http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 2, Pages 569–587 (2024)
DOI:10.31801/cfsuasmas.1327826
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received:July 15, 2023; Accepted: February 24, 2024

A SECOND-ORDER NUMERICAL METHOD FOR PSEUDO-PARABOLIC EQUATIONS HAVING BOTH LAYER BEHAVIOR AND DELAY PARAMETER

Baransel GUNES[1] and Hakkı DURU[2]

[1,2]Department of Mathematics, Faculty of Science, Van Yuzuncu Yil University, Van, TÜRKİYE

Abstract. In this paper, singularly perturbed pseudo-parabolic initial-boundary value problems with time-delay parameter are considered by numerically. Initially, the asymptotic properties of the analytical solution are investigated. Then, a discretization with exponential coefficient is suggested on a uniform mesh. The error approximations and uniform convergence of the presented method are estimated in the discrete energy norm. Finally, some numerical experiments are given to clarify the theory.

1. Introduction

Singularly perturbed problems are defined by a small parameter ε multiplying the highest order derivative term in the differential equation. The solutions of them typically include the boundary or interior layers depending on the situation of the problem. Because of the existence of the layers, the solution shows a multiscale character, i.e., the solution behaves stable and slowly away from the layer region while it behaves unstable and rapidly in the layer region. Therefore, the conventional numerical approaches do not produce the reliable results and ε-uniform computational techniques are required [19, 24, 35, 37, 43, 45, 47, 51] (see, also the references therein). To examine singular perturbation problems and their applications more comprehensively, one may refer in [19, 24, 35, 37, 43, 45, 47, 51].

Intercalarily, many mathematical models of real life situations in science are explained with the singularly perturbed delay differential equations (SPDDEs). Their

2020 *Mathematics Subject Classification.* 35B25, 65M06, 65M12, 65M15.

Keywords. Delay differential equation, difference scheme, error estimate, pseudo-parabolic problem, singular perturbation, uniform convergence.

1 baranselgunes23@gmail.com-Corresponding author; 0000-0002-3265-8881;

2 hakkiduru@gmail.com; 0000-0002-3179-3758.

applications can be found in processes for metal plates, spread of HIV and bacterial infections, control theory, population dynamics, neurobiology, thermo elasticity, hydrodynamics of liquid helium, mechanic systems, laser optics and financial mathematics [10, 20, 31, 38, 57, 61] (see, also the references therein). In the literature, SPDDEs have been investigated widely by many authors and different numerical methods have been introduced. These include: Reproducing kernel method [16,28,29], initial value technique [59], numerical integration method [27,50,57], Numerov method [15], the method of hybrid difference schemes [13, 14], discontinuous Galerkin method [64], collocation methods [39, 60, 63], Ritz-Galerkin method [34], $hp-$finite element method [46], fitted mesh technique [33, 42], domain decomposition approach [56], cubic spline methods [36], finite difference methods [5,7,9,22,53] and so on [11, 25, 44, 48, 49, 54].

In this paper, we consider the singularly perturbed linear initial-boundary value pseudo-parapolic problem with time-delay on the domain $\bar{D} = \bar{\Omega} \times [0,T]$; $\bar{\Omega} = [0,l]$, $\Omega = (0,l)$, $D = \Omega \times (0,T]$:

$$Lu \equiv L_1\left[\frac{\partial u}{\partial t}\right] + L_2 u + c\,(t)\,u\,(x,t-r) = f\,(x,t)\,, \quad (x,t) \in D, \tag{1}$$

$$u\,(x,t) = \varphi\,(x,t)\,, \quad (x,t) \in \bar{\Omega} \times [-r,0]\,, \tag{2}$$

$$u(0,t) = u(l,t) = 0,\ t \in (0,T]\,, \tag{3}$$

where

$$\begin{aligned} L_1\left[\frac{\partial u}{\partial t}\right] &= -\varepsilon\frac{\partial^3 u}{\partial x^2 \partial t} + a\,(x)\,\frac{\partial u}{\partial t}, \\ L_2\,[u(x,t)] &= -\varepsilon\frac{\partial^2 u}{\partial x^2} + b\,(x,t)\,u\,(x,t)\,, \end{aligned}$$

and $0 < \varepsilon \ll 1$ is the perturbation parameter; the functions a, b, c, f and φ are sufficiently smooth, $r > 0$ is delay parameter and $a(x) \geq \alpha > 0$. The problem (1)-(3) have been studied on Boglaev-type adaptive mesh by conducting linear basis functions and energy inequalities in [32]. Also, G. Amiraliyev and Y. Mamedov [4] have proposed an exponentially difference scheme for solving the problem (1)-(3) without delay parameter.

Pseudo-parabolic or Sobolev type problems have had an important role in the literature. For scientific background and existence-uniqueness results of pseudo-parabolic problems without singular perturbation and the delay parameter, one may refer in [55, 58]. I. Amirali et. al [1] have constructed two-level difference scheme for semilinear pseudo-parabolic initial-boundary value problems with delay parameter (Please, see also a series of the papers [2, 3, 8]). C. Zhang and Z. Tan [65] have used linearized compact finite difference methods for solving nonlinear delay Sobolev partial differential equations. On the other hand, latterly, various numerical schemes have been proposed for parabolic type problems with singular perturbation case. L. Govindarao and J. Mohapatra [30] have suggested a numerical scheme comprised of implicit-trapezoidal scheme on temporal direction and hybrid type scheme on spatial direction for solving singularly perturbed delay parabolic initial-boundary value problems. In the paper [6], a fully discrete scheme has been generated on Shishkin mesh to solve singularly perturbed Sobolev initial-boundary value problem with initial jump. S. Kumar and M. Kumar [40] have discretized singularly perturbed nonlinear delay parabolic type partial differential equations on a generalized Shishkin mesh by using quasilinearization techniques. M. M. Woldaregay et. al [61] have developed a numerical approach by using Crank-Nicolson technique for temporal discretization and exponentially fitted difference scheme for spatial discretization to analyse parabolic convection-diffusion problems with layer behavior. N. A. Mbroh et. al [41] have designed a numerical discretization using fitted operator finite difference method on spatially direction and Crank Nicolson finite difference approach on time direction. S. Yadav and P. Rai [62] have constructed a higher-order difference method consisting of hybrid scheme on Shishkin mesh and implicit Euler method on a uniform mesh to examine singularly perturbed delay parabolic turning point problems of convection-diffusion type. Authors in [10,12] have provided the standard finite difference scheme on piecewise uniform fitted mesh to analyze singularly perturbed delay parabolic initial-boundary value problems. L Govindarao et. al [31] have established a fourth-order numerical scheme on Shishkin-type mesh by using Richardson extrapolation to examine singularly perturbed delay parabolic reaction-diffusion problems. A. B. Chiyaneh and H. Duru [17,18] have formulated difference schemes to resolve singularly perturbed Sobolev initial-boundary value problems with time-delay parameter. S. Elango et. al [23] have provided finite difference scheme on the rectangular piecewise uniform mesh by using trapezoidal rule for solving singularly perturbed partial delay differential equations with integral boundary condition. F. W. Gelu and G. F. Duressa [26] have suggested B-spline collocation technique on Shishkin mesh to obtain a numerical approximation of singularly perturbed delay parabolic problems of reaction-diffusion type. In [21], singularly perturbed Sobolev type initial-boundary value problems with Robin boundary condition have been discretized on a uniform mesh.

Our focus in this study is to present a robust and stable finite difference scheme on a uniform mesh for solving problem (1)-(3). With in this mind, we use the interpolating quadrature rules and exponential basis functions (see [4]).

The rest of this paper is as follows: In Section 2, some priori estimates for the continuous problem are given. The finite difference scheme is constructed on a uniform mesh in Section 3. Section 4 presents the stability and convergence analysis of the proposed scheme in the discrete energy norm. Two numerical examples are solved and the computed results are tabulated in Section 5. Lastly, the paper ends with a brief conclusion.

2. A Priori Bounds

In this section, we give the asymptotic behavior of the analytical solution and its derivatives.

Lemma 1. *The solution $u(x,t)$ of the problem (1)-(3) satisfies that*

$$\varepsilon\left\|\frac{\partial u}{\partial x}\right\|^2+\alpha\|u\|^2\leq\left\{\left[\varepsilon\left\|\frac{\partial\varphi(x,0)}{\partial x}\right\|^2+\|\varphi(x,0)\|^2\right]e^{Ct}\right.$$

$$\left.+\int_0^t c^*\|\varphi(x,s)\|^2e^{Ct}ds+\int_0^t\|f\|^2e^{Ct}ds\right\}$$

where $\|.\|=\|.\|_{L_2(0,l)}$, C is a generic positive constant and $c^*=\max\limits_{t\in[0,T]}|c(t)|$.

Proof. The proof of the lemma can be found in the paper [32]. □

Lemma 2. *Under the assumptions $a\in C^2[0,l]$, $b\in C^2\left(\bar{D}\right)$, $f\in C\left(\bar{D}\right)$ and*

$$|a(0)-b(0,t)|\leq C\varepsilon,\quad |a(l)-b(l,t)|\leq C\varepsilon, \tag{4}$$

asymptotic expansion of the solution of the problem (1)-(3) can be written in the form

$$\begin{aligned}u(x,t)&=u_0(x,t)+\vartheta_0(\xi,t)+w_0(\eta,t)\\&+\sqrt{\varepsilon}\left[u_1(x,t)+\vartheta_1(\xi,t)+w_1(\eta,t)\right]+R^*(x,t),\end{aligned}\tag{5}$$

where the functions $u_0(x,t)$, $u_1(x,t)$, $\vartheta_0(\xi,t)$, $w_0(\eta,t)$, $\vartheta_1(\xi,t)$, $w_1(\eta,t)$ are the solutions of the following problems:

$$\begin{cases}a(x)\frac{\partial u_0}{\partial t}+b(x,t)u_0+c(t)u_0(x,t-r)=f(x,t),\\ u_0(x,t-r)=\varphi(x),\ -r\leq t\leq 0;\end{cases}$$

$$\begin{cases}a(x)\frac{\partial u_1}{\partial t}+b(x,t)u_1+c(t)u_1(x,t-r)=-\sqrt{\varepsilon}\left[\frac{\partial^3u_0}{\partial t\partial x^2}+\frac{\partial^2u_0}{\partial x^2}\right],\\ u_1(x,t)=0,-r\leq t\leq 0;\end{cases}$$

$$\begin{cases}-\varepsilon\frac{\partial^3\vartheta_0}{\partial t\partial\xi^2}+a(0)\frac{\partial\vartheta_0}{\partial t}-\varepsilon\frac{\partial^2\vartheta_0}{\partial\xi^2}+a(0)\vartheta_0+c(t)\vartheta_0(x,t-r)=0,\\ \vartheta_0(\xi,t)=0,\ -r\leq t\leq 0;\\ \vartheta_0(0,t)=-u_0(0,t);\ \vartheta_0(\frac{l}{\sqrt{\varepsilon}},t)=0,\end{cases}$$

$$\begin{cases}-\varepsilon\frac{\partial^3\vartheta_1}{\partial t\partial\xi^2}+a(0)\frac{\partial\vartheta_1}{\partial t}-\varepsilon\frac{\partial^2\vartheta_1}{\partial\xi^2}+a(0)\vartheta_1+c(t)\vartheta_1(x,t-r)\\ =-\xi\frac{\partial b}{\partial x}(0,t)\vartheta_0-\xi a'(0)\frac{\partial\vartheta_0}{\partial t},\\ \vartheta_1(\xi,t)=0,\quad -r\leq t\leq 0;\\ \vartheta_1(0,t)=-u_1(0,t);\ \vartheta_1(\frac{l}{\sqrt{\varepsilon}},t)=0,\end{cases}$$

$$\begin{cases}-\varepsilon\frac{\partial^3w_0}{\partial t\partial\eta^2}+a(l)\frac{\partial w_0}{\partial t}-\varepsilon\frac{\partial^2w_0}{\partial\eta^2}+a(l)w_0+c(t)w_0(x,t-r)=0,\\ w_0(\eta,t)=0,\quad -r\leq t\leq 0;\\ w_0(\frac{l}{\sqrt{\varepsilon}},t)=0;w_0(0,t)=-u_0(l,t),\end{cases}$$

$$\begin{cases} -\varepsilon\frac{\partial^3 w_1}{\partial t\partial \eta^2}+a(l)\frac{\partial w_1}{\partial t}-\varepsilon\frac{\partial^2 w_1}{\partial \eta^2}+a(l)w_1+c(t)w_1\left(x,t-r\right) \\ =-\eta\frac{\partial b}{\partial x}(l,t)\omega_0-\eta a'(l)\frac{\partial^2 w_0}{\partial t^2}, \\ w_1(\eta,t)=0, \quad -r\le t\le 0; \\ w_1(\frac{l}{\sqrt{\varepsilon}},t)=0\,;\ w_1(0,t)=-u_1\left(l,t\right), \end{cases}$$

where $\xi=\frac{x}{\sqrt{\varepsilon}}$ *and* $\eta=\frac{l-x}{\sqrt{\varepsilon}}$. *Additionally, the remainder term of the asymptotic expansion can be estimated as*

$$\varepsilon^s\left\|\frac{\partial^{k+s}R^*}{\partial t^k\partial x^s}\right\|\le C\varepsilon^{1-s/2}\quad k,\ s=0,1,2.$$

Proof. The proof can be shown by using a similar approach of [4,17,18,21,32]. □

Lemma 3. *Under the conditions of Lemma (2), using*

$$\left|\frac{\partial^{k+s}\vartheta_0}{\partial t^k\partial x^s}\right|\le C\varepsilon^{-s/2}e^{-x\sqrt{a(0)/\varepsilon}}$$

and

$$\left|\frac{\partial^{k+s}w_0}{\partial t^k\partial x^s}\right|\le C\varepsilon^{-s/2}e^{-(l-x)\sqrt{a(l)/\varepsilon}},$$

we have the following bound:

$$\left|\frac{\partial^{k+s}u}{\partial t^k\partial x^s}\right|\le C\left\{1+\varepsilon^{-s/2}\left[e^{-x\sqrt{a(0)/\varepsilon}}+e^{-(l-x)\sqrt{a(l)/\varepsilon}}\right]\right\}, \tag{6}$$

$$(x,t)\in\bar{D},\ k=0,\ 1,\ 2;\ s=0,\ 1,\ 2.$$

Proof. The proof of the lemma is similar to those of [4,17,18,21,32]. □

3. Spatial and Temporal Discretization

In this section, we propose the discretization for the problem (1)-(3). Let $\omega_{h\tau}=\omega_h\times\omega_\tau$ denote the mesh on D:

$$\omega_h=\{x_i=ih,\ i=1,2,...,N-1,\ h=l/N\}$$
$$\omega_\tau=\{t_j=j\tau,\ j=1,2,...,M;\tau=T/M\}$$

and

$$\bar{\omega}_h=\omega_h\cup\{x_0=0,x_N=l\}\,,\ \bar{\omega}_\tau=\omega_\tau\cup\{t=0\}\,.$$

For any mesh function $v(x)$ described on $\bar{\omega}_h$, we use the difference formulas in [52]:

$$v_i=v(x_i),\ v_{\bar{x},i}=\frac{v_i-v_{i-1}}{h},$$
$$v_{x,i}=\frac{v_{i+1}-v_i}{h},\ v_{\bar{x}x,i}=\frac{v_{i+1}-2v_i+v_{i-1}}{h^2}.$$

Also, for a function $w\equiv w_i^j\equiv w(x_i,t_j)$ defined on $\bar{\omega}_\tau$, we need (see [52])

$$w_{\bar{t},i}^j=\frac{w_i^j-w_i^{j-1}}{\tau}.$$

Now, we begin to establish the difference scheme according to the space variable. To formulate the difference method, the following integral identity is used:

$$h^{-1}\int_{x_{i-1}}^{x_{i+1}} L_1\left[\frac{\partial u}{\partial t}\right]\varphi_i(x)\,dx+h^{-1}\int_{x_{i-1}}^{x_{i+1}} L_2[u]\,\varphi_i(x)\,dx$$
$$+h^{-1}\int_{x_{i-1}}^{x_{i+1}} c(t)\,u(x,t-r)\,\varphi_i(x)\,dx=h^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x,t)\varphi_i(x)\,dx \qquad (7)$$

where the exponential basis function

$$\varphi_i(x)=\begin{cases}\varphi_i^{(1)}(x)\equiv\frac{\sinh\gamma_{i-0,5}(x-x_{i-1})}{\sinh\gamma_{i-0,5}h},\ x\in[x_{i-1},x_i],\\ \varphi_i^{(2)}(x)\equiv\frac{\sinh\gamma_{i+0,5}(x_{i+1}-x)}{\sinh\gamma_{i+0,5}h},\quad x\in[x_i,x_{i+1}],\\ 0,\quad x\notin(x_{i-1},x_{i+1}).\end{cases}$$

Also $\gamma_i=\sqrt{a_i/\varepsilon}$ $(i=1,2,...,N-1)$, and $a(x_{i\pm 0,5})=a\left(x_i\pm\frac{h}{2}\right)$. The functions $\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$ are the solutions of the following problems, respectively:

$$\begin{cases}-\varepsilon\varphi_i^{(1)''}(x)+a_{i-0.5}\varphi^{(1)}(x)=0,\quad x_{i-1}<x<x_i,\\ \varphi^{(1)}(x_{i-1})=0,\ \varphi^{(1)}(x_i)=1,\end{cases}$$

$$\begin{cases}-\varepsilon\varphi_i^{(2)''}(x)+a_{i+0.5}\varphi^{(2)}(x)=0,\ x_i<x<x_{i+1},\\ \varphi^{(2)}(x_i)=1,\quad \varphi^{(2)}(x_{i+1})=0.\end{cases}$$

For the first term of the equality (7), applying interpolating quadrature rules in [4] and some processes in [17], it is found that

$$h^{-1}\int_{x_{i-1}}^{x_{i+1}} L_1\left[\frac{\partial u}{\partial t}\right]\varphi_i(x)\,dx=h^{-1}\int_{x_{i-1}}^{x_{i+1}}\left\{-\varepsilon\frac{\partial^3 u}{\partial x^2\partial t}+a(x)\frac{\partial u}{\partial t}\right\}\varphi_i(x)\,dx$$
$$=h^{-1}\int_{x_{i-1}}^{x_i}\left\{-\varepsilon\frac{\partial^3 u}{\partial x^2\partial t}+a_{i-0.5}\frac{\partial u}{\partial t}\right\}\varphi_i^{(1)}(x)\,dx$$
$$+h^{-1}\int_{x_i}^{x_{i+1}}\left\{-\varepsilon\frac{\partial^3 u}{\partial x^2\partial t}+a_{i+0.5}\frac{\partial u}{\partial t}\right\}\varphi_i^{(2)}(x)\,dx$$
$$+h^{-1}\int_{x_{i-1}}^{x_i}[a(x)-a_{i-0.5}]\frac{\partial u}{\partial t}\varphi_i^{(1)}(x)\,dx+h^{-1}\int_{x_i}^{x_{i+1}}[a(x)-a_{i+0.5}]\frac{\partial u}{\partial t}\varphi_i^{(2)}(x)\,dx.$$

Then, we get

$$h^{-1}\int_{x_{i-1}}^{x_{i+1}} L_1\left[\frac{\partial u}{\partial t}\right]\varphi_i(x)\,dx=-\varepsilon\left(\theta_0\left[\frac{\partial u}{\partial t}\right]_{\bar{x}}\right)_{x,i}+A\theta_1\left(\frac{\partial u}{\partial t}\right)_i+R_{1,i}^*(t) \qquad (8)$$

where

$$A=\frac{1}{2}(a_{i-0.5}+a_{i+0.5}),$$
$$R_{1,i}^*(t)=h^{-1}\int_{x_{i-1}}^{x_i}[a(x)-a_{i-0.5}]\frac{\partial u}{\partial t}\varphi_i^{(1)}(x)\,dx+h^{-1}\int_{x_i}^{x_{i+1}}[a(x)-a_{i+0.5}]\frac{\partial u}{\partial t}\varphi_i^{(2)}(x)\,dx$$

$$+\left[\left(a_{i-0.5}-A\right)\theta_1^{(1)}+\left(a_{i+0.5}-A\right)\theta_1^{(2)}\right]\left(\frac{\partial u}{\partial t}\right)$$

and

$$\begin{aligned}\theta_0 &\equiv \left(\theta_0\right)_i=1+\varepsilon^{-1}a_{i-0.5}\int_{x_{i-1}}^{x_i}(x-x_i)\varphi_i^{(1)}(x)\,dx\\ &= \frac{\rho\sqrt{a_{i-0.5}}}{\sinh\left(\rho\sqrt{a_{i-0.5}}\right)},\ \left(\rho=h/\sqrt{\varepsilon}\right),\end{aligned}$$

$$\theta_1^{(1)}=h^{-1}\int_{x_{i-1}}^{x_i}\varphi_i^{(1)}(x)\,dx=\frac{1}{\rho\sqrt{a_{i-0.5}}}\tanh\frac{\rho\sqrt{a_{i-0.5}}}{2},$$

$$\theta_1^{(2)}=h^{-1}\int_{x_i}^{x_{i+1}}\varphi_i^{(2)}(x)\,dx=\frac{1}{\rho\sqrt{a_{i+0.5}}}\tanh\frac{\rho\sqrt{a_{i+0.5}}}{2},$$

$$\theta_{1,i}=\theta_{1,i}^{(1)}+\theta_{1,i}^{(2)},\quad A_i=\frac{1}{2}\left(a\left(x_{i-0.5}\right)+a\left(x_{i+0.5}\right)\right).$$

For the second term of the equation (7), we obtain

$$\begin{aligned}h^{-1}\int_{x_{i-1}}^{x_{i+1}}L_2[u]\,\varphi_i(x)\,dx &= h^{-1}\int_{x_{i-1}}^{x_{i+1}}\left(-\varepsilon\frac{\partial^2 u}{\partial x^2}+b(x,t)\,u(x,t)\right)\varphi_i(x)\,dx\\ &= -\varepsilon\left(\theta_0 u_{\bar{x}}\right)_{x,i}+B\theta_1 u\left(x_i,t\right)+\theta_1 R_{2,i}^*(t)\end{aligned}\tag{9}$$

where

$$B=\frac{1}{2}\left[b(x_{i-0.5},t)+b(x_{i+0.5},t)\right]$$

and

$$R_{2,i}^*(t)=\theta_1^{-1}\left[h^{-1}\int_{x_{i-1}}^{x_{i+1}}L_2[u]\,\varphi_i(x)\,dx+\left(\varepsilon\left(\theta_0 u_{\bar{x}}\right)_{x,i}-B\theta_1 u\left(x_i,t\right)\right)\right].$$

For the third term of the equation (7), it is found that

$$\begin{aligned}h^{-1}\int_{x_{i-1}}^{x_{i+1}}c(t)\,u(x,t-r)\,\varphi_i(x)\,dx &= h^{-1}c(t)\int_{x_{i-1}}^{x_{i+1}}u(x,t-r)\,\varphi_i(x)\,dx\\ &= c(t)\,u\left(x_i,t-r\right)h^{-1}\int_{x_{i-1}}^{x_{i+1}}\varphi_i(x)\,dx+R_{3,i}^*(t)\\ &= \theta_1 c(t)\,u\left(x_i,t-r\right)+R_{3,i}^*(t)\end{aligned}\tag{10}$$

where

$$R_{3,i}^*(t)=\theta_1^{-1}h^{-1}c(t)\int_{x_{i-1}}^{x_{i+1}}\left[u(x,t-r)-u(x_i,t-r)\right]\varphi_i(x)\,dx.$$

The term of the right side of the equation (7), we can write:

$$h^{-1}\int_{x_{i-1}}^{x_{i+1}}f(x,t)\varphi_i(x)\,dx=\theta_1 F_i-\theta_1 R_{4,i}^*(t)\tag{11}$$

where

$$R_{4,i}^{*}\left(t\right)=-h^{-1}\theta_1^{-1}\int_{x_{i-1}}^{x_{i+1}}\left[f(x,t)-F\left(x_i,t\right)\right]\varphi_i\left(x\right)dx$$

and

$$F\left(x_i,t\right)=\frac{1}{2}\left[f(x_{i-0.5},t)+f(x_{i+0.5},t)\right].$$

By combining (8),(9),(10) and (11), we have

$$-\varepsilon\left(\theta_0\left[\frac{\partial u}{\partial t}\right]_{\bar{x}}\right)_{x,i}+A\theta_1\left(\frac{\partial u}{\partial t}\right)-\varepsilon\left(\theta_0 u_{\bar{x}}\right)_{x,i}+B\theta_1 u\left(x_i,t\right)+\theta_1 c(t)u\left(x_i,t-r\right)+\theta_1 R_i^{*}=\theta_1 F_i$$

where

$$R_i^{*}=R_{1,i}^{*}\left(t\right)+R_{2,i}^{*}\left(t\right)+R_{3,i}^{*}\left(t\right)+R_{4,i}^{*}\left(t\right).$$

Then, to obtain the discretization for the time variable, we consider the integral equality in the form

$$\tau^{-1}\int_{t_{j-1}}^{t_j}\left[Lu-f\left(x_i,t\right)\right]dt=\tau^{-1}\int_{t_{j-1}}^{t_j}\left\{\varepsilon\left[\theta_0\left(\frac{\partial u}{\partial t}\right)_{\bar{x}}\right]_{x,i}+A\theta_1\left(\frac{\partial u}{\partial t}\right)_i-\varepsilon\left(\theta_0 u_{\bar{x}}\right)_{x,i}\right.$$
$$\left.+B\theta_1 u(x_i,t)+\theta_1 c(t)u(x_i,t-r)-\theta_1 F_i+\theta_1 R_i^{*}\right\}dt \qquad (12)$$

Applying the interpolating quadrature rules [4] to first two terms of the equation (12), we have

$$\tau^{-1}\int_{t_{j-1}}^{t_j}\left[-\varepsilon\left(\theta_0\left(\frac{\partial u}{\partial t}\right)_{\bar{x}}\right)_{x,i}+A\theta_1\left(\frac{\partial u}{\partial t}\right)_i\right]dt=-\varepsilon\left(\theta_0 u_{\bar{t}\bar{x}}\right)_x+\theta_1 A u_{\bar{t}}.$$

For the third and fourth terms of the equation (12), it is obtained that

$$\tau^{-1}\int_{t_{j-1}}^{t_j}\left[-\varepsilon\left(\theta_0 u_{\bar{x}}\right)_x+\theta_1 B_i u_i\right]dt=-\varepsilon\left(\theta_0 u_{\bar{x}}^{(\sigma)}\right)_x+\theta_1 B_i^j u_i^{(\sigma)}+\varepsilon\left(\theta_0 R^{(0)}\right)_{\bar{x}}+R_1^{(1)}$$

where

$$R^{(0)}=u^{(\sigma)}\left(x_i,t_j\right)-\tau^{-1}\int_{t_{j-1}}^{t_j}u\left(x_i,\eta\right)d\eta$$

and

$$R_1^{(1)}=\theta_1 B_i^j\left(\tau^{-1}\int_{t_{j-1}}^{t_j}u(x_i,t)dt-u_i^{(\sigma)}\right).$$

For the term involving the delay parameter, rewriting $\tau=T/M$ and $rM/T=M_0$, we have

$$\theta_1\tau^{-1}\int_{t_{j-1}}^{t_j}c(t)u\left(x_i,t-r\right)dt=\theta_1 c^j u_i^{j-M_0}+R_c^j$$

where

$$R_c^j=\theta_1\left\{\tau^{-1}\int_{t_{j-1}}^{t_j}\left[c(t)-c^j\right]u\left(x_i,t-r\right)dt\right.$$

$$+\tau^{-1}\int_{t_{j-1}}^{t_j} c^j\left(u\left(x_i,t-r\right)-u\left(x_i,t_j-r\right)\right)dt\Bigg\}.$$

For the term of the right side of the equation (12), we find

$$\tau^{-1}\int_{t_{j-1}}^{t_j}\theta_1 F(x_i,t)dt = \theta_1 F_i^j + R_f,$$

where

$$R_f = \tau^{-1}\int_{t_{j-1}}^{t_j}\theta_1 F(x_i,t)dt - \theta_1 F_i^j.$$

Thus, we can suggest the following difference scheme

$$\ell u_i^j := \ell_1\left(u_{\bar{t},i}^j\right) + \ell_2\left(u_i^j\right) + \theta_1 c^j u_i^{j-M_0} + R_i^j = \theta_1 F_i^j. \tag{13}$$

where

$$\ell_1\left(u_{\bar{t},i}^j\right) = -\varepsilon\left(\theta_0 u_{\bar{t}\bar{x}}^j\right)_{x,i} + \theta_1 A u_{\bar{t}}^j,$$

$$\ell_2\left(u_i^j\right) = -\varepsilon\left(\theta_0 u_{\bar{x}}^{(\sigma)}\right)_{x,i} + \theta_1 B_i^j u_i^{(\sigma)},$$

and the remainder term is denoted by

$$R_i^j = \varepsilon\left(\theta_0 R^{(0)}\right)_x + \theta_1 R^{(1)} + \theta_1 R_{c,j}$$

where

$$R^{(1)} = \tau^{-1}\int_{t_{j-1}}^{t_j} R_i^*(t)dt + R_1^{(1)} - R_f.$$

By omitting the remainder term R_i^j in (13), we can write for the approximate solution

$$-\varepsilon\left(\theta_0 y_{\bar{t}\bar{x}}^j\right)_{x,i} + \theta_1 A y_{\bar{t}}^j - \varepsilon\left(\theta_0 y_{\bar{x}}^{(\sigma)}\right)_{x,i} + \theta_1 B_i^j y_i^{(\sigma)} + \theta_1 c^j y_i^{j-M_0} = \theta_1 F_i^j, \tag{14}$$

$$y\left(x_i,t_j\right) = \varphi\left(x_i,t_j\right),\ -M_0 \le j \le 0,\ 0 \le i \le N, \tag{15}$$

$$y_0^j = y_N^j = 0. \tag{16}$$

4. Error Bounds

Let u_i^j be the solution of the problem (1)-(3) and let y_i^j be the solution of the problem (14)-(16). Then, the error function $z_i^j = y_i^j - u_i^j$ be the solution of the following discrete problem:

$$\ell_1\left(z_{\bar{t},i}^j\right) + \ell_2\left(z_i^j\right) + \theta_1 c^j z_i^{j-M_0} = R_i^j. \tag{17}$$

$$z\left(x_i,t_j\right) = 0,\quad 0 \le i \le N,\quad -M_0 \le j \le 0, \tag{18}$$

$$z_0^j = z_N^j = 0,\ t \in \bar{\omega}_\tau, \tag{19}$$

where

$$\ell_1\left(z_{\bar{t},i}^j\right) = -\varepsilon\left(\theta_0 z_{\bar{t}\bar{x}}^j\right)_{x,i} + \theta_1 A z_{\bar{t}}^j$$

and

$$\ell_2\left(z_i^j\right) = -\varepsilon\left(\theta_0 z_{\bar{x}}^{(\sigma)}\right)_{x,i} + \theta_1 B_i^j z_i^{(\sigma)}.$$

Lemma 4. *Under the conditions* $1+\sigma\tau > 0$ *and* $A + \sigma\tau B > 0$, *the following estimate is satisfied:*

$$\|z\|^2 \leq C\tau\sum_{k=1}^{j}\left\|\theta_1^{-1}R\right\|_{-*}^2.$$

where

$$\left\|\theta_1^{-1}R\right\|_{-*}^2 = (\theta_1)^{-1}\sup_v\frac{|(R,v)|^2}{(\theta_0 v_{\bar{x}}, v_{\bar{x}}) + (\theta_1 v, v)}$$
$$\leq (\theta_1)^{-1}\left\{\varepsilon\left(\theta_0 R^{(0)}, R^{(0)}\right) + \left(\theta_1 R^{(1)}, R^{(1)}\right) + (\theta_1 R_c, R_c)\right\}.$$

Proof. To carry out the error analysis, we use a similar approach in [4,17,32]. First of all, we consider the following equation for the discrete problem (17)-(19):

$$(\ell z, z) = (R, z).$$

From here, we get

$$\left(-\varepsilon\left(\theta_0 z_{\bar{t}\bar{x}}\right)_{x,i}, z\right) = \frac{\varepsilon}{2}\left(\theta_0 z_{\bar{t}\bar{x},i}, z_{\bar{x},i}\right) = \frac{\varepsilon}{2}\left(\theta_0 z_{\bar{x},i}, z_{\bar{x},i}\right)_{\bar{t}} + \frac{\varepsilon\tau}{2}\left(\theta_0 z_{\bar{t}\bar{x},i}, z_{\bar{t}\bar{x},i}\right)$$

and

$$\left(\theta_1 A z_{\bar{t}}, z\right) = \frac{1}{2}\left(A\theta_1 z, z\right)_{\bar{t}} + \frac{\tau}{2}\left(A\theta_1 z_{\bar{t}}, z_{\bar{t}}\right).$$

Then, to obtain the bound for the term $z^{(\sigma)}(x_i, t_j)$, we take the following equality into account:

$$z^{(\sigma)} = \sigma z + (1-\sigma)\check{z} = \sigma z(x_i, t_j) + (1-\sigma) z(x_i, t_{j-1}) \tag{20}$$

Hence, we acquire as

$$\begin{aligned}\left(-\varepsilon\left(\theta_0 z_{\bar{x}}^{(\sigma)}\right)_{x,i}, z\right) &= \varepsilon\left(\theta_0 \check{z}_{\bar{x}}, \check{z}_{\bar{x}}\right) + \varepsilon\tau\left(\theta_0 \check{z}_{\bar{x}}, z_{\bar{t}\bar{x}}\right)\\ &\quad + \frac{\varepsilon\sigma\tau}{2}\left(\theta_0 z_{\bar{x}}, z_{\bar{x}}\right)_{\bar{t}} + \frac{\varepsilon\sigma\tau^2}{2}\left(\theta_0 z_{\bar{t}\bar{x}}, z_{\bar{t}\bar{x}}\right)\end{aligned}$$

and

$$\begin{aligned}\left(\theta_1 B z^{(\sigma)}, z\right) &= \left(\theta_1 B\check{z}, \check{z}\right) + \tau\left(\theta_1 B\check{z}, z_{\bar{t}}\right)\\ &\quad + \frac{\sigma\tau}{2}\left(\theta_1 B z, z\right)_{\bar{t}} + \frac{\sigma\tau^2}{2}\left(\theta_1 B z_{\bar{t}}, z_{\bar{t}}\right).\end{aligned}$$

Furthermore, we can write the following estimates:

$$\left|\left(\theta_1 c^j z_i^{j-M_0}, z\right)\right| \leq c^* \mu_1 \left(\theta_1 z_i^{j-M_0}, z_i^{j-M_0}\right) + \frac{c^*}{4\mu_1} \left(\theta_1 z, z\right),$$

$$\left|\left(-\varepsilon \left(\theta_0 R^{(0)}\right)_x, z\right)\right| \leq \varepsilon \left|\left(\theta_0 R^{(0)}, z_{\bar{x}}\right)\right| \leq \varepsilon \mu_2 \left|\left(\theta_0 z_{\bar{x}}, z_{\bar{x}}\right)\right| + \frac{\varepsilon}{4\mu_2} \left(\theta_0 R^{(0)}, R^{(0)}\right),$$

$$\left|\left(\theta_1 R^{(1)}, z\right)\right| \leq \mu_3 \left|\left(\theta_1 z, z\right)\right| + \frac{1}{4\mu_3} \left(\theta_1 R^{(1)}, R^{(1)}\right)$$

and

$$\left|\left(\theta_1 R_c, z\right)\right| \leq \mu_4 \left|\left(\theta_1 R_c, R_c\right)\right| + \frac{1}{4\mu_4} \left(\theta_1 z, z\right).$$

Taking $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \frac{1}{2}$ and merging these results, we obtain

$$\begin{aligned}
&\frac{1}{2} \left\{\varepsilon \left(1 + \sigma\tau\right) \left(\theta_0 z_{\bar{x},i}, z_{\bar{x},i}\right) + \left(A + \sigma\tau B\right) \left(\theta_1 z, z\right)\right\}_{\bar{t}} \\
&+\frac{\tau}{2} \left\{\varepsilon \left(1 + \sigma\tau\right) \left(\theta_0 z_{\bar{t}\bar{x},i}, z_{\bar{t}\bar{x},i}\right) + \left(A + \sigma\tau B\right) \left(\theta_1 z_{\bar{t}}, z_{\bar{t}}\right)\right\} \\
&+\varepsilon \left(\theta_0 \check{z}_{\bar{x}}, \check{z}_{\bar{x}}\right) + \left(\theta_1 B \check{z}, \check{z}\right) + \varepsilon\tau \left(\theta_0 \check{z}_{\bar{x}}, z_{\bar{t}\bar{x}}\right) + \tau \left(\theta_1 B \check{z}, z_{\bar{t}}\right) \\
&\leq \frac{c^*}{2} \left(\theta_1 z_i^{j-M_0}, z_i^{j-M_0}\right) + \frac{c^*}{2} \left(\theta_1 z, z\right) + \frac{\varepsilon}{2} \left(\theta_0 z_{\bar{x}}, z_{\bar{x}}\right) \\
&+\frac{\varepsilon}{2} \left(\theta_0 R^{(0)}, R^{(0)}\right) + \left(\theta_1 z, z\right) + \frac{1}{2} \left(\theta_1 R^{(1)}, R^{(1)}\right) + \frac{1}{2} \left(\theta_1 R_c, R_c\right)
\end{aligned} \tag{21}$$

which concludes the proof of the lemma. □

Lemma 5. *Under the conditions of the Lemma (3) and rewriting $\sigma = 0.5$ in the relation (20), the remainder term R holds the following estimate:*

$$\|R\| \leq C \left(h + \tau^2\right).$$

Proof. The proof of the lemma is similar manner as in [4, 17]. □

5. Numerical Results

In this section, the numerical method is tested on two examples to validate the theory. To determine the reliability of the numerical approximation, we take into consideration the elimination method in [52]. Firstly, the difference equation (14) can be written explicitly:

$$\begin{aligned}
&-\varepsilon h^{-2} \tau^{-1} \left[\theta_{0,i+1} \left(y_{i+1}^j - y_{i+1}^{j-1} - y_i^j + y_i^{j-1}\right) - \theta_{0,i} \left(y_i^j - y_i^{j-1} - y_{i-1}^j + y_{i-1}^{j-1}\right)\right] \\
&+\tau^{-1} A \theta_1 \left(y_i^j - y_i^{j-1}\right) - \varepsilon h^{-2} \left[\theta_{0,i+1} \left(y_{i+1}^{(\sigma)} - y_i^{(\sigma)} - y_i^{(\sigma)} + y_{i-1}^{(\sigma)}\right)\right] \\
&+\theta_{1,i} B y_i^{(\sigma)} + \theta_1 c^j y_i^{j-M_0} = \theta_{1,i} F_i.
\end{aligned} \tag{22}$$

Secondly, we adapt the relation (22) according to the following difference equality:

$$D_i^* y_{i-1}^j - E_i^* y_i^j + G_i^* y_{i+1}^j = -H_i^*, \; i = 2,3,...,N-1, \quad j = 2,3,...,M-1.$$

Here, to express the term $y_i^{(\sigma)}$ in (22), we use

$$y^{(\sigma)} = \sigma y + (1-\sigma)\check{y} = \sigma y(x_i, t_j) + (1-\sigma) y(x_i, t_{j-1}). \tag{23}$$

Substituting $\sigma = \frac{1}{2}$ in the equation (23), we obtain

$$D_i^* = -\varepsilon h^{-2}\left(\theta_{0,i}\tau^{-1} - \frac{\theta_{0,i+1}}{2}\right), \; G_i^* = -\varepsilon h^{-2}\theta_{0,i+1}\left(\tau^{-1} + \frac{1}{2}\right),$$

$$E_i^* = -\varepsilon h^{-2}\theta_{0,i+1}\left(\tau^{-1}+1\right) + \theta_{0,i+1}\tau^{-1} - \theta_{1,i}\left(A_i\tau^{-1} + \frac{B_i}{2}\right)$$

and

$$\begin{aligned} H_i^* = -\theta_{1,i}F_i + \theta_{1,i}c^j y_i^{j-M_0} + y_{i-1}^{j-1}\left(-\varepsilon h^{-2}\left(\frac{\theta_{0,i+1}}{2} + \theta_{0,i}\tau^{-1}\right)\right) \\ + y_{i+1}^{j-1}\left(-\varepsilon h^{-2}\theta_{0,i+1}\left(\frac{1}{2} + \tau^{-1}\right)\right) \\ + y_i^{j-1}\left(\theta_{1,i}\frac{B_i}{2} - \varepsilon h^{-2}\left(\theta_{0,i+1} + \tau^{-1}(1+\theta_{0,i})\right) - \theta_{0,i+1}A_i\tau^{-1}\right). \end{aligned}$$

Thirdly, the coefficients of the elimination method [52] are indicated as

$$\alpha_{i+1} = \frac{G_i^*}{E_i^* - D_i^*\alpha_i}, \quad \beta_{i+1} = \frac{H_i^* + D_i^*\beta_i}{E_i^* - D_i^*\alpha_i},$$

and the output of the computational approach is calculated by

$$y_i^j = \alpha_{i+1}y_{i+1}^j + \beta_{i+1}, \quad i = 1,...,N-1.$$

In numerical calculations, we use the double-mesh approach [19,24]. The approximate errors and ε-uniform maximum pointwise errors are noted as

$$e_\varepsilon^N = \max_{\omega_h\times\omega_\tau}\left|y_i^{\varepsilon,N} - y_i^{\varepsilon,2N}\right|$$

and

$$e^N = \max_\varepsilon e_\varepsilon^N.$$

Additionally, the order of convergence and ε-uniform rate of convergence are calculated as

$$p_\varepsilon^N = \frac{\ln(e_\varepsilon^N/e_\varepsilon^{2N})}{\ln 2}, \; p^N = \frac{\ln(e^N/e^{2N})}{\ln 2}.$$

Example 1. *Consider the following singularly perturbed pseudo-parabolic initial-boundary value problem:*

$$\begin{aligned} -\varepsilon\frac{\partial^3 u}{\partial t\partial x^2} + (x^2(1-x)+1)\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + (3 + t\sin(\pi x t))u + (1+t^2)u(x,t-r) \\ = e^{-t}\sin t(x + \sin(\pi x)), \quad (x,t) \in (0,l)\times(0,T] \end{aligned},$$

subject to the conditions

$$u(x,t) = \varphi(x,t) = e^{-t}\sin(\pi x),\ (x,t) \in \bar{\Omega} \times [-r,0],$$
$$u(0,t) = u(1,t) = 0,\ t \in (0,T],$$

where $l = 1$, $r = 1$ and $T = 2$. The computed results are shown in Table 1.

Example 2. *Take into account the second test problem:*

$$-\varepsilon\frac{\partial^3 u}{\partial t \partial x^2} + (1 + \frac{x}{2}(1-x))\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + (3 + t\cos(\pi x t))u + (1 + \frac{t^2}{2})u(x,t-r)$$
$$= e^{-t}\sin t(x - \cos(\pi x)),\ (x,t) \in (0,l) \times (0,T] \quad ,$$

with

$$u(x,t) = \varphi(x,t) = e^{-t}\sin(2\pi x),\ (x,t) \in \bar{\Omega} \times [-r,0],$$
$$u(0,t) = u(1,t) = 0,\ t \in (0,T],$$

where $l = 1$, $r = 1$ and $T = 2$. The experimental results are presented in Table 2.

In Tables 1-2, the maximum nodal errors and order of convergence are presented for the values $N = 2^n$, $(n = 7, 8, ..., 11)$ and $\varepsilon = 2^{-2w}$, $(w = 1, 2, ..., 8)$. It is concluded that as the value N increases the maximum pointwise errors e^N, e^{2N} are decrease. This implies the reliability of the proposed scheme. Even though the presented numerical algorithm produce stable results, it can be further improved in terms of computational timing.

6. Discussion and Conclusion

In this paper, we have generated a new and efficient numerical scheme to solve initial-boundary value problems of singularly perturbed delay pseudo-parabolic equations. Using the energy estimates and difference analogues of integral inequalities, the error bounds and the parameter-uniform convergence of the proposed scheme have been analyzed. Two test problems have been solved and the experimental results have been reflected in Tables 1-2. From these results, it is observed that the order of convergence p^N is almost 2. In a nutshell, numerical applications agree with the theory. To improve the outlines of this study, the suggested approach can be carried out for more sophisticated problems involving higher dimensional equations, nonlinear functions, fractional derivatives.

Author Contribution Statements These authors contributed equally to this work.

Declaration of Competing Interests The authors declare no competing interests.

TABLE 1. Maximum pointwise errors e^N, e^{2N} and the order of convergence p^N on $\omega_h \times \omega_\tau$

ε				N		
		128	256	512	1024	2048
2^{-2}	e^N	0.00002343	0.00000586	0.00000146	0.00000037	0.00000009
	e^{2N}	0.00000586	0.00000146	0.00000037	0.00000009	0.00000002
	p^N	1.9997	2.0003	2.0002	2.0002	2.0000
2^{-4}	e^N	0.00009372	0.00002343	0.00000586	0.00000146	0.00000037
	e^{2N}	0.00002343	0.00000586	0.00000146	0.00000037	0.00000009
	p^N	1.9997	2.0003	2.0002	2.0001	2.0001
2^{-6}	e^N	0.00037489	0.00009374	0.00002346	0.00000586	0.00000146
	e^{2N}	0.00009374	0.00002343	0.00000586	0.00000146	0.00000037
	p^N	1.9998	2.0003	2.0002	2.0001	2.0001
2^{-8}	e^N	0.00150000	0.00037497	0.00009371	0.00002342	0.00000586
	e^{2N}	0.00037497	0.00009371	0.00002342	0.00000586	0.00000146
	p^N	2.0001	2.0004	2.0003	2.0001	2.0000
2^{-10}	e^N	0.00600672	0.00150030	0.00037488	0.00009370	0.00002342
	e^{2N}	0.00150030	0.00037488	0.00009370	0.00002342	0.00000585
	p	2.0013	2.0007	2.0003	2.0001	2.0001
2^{-12}	e^N	0.02413597	0.00600810	0.00149997	0.00037481	0.00009369
	e^{2N}	0.00600810	0.00149997	0.00037481	0.00009369	0.00002342
	p^N	2.0062	2.0019	2.0006	2.0002	2.0001
2^{-14}	e^N	0.09832477	0.02414373	0.00600685	0.00149968	0.00037477
	e^{2N}	0.02414320	0.00600685	0.00149968	0.00037477	0.00009368
	p^N	2.0259	2.0069	2.0019	2.0005	2.0001
2^{-16}	e^N	0.42375898	0.09838510	0.02413962	0.00600574	0.00149951
	e^{2N}	0.09838312	0.02413933	0.00600574	0.00149951	0.00037474
	p^N	2.1067	2.0270	2.0069	2.0018	2.0005

REFERENCES

[1] Amirali, I., Amiraliyev, G. M., Cakir, M., Cimen, E., Explicit finite difference methods for the delay pseudoparabolic equations, *The Scientific World Journal*, 2014 (2014). https://doi.org/10.1155/2014/497393

[2] Amirali, I., Analysis of higher order difference method for a pseudo-parabolic equation with delay, *Miskolc Math. Notes*, 20(2) (2019), 755-766. DOI: 10.18514/MMN.2019.2895

[3] Amirali, I., Amiraliyev, G. M., Three layer difference method for linear pseudo-parabolic equation with delay, *J. Comput. Appl. Math.*, 401 (2022), 113786. https://doi.org/10.1016/j.cam.2021.113786

[4] Amiraliyev, G. M., Mamedov, Y. D., Difference schemes on the uniform mesh for singularly perturbed pseudo-parabolic equations, *Turk. J. Math.*, 19(3) (1995), 207-222.

TABLE 2. Maximum pointwise errors e^N, e^{2N} and the order of convergence p^N on $\omega_h \times \omega_\tau$

ε				N		
		128	256	512	1024	2048
2^{-2}	e^N	0.00002862	0.00000715	0.00000179	0.00000045	0.00000011
	e^{2N}	0.00000715	0.00000179	0.00000045	0.00000011	0.00000003
	p^N	2.0013	2.0008	2.0004	2.0002	2.0001
2^{-4}	e^N	0.00011449	0.00002860	0.00000714	0.00000179	0.00000045
	e^{2N}	0.00002859	0.00000714	0.00000179	0.00000045	0.00000011
	p^N	2.0013	2.0008	2.0004	2.0002	2.0001
2^{-6}	e^N	0.00045795	0.00011438	0.00002858	0.00000714	0.00000179
	e^{2N}	0.00011438	0.00002858	0.00000714	0.00000179	0.00000045
	p^N	2.0013	2.0008	2.0004	2.0002	2.0001
2^{-8}	e^N	0.00183193	0.00045753	0.00011432	0.00002857	0.00000714
	e^{2N}	0.00045752	0.00011432	0.00002857	0.00000714	0.00000179
	p^N	2.0014	2.0008	2.0004	2.0002	2.0002
2^{-10}	e^N	0.00732988	0.00183027	0.00045727	0.00011428	0.00002857
	e^{2N}	0.00183020	0.00045727	0.00011428	0.00002857	0.00000714
	p^N	2.0017	2.0009	2.0004	2.0002	2.0001
2^{-12}	e^N	0.02935407	0.00732329	0.00182923	0.00045713	0.00011426
	e^{2N}	0.00732307	0.00182922	0.00045713	0.00011426	0.00002856
	p^N	2.0030	2.0012	2.0005	2.0002	2.0001
2^{-14}	e^N	0.11797011	0.02932844	0.00731919	0.00182867	0.00045706
	e^{2N}	0.02932840	0.00731914	0.00182866	0.00045706	0.00011425
	p^N	2.0080	2.0025	2.0008	2.0003	2.0001
2^{-16}	e^N	0.48115439	0.11789250	0.02931321	0.00731696	0.00182837
	e^{2N}	0.11789250	0.02931274	0.00731695	0.00182837	0.00045702
	p^N	2.0290	2.0078	2.0022	2.0006	2.0002

[5] Amiraliyev, G. M., Erdogan, F., Uniform numerical method for singularly perturbed delay differential equations, *Comput. Math. Appl.*, 53(8) (2007), 1251-1259. https://doi.org/10.1016/j.camwa.2006.07.009

[6] Amiraliyev, G. M., Duru, H., Amiraliyeva, I. G., A parameter-uniform numerical method for a Sobolev problem with initial layer, *Numer. Algorithms*, 44 (2007), 185-203. https://doi.org/10.1007/s11075-007-9096-0

[7] Amiraliyev, G. M., Cimen, E., Numerical method for a singularly perturbed convection-diffusion problem with delay, *Appl. Math. Comput.*, 216(8) (2010), 2351-2359. https://doi.org/10.1016/j.amc.2010.03.080

[8] Amiraliyev, G. M., Cimen, E., Amirali, I., Cakir, M., High-order finite difference technique for delay pseudo-parabolic equations, *J. Comput. Appl. Math.*, 321 (2017), 1-7. https://doi.org/10.1016/j.cam.2017.02.017

[9] Amiraliyeva, I. G., Erdogan, F., Amiraliyev, G. M., A uniform numerical method for dealing with a singularly perturbed delay initial value problem, *Appl. Math. Lett.*, 23(10) (2010), 1221-1225. https://doi.org/10.1016/j.aml.2010.06.002

[10] Ansari, A. R., Bakr, S. A., Shishkin, G. I., A parameter-robust finite difference method for singularly perturbed delay parabolic partial differential equations, *J. Comput. Appl. Math.*, 205(1) (2007), 552-566. https://doi.org/10.1016/j.cam.2006.05.032

[11] Arslan, D., A novel hybrid method for singularly perturbed delay differential equations, *Gazi Univ. J. Sci.*, 32(1) (2019), 217-223.

[12] Bansal, K., Sharma, K. K., Parameter-robust numerical scheme for time-dependent singularly perturbed reaction-diffusion problem with large delay, *Numer. Funct. Anal. Optim.*, 39(2) (2018), 127-154. https://doi.org/10.1080/01630563.2016.1277742

[13] Bawa, L. K., Lal, A. K., Kumar, V., An $\varepsilon-$ uniform hybrid scheme for singularly perturbed delay differential equations, *Appl. Math. Comput.*, 217(21) (2011), 8216-8222. https://doi.org/10.1016/j.amc.2011.02.089

[14] Cen, Z., A hybrid finite difference scheme for a class of singularly perturbed delay differential equations, *Neural Parallel Sci. Comput.*, 16(3) (2008), 303-308.

[15] Chakravarthy, P.P., Kumar, K., A novel method for singularly perturbed delay differential equations of reaction-diffusion type, *Differ. Equ. Dyn. Syst.*, 29 (2021), 723-734. https://doi.org/10.1007/s12591-017-0399-x

[16] Chen, S-B., Soradi-Zeid, S., Dutta, H., Mesrizadeh M., Johanshahi H., Chu Y-M., Reproducing kernel Hilbert space method for nonlinear second order singularly perturbed boundary value problems with time-delay, *Chaos Solit.*, 144 (2021), 110674. https://doi.org/10.1016/j.chaos.2021.110674

[17] Chiyaneh, A. B., Duru, H., Uniform difference method for singularly pertubated delay Sobolev problems, *Quaest. Math.*, 43(12) (2020), 1713-1736. https://doi.org/10.2989/16073606.2019.1653395

[18] Chiyaneh, A. B., Duru, H., A numerical scheme on S-mesh for the singularly perturbed initial boundary value Sobolev problems with large time delay, *Journal of Mathematics, Mechanics and Computer Science*, 117(1) (2023), 93-111. https://doi.org/10.26577/JMMCS.2023.v117.i1.08

[19] Doolan, E. P., Miller, J. J. H., Schilders, W.H.A., Uniform Numerical Methods for Problems with Initial and Boundary Layers, Boole Press, Dublin, 1980.

[20] Ducrot, A., Magal, P., Seydi, O., A singularly perturbed delay differential equation modeling nosomial infections, *Differ. Integral Equ.*, 29(3-4) (2016), 321-358. DOI: 10.57262/die/1455806027

[21] Duru, H., Gunes, B., The stability and convergence analysis for singularly perturbed Sobolev problems with Robin type boundary condition, *Georgian Math. J.*, 30(3) (2023), 349-363. https://doi.org/10.1515/gmj-2023-2004

[22] Erdogan, F., An exponentially fitted method for singularly perturbed delay differential equations, *Adv. Differ. Equ.*, 2009 (2009), 1-9. doi:10.1155/2009/781579

[23] Elango, S., Tamilselvan, A., Vadivel, R., Gunasekaran, N., Zhu, H., Cao, J., Li, X., Finite difference scheme for singularly perturbed reaction diffusion problem of partial delay differential equation with nonlocal boundary condition, *Adv. Differ. Equ.*, 2021 (2021), 1-20. https://doi.org/10.1186/s13662-021-03296-x

[24] Farrell, P., Hegarty, A. F., Miller, J. J. H., O'Riordan, E., Shishkin, G. I., Robust Computational Techniques for Boundary Layers, Chapman and Hall/CRC, 2000.

[25] Feng, T., Ni, M., Internal layers for a quasi-linear singularly perturbed delay differential equation, *J. Appl. Anal. Comput.*, 10(4) (2020), 1666-1682. DOI:10.11948/20190337

[26] Gelu, F. W., Duressa, G. F., A uniformly convergent collocation method for singularly perturbed delay parabolic reaction-diffusion problem, *Abstr. Appl. Anal.*, 2021, (2021), 1-11. https://doi.org/10.1155/2021/8835595

[27] Gemechis, F., Gashu, G., Tesfaye, A., Reddy, Y. N., Numerical solution of singularly perturbed delay reaction-diffusion equations with layer or oscillatory behavior, *Am. J. Numer. Anal.*, 5(1) (2017), 1-10. DOI:10.12691/ajna-5-1-1

[28] Geng, F. Z., Qian, S. P., Piecewise reproducing kernel methods for singularly perturbed delay initial value problems, *Appl. Math. Lett.*, 37 (2014), 67-71. https://doi.org/10.1016/j.aml.2014.05.014

[29] Geng, F. Z., Qian, S. P., Modified reproducing kernel method for singularly perturbed boundary value problems with a delay, *Appl. Math. Model.*, 39(18) (2015), 5592-5597. https://doi.org/10.1016/j.apm.2015.01.021

[30] Govindarao, L., Mohapatra, J., A second order numerical method for singularly perturbed delay parabolic partial differential equation, *Eng. Comput.*, 36(2) (2019), 420-444. https://doi.org/10.1108/EC-08-2018-0337

[31] Govindarao, L., Mohapatra, J., Das, A., A fourth-order numerical scheme for singularly perturbed delay parabolic problem arising in population dynamics, *J. Appl. Math. Comput.*, 63(1) (2020), 171-195. https://doi.org/10.1007/s12190-019-01313-7.

[32] Gunes, B., Duru, H., A computational method for the singularly perturbed delay pseudo-parabolic differential equations on adaptive mesh, *Int. J. Comput. Math.*, 100(8) (2023), 1667-1682. https://doi.org/10.1080/00207160.2023.2208681

[33] Kadalbajoo, M. K., Sharma, K. K., Parameter-uniform fitted mesh method for singularly perturbed delay differential equations with layer behavior, *Electron. Trans. Numer. Anal.*, 23 (2006), 180-201.

[34] Kadalbajoo, M. K., Yadaw, A. S., An $\varepsilon-$uniform Ritz-Galerkin finite element method for numerical solution of singularly perturbed delay differential equations, *Int. J. Pure Appl. Math.*, 55(2) (2009), 265-286.

[35] Kadalbajoo, M. K., Gupta, V., A brief survey on numerical methods for solving singularly perturbed problems, *Appl. Math. Comput.*, 217(8) (2010), 3641-3716. https://doi.org/10.1016/j.amc.2010.09.059

[36] Kanth, A. R., Murali M. K. P., A numerical technique for solving nonlinear singularly perturbed delay differential equations, *Math. Model. Anal.*, 23(1) (2018), 64-78. https://doi.org/10.3846/mma.2018.005

[37] Kevorkian J., Cole J. D., Perturbation Methods in Applied Mathematics, Springer, New York, 1981.

[38] Kiltu, G. G., Duressa, G. F., Aga Bullo, T., Numerical treatment of singularly perturbed delay reaction-diffusion equations, *Int. J. Eng. Sci.*, 12(1) (2020), 15-24. doi:10.4314/ijest.v12i1.2

[39] Kumar, D., Kadalbajoo, M. K., Numerical treatment of singularly perturbed delay differential equations using B-spline collocation method on Shishkin mesh, *J. Numer. Anal. Ind.*, 7(3-4) (2012), 73-90.

[40] Kumar, S., Kumar, M., High order parameter-uniform discretization for singularly perturbed parabolic partial differential equations with time delay, *Comput. Math. Appl.*, (68)(10) (2014), 1355-1367. https://doi.org/10.1016/j.camwa.2014.09.004

[41] Mbroh, N. A., Noutchie, S. C. O., Massoukou R. Y. M., A robust method of lines solution for singularly perturbed delay parabolic problem, *Alex. Eng. J.*, 59(4) (2020), 2543-2554. https://doi.org/10.1016/j.aej.2020.03.042

[42] Melesse, W. G., Tiruneh, A. A., Derese, G. A., Fitted mesh method for singularly perturbed delay differential turning point problems exhibiting twin boundary layers, *J. Appl. Math. Inform.*, 38(1-2) (2020), 113-132. https://doi.org/10.14317/jami.2020.113

[43] Miller, J. J. H., O'riordan, E., Shishkin, G. I., Fitted Numerical Methods for Singular Perturbation Problems: Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions, World Scientific, 1996.

[44] Mohapatra, J., Natesan, S., Uniform convergence analysis of finite difference scheme for singularly perturbed delay differential equation on an adaptively generated grid, *Numer. Math. Theor. Meth. Appl.*, 3(1) (2010), 1-22. doi: 10.4208/nmtma.2009.m8015
[45] Nayfeh, A. H., Introduction to Perturbation Techniques, Wiley, New York, 1993.
[46] Nicaise, S., Xenophontos, C., Robust approximation of singularly perturbed delay differential equations by the *hp* finite element method, *Comput. Methods Appl. Math.*, 13(1) (2013), 21-37. https://doi.org/10.1515/cmam-2012-0001
[47] O'Malley, R. E., Singular perturbation methods for ordinary differential equations, Springer-Verlag, New York, 1991.
[48] Podila, P. C., Gupta, T., Vigo-Aguiar, J., A numerical scheme for a weakly coupled system of singularly perturbed delay differential equations on an adaptive mesh, *Comp. and Math. Methods.*, 3(3) (2021), e1104. https://doi.org/10.1002/cmm4.1104
[49] Rai, P., Sharma, K. K., Numerical approximation for a class of singularly perturbed delay differential equations with boundary and interior layer(s), *Numer. Algorithms*, 85(1) (2020), 305-328. https://doi.org/10.1007/s11075-019-00815-6
[50] Reddy, Y. N., Soujana, G. B. S. L., Phaneendra, K., Numerical integration method for singularly perturbed delay differential equations, *Int. J. Appl. Sci.*, 10(3) (2012), 249-261. https://doi.org/10.6703/IJASE.2012.10(3).249
[51] Roos H. G., Stynes, M., Tobiska, L., Numerical Methods for Singularly Perturbed Differential Equations, Springer-Verlag, Berlin, 1996.
[52] Samarskii, A. A., The Theory of Difference Schemes, Marcel Dekker, New York, 2001.
[53] Sekar, E., Tamilselvan, A., Finite difference scheme for singularly perturbed system of delay differential equations with integral boundary conditions, *J.KSIAM*, 22(3) (2018), 201-215. http://dx.doi.org/10.12941/jksiam.2018.22.201
[54] Sharma, K. K., Rai, P., Mishra, P., A robust numerical scheme for singularly perturbed delay differential equations with turning point, *Int. J. Comput. Methods Eng. Sci. Mech.*, 20(5) (2019), 423-433. https://doi.org/10.1080/15502287.2019.1687608
[55] Showalter, R. E., The Sobolev equation II, Appl. Anal., 5(2) (1975), 81-99. https://doi.org/10.1080/00036817508839111
[56] Singh, J., Kumar, S., Kumar, M., A domain decomposition method for solving singularly perturbed parabolic reaction-diffusion problems with time delay, *Numer. Methods Partial Differ. Equ.*, 34(5) (2018), 1849-1866. https://doi.org/10.1002/num.22256
[57] Sirisha, C. L., Reddy, Y. N., Numerical integration of singularly perturbed delay differential equations using exponential integrating factor, *Math. Commun.*, 22(2) (2017), 251-264. https://hrcak.srce.hr/185984
[58] Sobolev, S. L., About new problems in mathematical physics, *Izv. Acad. Sci. USSR Math.* 18(1) (1954), 3-50.
[59] Subburayan, V., Ramanujam, N., Asymptotic initial value technique for singularly perturbed convection-diffusion delay problems with boundary and weak interior layers, *Appl. Math. Lett.*, 25(12) (2012), 2272-2278. https://doi.org/10.1016/j.aml.2012.06.016
[60] Vaid, M. K., Arora, G., Solution of second order singular perturbed delay differential equation using trigonometric B-spline, *Int. J. Math. Eng. Manag. Sci.*, 4(2) (2019), 349-360.
[61] Woldaregay, M. M., Aniley, W. T., Duressa G. F., Novel numerical scheme for singularly perturbed time delay convection-diffusion equation, *Adv. Math. Phys.*, 2021 (2021), 1-13. https://doi.org/10.1155/2021/6641236
[62] Yadav, S., Rai, P., A higher order scheme for singularly perturbed delay parabolic turning point problem, *Eng. Comput.*, 38(2) (2021), 819-851. https://doi.org/10.1108/EC-03-2020-0172
[63] Yüzbaşı, Ş., Sezer, M., Exponential collocation method for solutions of singularly perturbed delay differential equations, *Abstr. Appl. Anal.*, 2013 (2013), Article ID: 493204. https://doi.org/10.1155/2013/493204

[64] Zarin, H., On discontinuous Galerkin finite element method for singularly perturbed delay differential equations, *Appl. Math. Lett.*, 38 (2014), 27-32. https://doi.org/10.1016/j.aml.2014.06.013

[65] Zhang C., Tan, Z., Linearized compact difference methods combined with Richardson extrapolation for nonlinear delay Sobolev equations, *Commun. Nonlinear Sci. Numer. Simul.*, 91 (2020), 105461. https://doi.org/10.1016/j.cnsns.2020.105461