

Proof. When $\lim\inf_{u\to\infty}\frac{f(u)}{u}>0$ for $u>0$, it means that there exists a positive number c such that $f(u)>cu$ holds $u>0$. Let $Z\in w_\alpha^\beta(\rho,F,q,f)$. Therefore, we have

$$
\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_k,Z_0)\right)\right]^{q}\right)^{\beta}\ge\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}[cd(Z_k,Z_0)]^{q}\right)^{\beta}=\frac{c^{q\beta}}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}[d(Z_k,Z_0)]^{q}\right)^{\beta}.
$$

Consequently, we have obtained $w_\alpha^\beta(\rho,F,q,f)\subset w_\alpha^\beta(\rho,F,q)$.

Theorem 3.5. Let $Z=(Z_k)$ be a sequence of fuzzy numbers, f be a modulus function and $\lim q_k>0$. If the sequence $Z=(Z_k)$ is strongly $w_\alpha^\beta(\rho,F,q,f)-$summable to the fuzzy number Z_0, then the limit is unique.

Proof. Let $w_\alpha^\beta(\rho,F,q,f)-\lim Z_k=Z_0$, $w_\alpha^\beta(\rho,F,q,f)-\lim Z_k=Z_0'$ and $\lim q_k=t>0$. In this case, we can obtain the following:

$$
\lim_{n\to\infty}\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}\right)^{\beta}=0
$$

and

$$\lim_{n\to\infty}\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_k,Z_0')\right)\right]^{q_k}\right)^{\beta}=0.$$

Thus, from the definition of f and (1), for $\sup_k q_k = K$, $0<\alpha\le\beta\le 1$ ve $N=\max(1,2^{K-1})$, we can write:

$$\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_0,Z_0')\right)\right]^{q_k}\right)^{\beta}\le\frac{N}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}+\sum_{k=1}^{n}\left[f\left(d(Z_k,Z_0')\right)\right]^{q_k}\right)^{\beta}$$
$$\le\frac{N}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}\right)^{\beta}+\frac{N}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_k,Z_0')\right)\right]^{q_k}\right)^{\beta}.$$

Therefore, we have:

$$\frac{1}{\rho_n^{\alpha}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_0,Z_0')\right)\right]^{q_k}\right)^{\beta}=0.$$

Since $\lim q_k = t$, we can conclude that $Z_0-Z_0'=0$. Thus, the limit is unique.

Theorem 3.6. Let $Z=(Z_k)$ be a sequence of fuzzy numbers, f be a modulus function and $0<\alpha_1\le\alpha_2\le\beta_1\le\beta_2\le 1$. Let $\rho=(\rho_n)$ and $\tau=(\tau_n)$ be two sequences such that $\rho_n\le\tau_n$ for every $n\in\mathbb{N}$. In this case:

(i) If $\lim\inf_{n\to\infty}\frac{\rho_n^{\alpha_1}}{\tau_n^{\alpha_2}}>0$, then $w_{\alpha_2}^{\beta_2}(\tau,F,q,f)\subset w_{\alpha_1}^{\beta_1}(\rho,F,q,f)$. (2)

(ii) If $\lim\sup_{n\to\infty}\frac{\rho_n^{\alpha_1}}{\tau_n^{\alpha_2}}<\infty$, then $w_{\alpha_1}^{\beta_2}(\rho,F,q,f)\subset w_{\alpha_2}^{\beta_1}(\tau,F,q,f)$. (3)

Proof. (i) Let $Z=(Z_k)\in w_{\alpha_2}^{\beta_2}(\tau,F,q,f)$ be a sequence of fuzzy numbers satisfying (2). In this case:

$$\frac{1}{\tau_n^{\alpha_2}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}\right)^{\beta_2}\ge\frac{\rho_n^{\alpha_1}}{\tau_n^{\alpha_2}}\frac{1}{\rho_n^{\alpha_1}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}\right)^{\beta_1}$$

Thus, if $Z\in w_{\alpha_2}^{\beta_2}(\tau,F,q,f)$, then $Z\in w_{\alpha_1}^{\beta_1}(\rho,F,q,f)$.

(ii) Let $Z=(Z_k)\in w_{\alpha_1}^{\beta_2}(\rho,F,q,f)$ and (3) holds. In this case, since $\rho_n\le\tau_n$ for every $n\in\mathbb{N}$:

$$\frac{1}{\tau_n^{\alpha_2}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}\right)^{\beta_1}\le\frac{1}{\tau_n^{\alpha_2}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}\right)^{\beta_2}=\frac{\rho_n^{\alpha_1}}{\tau_n^{\alpha_2}}\frac{1}{\rho_n^{\alpha_1}}\left(\sum_{k=1}^{n}\left[f\left(d(Z_k,Z_0)\right)\right]^{q_k}\right)^{\beta_2}.$$

Therefore, $w_{\alpha_1}^{\beta_2}(\rho,F,q,f)\subset w_{\alpha_2}^{\beta_1}(\tau,F,q,f)$.

Ethical statement:

The author declares that this document does not require ethics committee approval or any special permission. Our study does not cause any harm to the environment.

Conflict of interest:

The author declares no potential conflicts of interest related to this article's research, authorship, and publication.

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