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RESEARCH ARTICLE

Quasi Bi-slant Submanifolds of Locally Metallic Riemannian Manifolds

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ABSTRACT

In this article we investigate quasi bi-slant submanifolds of locally metallic Riemannian manifolds. The main objective is to determine the conditions under which the distributions used in defining these submanifolds are integrable. We also establish the necessary and sufficient conditions for quasi bi-slant submanifold to be a totally geodesic foliation.

Keywords: *Metallic Riemannian manifolds, quasi bi-slant submanifolds, distribution.*
AMS Subject Classification (2020): *Secondary: 53B20; 53B25; 53C15; 53C421.*

1. Introduction

V. W. de Spinadel [25] introduced the metallic mean family, a generalization of the golden mean, which consists of positive solutions to quadratic equation of the form $x^2 = px + q$, where p and q are positive integers. These solutions, known as (p, q)-metallic numbers denoted by $\sigma_{p,q}$, are associated with various metals such as the golden mean, silver mean, subtle mean, bronze mean, copper mean, nickel mean, and others. Well-known metallic means include the golden mean ($\sigma_{1,1} = \Phi = \frac{(1+\sqrt{5})}{2}$ for p = 1 and q = 1), the silver mean ($\sigma_{2,1} = \sigma_{Ag} = 1 + \sqrt{2}$ for p = 2 and q = 1), and the bronze mean ($\sigma_{3,1} = \sigma_{Br} = \frac{(3+\sqrt{13}}{2}$ for p = 3 and q = 1).

Crasmareanu and Hretcanu [17] introduced the concept of a metallic structure on C^∞-differentiable real manifolds, which is a specific type of polynomial structure defined by Goldberg in [16]. This metallic structure is a generalization of geometric structures such as golden, silver, bronze, subtle, copper, and nickel structures on C^∞-differentiable manifolds.

In recent years, slant submanifolds have gained importance in differential geometry. They were introduced by Chen [6, 7] as a generalization of invariant and anti-invariant submanifolds of Kaehler manifolds. Chen provided initial results and examples of slant submanifolds in his book [6]. A submanifold N of an almost Hermitian manifold, involving an almost complex structure J, is called a slant submanifold if the angle between JX_p and X_p is independent of the choice of point p in N and every non-zero tangent vector X_p. Since their introduction, slant submanifolds have attracted significant attention and have been studied in various space forms with complex, contact, and product structures ([1]-[3], [21]-[24]). Slant distributions were introduced using the concept of slanting, and Carriazo [10] defined bi-slant submanifolds of almost Hermitian manifolds based on slant distributions. More recently, Prasad et al. [23] studied quasi bi-slant submanifolds of almost Hermitian manifolds.

Motivated by the desire to understand the geometry and topology of slant submanifolds in metallic Riemannian manifolds, several research works have been conducted in this area. Various types of submanifolds, including invariant, anti-invariant, slant, semi-slant, hemi-slant, and bi-slant submanifolds, have been studied in metallic and golden Riemannian manifolds. Different integrability conditions for the distributions involved in defining these submanifolds have been obtained through these studies (see, for example, [4, 5], [8, 9], [11]-[15], [17]-[20], and the references therein). Based upon the aforementioned

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articles, we introduce the concept of quasi bi-slant submanifolds in locally metallic Riemannian manifolds, encompassing slant, semi-slant, hemi-slant, and bi-slant submanifolds as special cases.

2. Preliminaries

If a $(1,1)$-type tensor field $\tilde{\Psi}$ on a C^∞-differentiable manifold $\tilde{N}$ satisfies the equation

$$\tilde{\Psi}^2 = p\tilde{\Psi} + qI, \tag{2.1}$$

then $\tilde{\Psi}$ is referred to as a metallic structure on $\tilde{N}$, where I denotes the identity transformation. Consider a Riemannian manifold $(\tilde{N}, \tilde{g})$ and let $\tilde{\Psi}$ be a metallic structure on $\tilde{N}$, if $\tilde{\Psi}$ satisfies the following equation

$$\tilde{g}(\tilde{\Psi}X_1, Y_1) = \tilde{g}(X_1, \tilde{\Psi}Y_1), \tag{2.2}$$

then $(\tilde{N}, \tilde{g}, \tilde{\Psi})$ is said to be $\tilde{\Psi}$ compatible metallic Riemannian manifold. Furthermore if $\tilde{N}$ is locally metallic Reimannian manifold then we have

$$(\tilde{\nabla}_{X_1}\tilde{\Psi})Y_1 = 0. \tag{2.3}$$

From (2.1) and (2.2) one can easily obtain

$$\tilde{g}(\tilde{\Psi}X_1, \tilde{\Psi}Y_1) = p\tilde{g}(\tilde{\Psi}X_1, Y_1) + q\tilde{g}(X_1, Y_1). \tag{2.4}$$

for all $X_1, Y_1 \in \Gamma(T\tilde{N})$.

If N is submanifold of $\tilde{N}$ and If ∇ is the induced connection on $\tilde{N}$, then the Gauss and Weingarten formulae are given by

$$\tilde{\nabla}_{X_1}Y_1 = \nabla_{X_1}Y_1 + h(X_1, Y_1) \tag{2.5}$$

$$\tilde{\nabla}_{X_1}V_1 = -A_{V_1}X_1 + \nabla^\perp_{X_1}Y_1, \tag{2.6}$$

for all $X_1, Y_1 \in \Gamma(TN)$ and $V_1 \in \Gamma(T^\perp N)$, where A and h are shape operator and second fundamental form and $\nabla^\perp$ is the connection on the normal bundle of N. Furthermore the shape operator and second fundamental form are related by

$$\tilde{g}(A_{V_1}X_1, Y_1) = \tilde{g}(h(X_1, Y_1), V_1).$$

Let us consider for any $X_1 \in \Gamma(TN)$ and $V_1 \in \Gamma(T^\perp N)$, the decomposition of $\tilde{\Psi}(X_1)$ and $\tilde{\Psi}(V_1)$ into tangential and normal components as

$$\tilde{\Psi}(X_1) = fX_1 + \phi X_1, \tag{2.7}$$

$$\tilde{\Psi}(V_1) = BV_1 + CV_1, \tag{2.8}$$

where $fX_1 := (\tilde{\Psi}X_1)^\top$, $\phi X_1 := (\tilde{\Psi}X_1)^\perp$, $BV_1 := (\tilde{\Psi}V_1)^\top$ and $CV_1 := (\tilde{\Psi}V_1)^\perp$.

The covariant derivative of projection morphisms in foregoing equations are defined as

$$(\nabla_{X_1}f)Y_1 = \nabla_{X_1}fY_1 - f(\nabla_{X_1}Y_1), \tag{2.9}$$
$$(\tilde{\nabla}_{X_1}\phi)Y_1 = \nabla^\perp_{X_1}\phi Y_1 - \phi(\nabla_{X_1}Y_1), \tag{2.10}$$
$$\nabla_{X_1}BV_1 = \nabla_{X_1}BV_1 - B(\nabla_{X_1}Y_1), \tag{2.11}$$
$$(\tilde{\nabla}_{X_1}C)V_1 = \nabla^\perp_{X_1}CY_1 - C(\nabla^\perp_{X_1}V_1), \tag{2.12}$$

for any $X_1, Y_1 \in \Gamma(TN)$ and $V_1 \in \Gamma(T^\perp N)$.

3. Quasi bi-slant submanifolds of locally metallic Riemannian manifolds.

In this section, we introduce the concept of quasi bi-slant submanifolds in metallic Riemannian manifolds and provide the necessary and sufficient conditions for their integrability. Before delving into the study of quasi bi-slant submanifolds, we review some key results from [18]. We begin by defining slant submanifolds in metallic Riemannian manifolds:

Definition 3.1. A submanifold N in a metallic Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{\Psi})$ is called a slant submanifold if it satisfies the condition $\theta(X_x) =$ constant, where $\theta(X_x)$ is the angle between $\tilde{\Psi}X_x$ and T_xN for any $x \in N$ and $X_x \in T_xN$ with $\tilde{\Psi}X_x \neq 0$. The constant angle θ is referred to as the slant angle of N in $\tilde{N}$ and is given by

$$\cos(\theta) = \frac{\|fX_1\|}{\|\tilde{\Psi}X_1\|},$$

where fX_1 represents the orthogonal projection of $\tilde{\Psi}X_1$ onto T_xN. An immersion $i : N \to \tilde{N}$ satisfying these conditions is called a slant immersion of N in $\tilde{N}$.

We note that invariant and anti-invariant submanifolds are special cases of slant submanifolds with slant angles $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. Slant submanifolds that are neither invariant nor anti-invariant are referred to as proper slant submanifolds. The following proposition provides a key result for slant submanifolds:

Proposition 3.1. *If N is a slant submanifold with a slant angle θ that is isometrically immersed in the metallic Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{\Psi})$, then the following equations hold for any $X_1, Y_1 \in \Gamma(TN)$:*

$$\tilde{g}(fX_1, fY_1) = \cos^2\theta \tilde{g}(X_1, pfY_1 + qY_1),$$
$$\tilde{g}(\phi X_1, \phi Y_1) = \sin^2\theta \tilde{g}(X_1, pfY_1 + qY_1),$$

where $\tilde{g}$ denotes the metric on $\tilde{N}$.

Definition 3.2. [18] Let N be an immersed submanifold in a metallic Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{\Psi})$. We say that N is a bi-slant submanifold of $\tilde{N}$ if there exist two orthogonal differentiable distributions Δ_1 and Δ_2 on N satisfying the following conditions:

1. TN can be decomposed orthogonally as $TN = \Delta_1 \oplus \Delta_2$,
2. $\tilde{\Psi}(\Delta_1) \perp \Delta_2$ and $\tilde{\Psi}(\Delta_2) \perp \Delta_1$,
3. The distributions Δ_1 and Δ_2 are slant with slant angles $\theta_1 \neq \theta_2$.

We observe that if N is a bi-slant submanifold in a metallic Riemannian manifold $(\tilde{N}, \tilde{g}, \tilde{\Psi})$ with the decomposition $TN = \Delta_1 \oplus \Delta_2$ and $dim(\Delta_1) \cdot dim(\Delta_2) \neq 0$, where Δ_2 is the slant distribution with slant angle θ, then the following cases hold:

1. N is an invariant submanifold if $\theta = 0$ and Δ_1 is invariant,
2. N is an anti-invariant submanifold if $\theta = \frac{\pi}{2}$ and Δ_1 is anti-invariant,
3. N is a proper semi-invariant submanifold if Δ_1 is invariant and Δ_2 is anti-invariant. Semi-invariant submanifolds are a particular case of semi-slant submanifolds (hemi-slant submanifolds) with the slant angle $\theta = \frac{\pi}{2}$ $(\theta = 0)$, respectively.

These results set the stage for the introduction and study of quasi bi-slant submanifolds in metallic Riemannian manifolds. Quasi bi-slant submanifold are defined as a submanifolds of a Riemannian manifold that satisfies certain conditions. These conditions involve the existence of specific distributions Δ, Δ_1, and Δ_2 on N, as well as the properties of the tangent spaces and angles between subspaces.

Definition 3.3. A submanifold N of metallic Riemannian manifold $\tilde{N}$ is called quasi bi-slant submanifold if there exists distributions Δ, Δ_1 and Δ_2 such that:

1. The tangent bundle TN can be decomposed orthogonally as:

$$TN = \Delta \oplus \Delta_1 \oplus \Delta_2$$

.

2. The distribution Δ is invariant under the metallic structure $\tilde{\Psi}$, i.e., $\tilde{\Psi}(\Delta) = \Delta$.
3. The transformed distribution $\tilde{\Psi}(\Delta_1)$ is orthogonal to the distribution Δ_2, i.e., $\tilde{\Psi}(\Delta_1) \perp \Delta_2$.
4. For any non-zero vector field $X_1 \in (\Delta_1)_x$, where x is a point in N, the angle θ_1 between $\tilde{\Psi}X_1$ and $(\Delta_1)_x$ remains constant and does not depend on the specific choice of x and X_1.
5. For any non-zero vector field $Z_1 \in (\Delta_2)_y$, where y is a point in N, the angle θ_2 between $\tilde{\Psi}Z_1$ and $(\Delta)_y$ remains constant and does not depend on the specific choice of y and Z_1.

Remark 3.1. Based on the dimensions of the distributions and the values of the slant angles θ_1 and θ_2, different cases can be identified:

(i) If $\dim(\Delta) \neq 0$, $\dim(\Delta_1) = 0$, and $\dim(\Delta_2) = 0$, the submanifold N is classified as an invariant submanifold.

(ii) If $\dim(\Delta) \neq 0$, $\dim(\Delta_1) \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$, and $\dim(\Delta_2) = 0$, the submanifold N is considered a proper semi-slant submanifold.

(iii) If $\dim(\Delta) = 0$, $\dim(\Delta_1) \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$, and $\dim(\Delta_2) = 0$, the submanifold N is classified as a slant submanifold with a slant angle of θ_1.

(iv) If $\dim(\Delta) = 0$, $\dim(\Delta_1) = 0$, and $\dim(\Delta_2) \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, the submanifold N is considered a slant submanifold with a slant angle of θ_2.

(v) If $\dim(\Delta) = 0$, $\dim(\Delta_1) \neq 0$, $\theta_1 = \frac{\pi}{2}$, and $\dim(\Delta_2) = 0$, the submanifold N is classified as an anti-invariant submanifold.

(vi) If $\dim(\Delta) \neq 0$, $\dim(\Delta_1) \neq 0$, $\theta_1 = \frac{\pi}{2}$, and $\dim(\Delta_2) = 0$, the submanifold N is considered a semi-invariant submanifold.

(vii) If $\dim(\Delta) = 0$, $\dim(\Delta_1) \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$, and $\dim(\Delta_2) \neq 0$, $\theta_2 = \frac{\pi}{2}$, the submanifold N is classified as a hemi-slant submanifold.

(viii) If $\dim(\Delta) = 0$, $\dim(\Delta_1) \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$, and $\dim(\Delta_2) \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, the submanifold N is considered a bi-slant submanifold.

(ix) If $\dim(\Delta) \neq 0$ and $0 < \theta_1 = \theta_2 < \frac{\pi}{2}$, the submanifold N is classified as a proper semi-slant submanifold.

(x) If $\dim(\Delta) \neq 0$, $\dim(\Delta_1) \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$, and $\dim(\Delta_2) \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, the submanifold N is considered a proper quasi bi-slant submanifold.

Thus quasi bi-slant submanifolds are generalisation of invariant, anti-invariant, slant, semi-slant, hemi-slant and bi-slant submanifolds. In the context of a quasi bi-slant submanifold N in a metallic Riemannian manifold $\tilde{N}$, let $X_1 \in \Gamma(TN)$, and denote the projections of X_1 onto the distributions Δ, Δ_1, and Δ_2 by P, P_1, and P_2, respectively. Then, for any $X_1 \in \Gamma(TN)$, we can express this projection as follows:

$$X_1 = PX_1 + P_1X_1 + P_2X_1 \tag{3.1}$$

On applying $\tilde{\Psi}$ on both sides and using (2.7) we obtain

$$\tilde{\Psi}X_1 = fPX_1 + \phi PX_1 + fP_1X_1 + \phi P_1X_1 + fP_2X_1 + \phi P_2X_1.$$

As $\tilde{\Psi}\Delta = \Delta$, we have $\phi PX_1 = 0$, therefore we get

$$\tilde{\Psi}X_1 = fPX_1 + fP_1X_1 + \phi P_1X_1 + fP_2X_1 + \phi P_2X_1.$$

Furthermore from the foregoing equation, it is easy to verify that

$$\begin{aligned} fX_1 &= fPX_1 + fP_1X_1 + fP_2X_1, \\ \phi X_1 &= \phi P_1X_1 + \phi P_2X_1. \end{aligned}$$

This leads us to the following decomposition

$$\tilde{\Psi}(TN) \subset \Delta \oplus f\Delta_1 \oplus \phi\Delta_1 \oplus \phi\Delta_2 \oplus f\Delta_2.$$

As $\phi\Delta_1$ and $\phi\Delta_2$ are in $T^{\perp}N$, we have

$$T^{\perp}N = \phi\Delta_1 \oplus \phi\Delta_2 \oplus \nu,$$

where ν is the orthogonal complement of $\phi\Delta_1 + \phi\Delta_2$ in $T^{\perp}N$ and $\tilde{\Psi}(\nu) = \nu$. For any vector field $W_1 \in \Gamma(T^{\perp}N)$, we put

$$\tilde{\Psi}W_1 = BW_1 + CW_1, \tag{3.2}$$

where $BW_1 \in \Gamma(TN)$ and $CW_1 \in \Gamma(T^{\perp}N)$.

Lemma 3.1. *Let N be quasi bi-slant submanifold of metallic Reimannian manifold $\tilde{N}$. Then the endomorphism f and projection morphisms ϕ, T and Q, satisfy the following identities*
(i) $f^2 + B\phi = pf + qI$ on TN
(ii) $\phi f + C\phi = p\phi$ on TN
(iii) $\phi B + C^2 = pC + qI$ on $T^{\perp}N$
(iv) $fB + BC = pB$ on $T^{\perp}N$

Proof. From (2.7), (3.2) and using the fact that $\tilde{\Psi}^2 = p\tilde{\Psi} + qI$, then comparing tangential and normal component, we get the desired identities. □

Lemma 3.2. *Let N be quasi bi-slant submanifold of metallic Riemannian manifold $\tilde{N}$, then*
(i) $f^2X_1 = \cos^2\theta_1(pfX_1 + qX_1)$
(ii) $\tilde{g}(fX_1, fY_1) = \cos^2\theta\tilde{g}(pfX_1 + qX_1, Y_1)$
(iii)$\tilde{g}(\phi X_1, \phi Y_1) = \sin^2\theta\tilde{g}(pfX_1 + qX_1, Y_1)$, for any $X_1, Y_1 \in \Gamma(\Delta_1)$, where θ_1 denotes the slant angle of Δ_1.

Proof. (i) For any $X_1 \in \Gamma(\Delta_1)$, we have

$$\cos\theta_1 = \frac{\tilde{g}(\tilde{\Psi}X_1, fX_1)}{\|\tilde{\Psi}X_1\|.\|fX_1\|}.$$

Using equation (2.2) and as $\cos\theta_1 = \frac{\|fX_1\|}{\|\tilde{\Psi}X_1\|}$ we get

$$\cos^2\theta_1 = \frac{\tilde{g}(X_1, f^2X_1)}{\tilde{g}(pfX_1 + qX_1, X_1)}$$

from the foregoing equation it is easy to see that

$$f^2X_1 = \cos^2\theta_1(pfX_1 + qX_1)$$

(ii) From (2.7) we have

$$fX_1 = \tilde{\Psi}X_1 - \phi X_1.$$

Taking innerproduct of foregoing equation with fX_1, then using (2.2) and part (i) of the lemma we get the desired identity.
(iii) Using equation (2.2), (2.7) and part (ii) of the lemma we get the desired result. □

Lemma 3.3. *Let N be a quasi bi-slant submanifold of metallic Riemannian manifold $\tilde{N}$, then*
(i) $f^2W_1 = \cos^2\theta_1(pfW_1 + qW_1)$
(ii) $\tilde{g}(fZ_1, fW_1) = \cos^2\theta\tilde{g}(pfZ_1 + qZ_1, W_1)$
(iii)$\tilde{g}(\phi Z_1, \phi W_1) = \sin^2\theta\tilde{g}(pfZ_1 + qZ_1, W_1)$,
for any $Z_1, W_1 \in \Gamma(\Delta_2)$, where θ_2 denotes the slant angle of Δ_2.

Lemma 3.4. *Let N be submanifold of locally metallic Riemannian manifold then for any $X_1, Y_1 \in \Gamma(TN)$, we have*

$$\begin{aligned} \nabla_{X_1} fY_1 - A_{\phi Y_1}X_1 - f\nabla_{X_1}Y_1 - Bh(X_1, Y_1) &= 0, \\ h(X_1, fY_1) + \nabla^{\perp}_{X_1}\phi Y_1 - \phi(\nabla_{X_1}Y_1) - Ch(X_1, Y_1) &= 0. \end{aligned}$$

Proof. Using (2.3), (2.5), (2.6), (2.7), (2.8), then comparing tangential and normal components we have the lemma. □

Lemma 3.5. *Let N be quasi slant-submanifold of locally metallic Riemannian manifold then for any $X_1, Y_1 \in \Gamma(TN)$, we have*

$$\begin{aligned} (\nabla_{X_1} f)Y_1 &= A_{\phi Y_1}X_1 + Bh(X_1, Y_1), \\ (\tilde{\nabla}_{X_1}\phi)Y_1 &= Ch(X_1, Y_1) - h(X_1, fY_1). \end{aligned}$$

Proof. Using equation (2.9) and (2.10) in forgoing lemma we get the desired identities. □

Example 3.1. Consider an Euclidean space R^5 with Euclidean metric, define an immersion $i : N \to R^5$ by

$$i(x,y,z) = (x, y\cos t_1, y\frac{\sigma}{\sqrt{q}}\sin t_1, z\frac{\sqrt{q}}{\sigma}\cos t_1, z\sin t_2).$$

where $N = \{(x,y,z)|x,y,z > 0\}$, $\theta_1, \theta_2 \in (0, \frac{\pi}{2})$. We can easily find an orthogonal frame as

$$W_1 = \frac{\partial}{\partial x_1}, \quad W_2 = \cos t_1\frac{\partial}{\partial x_2} + \frac{\sigma}{\sqrt{q}}\sin t_1\frac{\partial}{\partial x_3}, and$$
$$W_3 = \frac{\sqrt{q}}{\sigma}\cos t_2\frac{\partial}{\partial x_4} + \sin t_2\frac{\partial}{\partial x_5}.$$

Now we define an endomorphism $\tilde{\Psi} : R^5 \to R^5$ by

$$\tilde{\Psi}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}\right) = \left(\sigma\frac{\partial}{\partial x_1}, \sigma\frac{\partial}{\partial x_2}, \tilde{\sigma}\frac{\partial}{\partial x_3}, \sigma\frac{\partial}{\partial x_4}, \tilde{\sigma}\frac{\partial}{\partial x_5}\right).$$

It is easy to see that $\tilde{\Psi}$ is metallic structure.

Now as

$$\begin{aligned}
\tilde{\Psi}W_1 &= \sigma\frac{\partial}{\partial_{x_1}} \\
\tilde{\Psi}W_2 &= \sigma\cos t_1\frac{\partial}{\partial x_2} - \sqrt{q}\sin t_1\frac{\partial}{\partial x_3}, and \\
\tilde{\Psi}W_2 &= \sqrt{q}\cos t_2\frac{\partial}{\partial x_4} + \tilde{\sigma}\sin t_2\frac{\partial}{\partial x_5}.
\end{aligned}$$

Now consider the distributions $\Delta = span\{W_1\}$, $\Delta_1 = span\{W_2\}$ and $\Delta_2 = span\{W_3\}$. It is easy to verify that the distribution Δ is invariant and Δ_1 and Δ_2 are slant distributions with slant angles $\theta_1 = \cos^{-1}\left(\frac{\sigma\sqrt{q}\cos 2t_1}{\sqrt{\sigma(q+p\sigma\sin^2 t_1)(\sigma-p\sin^2 t_1)}}\right)$ and $\theta_2 = \cos^{-1}\left(\frac{\sigma q\cos 2t_2}{\sqrt{\tilde{\sigma}(q+p\sigma\sin^2 t_2)(\tilde{\sigma}-p\cos^2 t_2)}}\right)$.

Theorem 3.1. *Let N be quasi bi-slant submanifold of locally metallic Riemannian manifold $\tilde{N}$. Then the invaraint distribution Δ is integrable if and only if*

$$\begin{aligned}
&\tilde{g}(\nabla_{W_1}fZ_1 - \nabla_{Z_1}fZ_1, fP_1X_1 + fP_2X_1 - pX_1) \\
&= -\tilde{g}(h(Z_1, fW_1) - h(W_1, fZ_1), \phi P_1X_1 + \phi P_2X_1),
\end{aligned}$$

for any $W_1, Z_1 \in \Gamma(\Delta)$ and $X_1 = P_1X_1 + P_2X_1 \in \Gamma(\Delta_1 \oplus \Delta_2)$,

Proof. For any $W_1, Z_1 \in \Gamma(\Delta)$ and $X_1 = P_1X_1 + P_2X_1 \in \Gamma(\Delta_1 \oplus \Delta_2)$, from (2.4) we have

$$\begin{aligned}
q\tilde{g}([W_1, Z_1], X_1) &= \tilde{g}(\tilde{\Psi}[W_1, Z_1], \tilde{\Psi}X_1) - pg(\tilde{\Psi}[W_1, Z_1], X_1) \\
&= \tilde{g}(\tilde{\nabla}_{W_1}fZ_1, \tilde{\Psi}X_1) - \tilde{g}(\tilde{\nabla}_{Z_1}fW_1, \tilde{\Psi}X_1) \\
&- p\tilde{g}(\tilde{\nabla}_{W_1}fZ_1, X_1) - \tilde{g}(\tilde{\nabla}_{Z_1}fW_1, X_1).
\end{aligned}$$

On further solvation we get

$$\begin{aligned}
q\tilde{g}([W_1, Z_1], X_1) &= \tilde{g}(\nabla_{W_1}fZ_1 - \nabla_{Z_1}fW_1, fP_1X_1 + fP_2X_1) \\
&+ \tilde{g}(h(W_1, fZ_1) - h(Z_1, fW_1), \phi P_1X_1 + \phi P_2X_1) \\
&- p\tilde{g}(\nabla_{W_1}fZ_1 - \nabla_{Z_1}fW_1, X_1).
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let N be a proper quasi bi-slant submanifold of locally metallic Riemannian manifold. Then the distribution Δ_1 is integrable if and only if*

$$\begin{aligned}
\tilde{g}(A_{\phi W_1}Z_1 - A_{\phi Z_1}W_1, fX_1) &= \tilde{g}(A_{\phi fZ_1}W_1 - A_{\phi fW_1}Z_1, X_1) \\
&- \tilde{g}(\nabla^{\perp}_{W_1}\phi Z_1 - \nabla^{\perp}_{Z_1}\phi W_1, \phi P_2X_1) \\
&+ p\tilde{g}(\nabla_{W_1}Z_1 - \nabla_{Z_1}W_1, fX_1),
\end{aligned}$$

for any $W_1, Z_1 \in \Gamma(\Delta_1)$ and $X_1 \in \Gamma(\Delta \oplus \Delta_2)$

Proof. For any $W_1, Z_1 \in \Gamma(\Delta_1)$ and $X_1 = PX_1 + P_2X_1 \in \Gamma(\Delta \oplus \Delta_2)$, using (2.4), (2.3) and (2.7), we have

$$\begin{aligned}
q\tilde{g}([W_1, Z_1], X_1) &= \tilde{g}(\tilde{\Psi}[W_1, Z_1], \tilde{\Psi}X_1) - p\tilde{g}(\tilde{\Psi}([W_1, Z_1], X_1) \\
&= \tilde{g}(\tilde{\nabla}_{W_1} fZ_1, \tilde{\Psi}X_1) + \tilde{g}(\tilde{\nabla}_{W_1}\phi Z_1, \tilde{\Psi}X_1) \\
&- \tilde{g}(\tilde{\nabla}_{Z_1} fW_1, \tilde{\Psi}X_1) - \tilde{g}(\tilde{\nabla}_{Z_1}\phi W_1, \tilde{\Psi}X_1) \\
&- p\tilde{g}(\tilde{\nabla}_{W_1} Z_1, \tilde{\Psi}X_1) + (\tilde{\nabla}_{Z_1} W_1, \tilde{\Psi}X_1).
\end{aligned}$$

Using (2.2) and part(i) of lemma (4.4) we get

$$\begin{aligned}
q\tilde{g}([W_1, Z_1], X_1) &= cos^2\theta(pf + qI)\tilde{g}([W_1, Z_1], X_1) \\
&- \tilde{g}(A_{\phi fZ_1}W_1 - A_{\phi fW_1}Z_1, X_1) \\
&- \tilde{g}(A_{\phi Z_1}W_1 - A_{\phi W_1}Z_1, \tilde{\Psi}X_1) \\
&+ \tilde{g}(\nabla^{\perp}_{W_1}\phi W_1 - \nabla^{\perp}_{Z_1}\phi W_1, \tilde{\Psi}X_1), \\
&- p\tilde{g}(\nabla_{W_1} Z_1 - \nabla_{Z_1} W_1, \tilde{\Psi}X_1)
\end{aligned}$$

which leads to

$$\begin{aligned}
(\cos^2\theta_1 pfI - \sin^2\theta_1 qI)\tilde{g}([W_1, Z_1], X_1) &= \tilde{g}(A_{\phi fZ_1}W_1 - A_{\phi fW_1}Z_1, X_1) \\
&+ \tilde{g}(A_{\phi Z_1}W_1 - A_{\phi W_1}Z_1, fX_1) \\
&- \tilde{g}(\nabla^{\perp}_{W_1}\phi Z_1 - \nabla^{\perp}_{Z_1}\phi W_1, \phi P_2X_1) \\
&+ p\tilde{g}(\nabla_{W_1} Z_1 - \nabla_{Z_1} W_1, fX_1).
\end{aligned}$$

Thus the proof follows. □

Corollary 3.1. *Let N be a proper quasi bi-slant submanifold of locally metallic Riemannian manifold. If*

$$\begin{aligned}
\nabla^{\perp}_{W_1}\phi Z_1 - \nabla^{\perp}_{Z_1}\phi W_1 &\in \phi\Delta_1 \oplus \nu, \\
A_{\phi fZ_1}W_1 - A_{\phi fW_1}Z_1 &\in \Delta_1, \\
A_{\phi Z_1}W_1 - A_{\phi W_1}Z_1 &\in \Delta_1, and \\
\nabla_{W_1} Z_1 - \nabla_{Z_1} W_1 &\in \Delta_1,
\end{aligned}$$

for any $W_1, Z_1 \in \Gamma(\Delta_1)$, then slant distribution Δ_1 is integrable.

From the forgoing theorem we obtain the following theorem

Theorem 3.3. *Let N be a proper quasi bi-slant submanifold of locally metallic Riemannian manifold. Then the distribution Δ_2 is integrable if and only if*

$$\begin{aligned}
\tilde{g}(A_{\phi W_1}Z_1 - A_{\phi Z_1}W_1, fX_1) &= \tilde{g}(A_{\phi fZ_1}W_1 - A_{\phi fW_1}Z_1, X_1) \\
&- \tilde{g}(\nabla^{\perp}_{W_1}\phi Z_1 - \nabla^{\perp}_{Z_1}\phi W_1, \phi P_1X_1) \\
&+ p\tilde{g}(\nabla_{W_1} Z_1 - \nabla_{Z_1} W_1, fX_1)
\end{aligned}$$

for any $W_1, Z_1 \in \Gamma(\Delta_2)$ and $X_1 \in \Gamma(\Delta \oplus \Delta_1)$.

Corollary 3.2. *Let N be a proper quasi bi-slant submanifold of locally metallic Riemannian manifold. If*

$$\begin{aligned}
\nabla^{\perp}_{W_1}\phi Z_1 - \nabla^{\perp}_{Z_1}\phi W_1 &\in \phi\Delta_2 \oplus \nu, \\
A_{\phi fZ_1}W_1 - A_{\phi fW_1}Z_1 &\in \Delta_2, \\
A_{\phi Z_1}W_1 - A_{\phi W_1}Z_1 &\in \Delta_2, and \\
\nabla_{W_1} Z_1 - \nabla_{Z_1} W_1 &\in \Delta_2
\end{aligned}$$

for any $W_1, Z_1 \in \Gamma(\Delta_2)$, then slant distribution Δ_2 is integrable.

4. Foliation determined by the distributions.

In this section we present some results regarding the foliations determined by the distributions.

Theorem 4.1. *Let N be a proper quasi bi-slant submanifold of locally metallic Riemannian manifold $\tilde{N}$. Then N is totally geodesic if and only if*

$$\begin{aligned}(1-p)\tilde{g}(h(X_1, PY_1), V_1) &+ \cos^2\theta_1(pf+qI)\tilde{g}(h(X_1, P_1Y_2), V_1)\\ &+ \cos^2\theta_2(pf+qI)\tilde{g}(h(X_1, P_2Y_1), V_1)\\ &= -\tilde{g}(\nabla^{\perp}_{X_1}\phi f P_1Y_1 + \nabla^{\perp}_{X_1}\phi f P_2Y_1, V_1)\\ &+ \tilde{g}(A_{\phi Y_1}, BV_1) - \tilde{g}(\nabla^{\perp}_{X_1}\phi Y_1, CV_1 - pV_1)\\ &+ p\tilde{g}\left(h(X_1, fP_1Y_1) + h(X_1, fP_2Y_1), V_1)\right),\end{aligned}$$

for any $X_1, Y_1 \in \Gamma(TN)$ and $V_1 \in \Gamma(T^{\perp}N)$.

Proof. For any $X_1, Y_1 \in \Gamma(TN)$ and $V_1 \in \Gamma(T^{\perp}N)$, using (3.1) we have

$$\tilde{g}(\tilde{\nabla}_{X_1}Y_1, V_1) = \tilde{g}(\tilde{\nabla}_{X_1}PY_1, V_1) + \tilde{g}(\tilde{\nabla}_{X_1}P_1Y_1, V_1) + \tilde{g}(\tilde{\nabla}_{X_1}P_2Y_1, V_1). \tag{4.1}$$

Using (2.4), (2.3) and (2.2) we get

$$\tilde{g}(\tilde{\nabla}_{X_1}PY_1, V_1) = \frac{(1-p)}{q}\tilde{g}(h(X_1, PY_1), V_1), \tag{4.2}$$

Simmilarly from (2.4), (2.3), (2.2) and (2.8) we have

$$\begin{aligned}q\tilde{g}(\tilde{\nabla}_{X_1}P_1Y_1, V_1) &= \cos^2\theta_1(pf+qI)\tilde{g}(h(X_1, P_1Y_1), V_1) + \tilde{g}(\nabla^{\perp}_{X_1}\phi fP_1Y_1, V_1)\\ &+ \tilde{g}(-A_{\phi P_1Y_1}, BV_1) + \tilde{g}(\nabla^{\perp}_{X_1}\phi P_1Y_1, CV_1 - pV_1)\\ &- p\tilde{g}(h(X_1, fP_1Y_1, V_1),\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}q\tilde{g}(\tilde{\nabla}_{X_1}P_2Y_1, V_1) &= \cos^2\theta_2(pf+qI)\tilde{g}(h(X_1, P_2Y_1), V_1) + \tilde{g}(\nabla^{\perp}_{X_1}\phi fP_2Y_1, V_1)\\ &+ \tilde{g}(-A_{\phi P_2Y_1}, BV_1) + \tilde{g}(\nabla^{\perp}_{X_1}\phi P_2Y_1, CV_1 - pV_1)\\ &- p\tilde{g}(h(X_1, fP_2Y_1, V_1).\end{aligned} \tag{4.4}$$

Using (4.2), (4.3) and (4.4) in (4.1) we get the desired result. □

Theorem 4.2. *Let N be a proper quasi bi-slant submanifold of locally metallic Riemannian manifold $\tilde{N}$. Then the distribution Δ defines a totally geodesic foliation on N if and only if*

$$\begin{aligned}\tilde{g}(\nabla_{X_1}fY_1, fW_1 - pW_1) &= -\tilde{g}(h(X_1, fY), \phi W_1),\\ \tilde{g}(\nabla_{X_1}fY_1, B\xi) &= \tilde{g}(h(X_1, fY_1), C\xi - p\xi),\end{aligned} \tag{4.5}$$

for any $X_1, Y_1 \in \Gamma(\Delta)$, $W_1 = P_1W_1 + P_2W_1 \in \Gamma(\Delta_1 \oplus \Delta_2)$ and $\xi \in \Gamma(T^{\perp}N)$.

Proof. For any vector field $X_1, Y_1 \in \Gamma(\Delta)$, $W_1 = P_1W_1 + P_2W_1 \in \Gamma(\Delta_1 \oplus \Delta_2)$, using (2.4) and $\phi Y_1 = 0$ we have

$$\begin{aligned}q\tilde{g}(\tilde{\nabla}_{X_1}Y_1, W_1) &= \tilde{g}(\nabla_{X_1}fY_1, fW_1) + h(X_1, fY_1), \phi W_1)\\ &- p\tilde{g}(\nabla_{X_1}fY_1, W_1).\end{aligned}$$

Which proves (4.5). Now for any $\xi \in \Gamma(T^{\perp}N)$ and $X_1, Y_1 \in \Gamma(\Delta)$, we have

$$q\tilde{g}(\tilde{\nabla}_{X_1}Y_1, \xi) = \tilde{g}(\tilde{\nabla}_{X_1}fY_1, \tilde{\Psi}\xi) - p\tilde{g}(\tilde{\nabla}_{X_1}Y_1, \tilde{\Psi}\xi).$$

Using (2.8) we get

$$\begin{aligned}q\tilde{g}(\tilde{\nabla}_{X_1}Y_1, \xi) &= \tilde{g}(\nabla_{X_1}fY_1, B\xi) + h(X_1, fY_1), C\xi)\\ &- p\tilde{g}(h(X_1, fY_1), \xi).\end{aligned}$$

Thus the proof follows. □

Theorem 4.3. *Let N be a proper quasi bi-slant submanifold of locally metallic Riemannian manifold $\tilde{N}$. Then the slant distribution Δ_1 defines a totally geodesic foliation on $\tilde{N}$ if and only if*

$$\begin{aligned}\tilde{g}(\nabla^{\perp}_{X_1}\phi Y_1, \phi P_2 W_1) &= \tilde{g}(A_{\phi f Y_1}X_1, W_1) + \tilde{g}(A_{\phi Y_1}X_1, fW_1)\\ &+ p\tilde{g}(\nabla_{X_1} fY_1 - A_{\phi Y_1}X_1, W_1),\end{aligned}$$

$$\begin{aligned}\tilde{g}(A_{\phi Y_1}X_1, B\xi) &= \tilde{g}(\nabla^{\perp}_{X_1}\phi f Y_1, \xi) + \tilde{g}(\nabla^{\perp}_{X_1}\phi Y_1, C\xi)\\ &- p\tilde{g}(\nabla^{\perp}_{X_1}\phi Y_1 + h(X_1, fY_1), \xi),\end{aligned}$$

for any $X_1, Y_1 \in \Gamma(\Delta_1), W_1 \in \Gamma(\Delta \oplus \Delta_2)$ and $\xi \in \Gamma(T^{\perp}N)$.

Proof. for any $X_1, Y_1 \in \Gamma(\Delta_1)$, $W_1 = PW_1 + P_2W_1 \in \Gamma(\Delta \oplus \Delta_2)$ and using (2.2), (2.4), and (2.7), we get

$$\begin{aligned}q\tilde{g}(\tilde{\nabla}_{X_1}Y_1, W_1) &= \tilde{g}(\tilde{\nabla}_{X_1} f^2 Y_1, W_1) - \tilde{g}(A_{\phi f Y_1}X_1, W_1) - \tilde{g}(A_{\phi Y_1}X_1, \tilde{\Psi}W_1)\\ &+ \tilde{g}(\nabla^{\perp}_{X_1}\phi Y_1, \tilde{\Psi}W_1) - p\tilde{g}(\nabla_{X_1} fY_1 - A_{\phi Y_1}X_1, W_1)\end{aligned}$$

On further solvation we get

$$\begin{aligned}(\sin^2\theta_1 qI - \cos^2\theta_1 pfI)\tilde{g}(\tilde{\nabla}_{X_1}Y_1, W_1) &= \tilde{g}(\nabla^{\perp}_{X_1}\phi Y_1, \phi P_2 W_1) - \tilde{g}(A_{\phi f Y_1}X_1, W_1)\\ &- p\tilde{g}(\nabla_{X_1} fY_1 - A_{\phi Y_1}X_1, W_1)\\ &- \tilde{g}(A_{\phi Y_1}X_1, fW_1)\end{aligned}$$

Simmilarly we obtain

$$\begin{aligned}(\sin^2\theta_1 qI - \cos^2\theta_1 pfI)\tilde{g}(\tilde{\nabla}_{X_1}Y_1, \xi) &= \tilde{g}(\nabla^{\perp}_{X_1}\phi f Y_1, \xi) - \tilde{g}(A_{\phi Y_1}X_1, B\xi)\\ &- p\tilde{g}(\nabla^{\perp}_{X_1}\phi Y_1 + h(X_1, fY_1), \xi)\\ &+ \tilde{g}(\nabla^{\perp}_{X_1}\phi Y_1, C\xi)\end{aligned}$$

Thus the proof follows from foregoing equations. □

Theorem 4.4. *Let N be a proper quasi bi-slant submanifold of locally metallic Riemannian manifold $\tilde{N}$. Then the slant distribution Δ_2 defines a totally geodesic foliation on $\tilde{N}$ if and only if*

$$\begin{aligned}\tilde{g}(\nabla^{\perp}_{X_1}\phi Y_1, \phi P_1 W_1) &= \tilde{g}(A_{\phi f Y_1}X_1, W_1) + p\tilde{g}(\nabla_{X_1} fY_1 - A_{\phi Y_1}X_1, W_1)\\ &+ \tilde{g}(A_{\phi Y_1}X_1, fW_1),\\ \tilde{g}(A_{\phi Y_1}X_1, B\xi) &= \tilde{g}(\nabla^{\perp}_{X_1}\phi f Y_1, \xi) - p\tilde{g}(\nabla^{\perp}_{X_1}\phi Y_1 + h(X_1, fY_1), \xi)\\ &+ \tilde{g}(\nabla^{\perp}_{X_1}\phi Y_1, C\xi),\end{aligned}$$

for any $X_1, Y_1 \in \Gamma(\Delta_2), W_1 \in \Gamma(\Delta \oplus \Delta_1)$ and $\xi \in \Gamma(T^{\perp}N)$.

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Author's contributions

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