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4- BOYUTLU ÖKLİD UZAYINDA ÜSTEL HOMOTETİK HAREKETLER VE TESSARİNESLER

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ÖZET

Bu çalışma tessarinesler düşünülerek, üstel homotetik hareketler üzerine detaylı bir çalışmadır. Bunu yapabilmek için, tessarines çarpım ve toplam kuralları kullanılarak bir matris tanımladık ve bu matrisin 4- boyutlu Öklid uzayında çeşitli cebirsel özelliklerini verdik. Daha sonra üstel hareketin üstel homotetik hareket olabilmesini ispatladık.

Bu süreç hızları, pol noktaları ve pol eğrileri hakkında bazı teoremler tanımlamamıza izin verdi. Sonunda, her t anında, bir M hiperyüzeyi üzerinde eğrilerin türevleri ve n' inci dereceden regular eğriler tarafından tanımlanan üstel hareketin sadece $(n-1)$' inci derecen bir hız merkezine sahip olduğu bulundu.

Anahtar Kelimeler: *Tessarineler, Homotetik üstel hareket, Hiperyüzey, Regüler eğri.*

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TESSARINES AND HOMOTHETIC EXPONENTIAL MOTIONS IN 4-DIMENSIONAL EUCLIDEAN SPACE

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ABSTRACT

This paper is a detailed study on homothetic exponential motions by considering tessarines. To do this, we introduce a matrix by using tessarines product and addition rules and give a variety of algebraic properties of this matrix in four dimensional Euclidean space E^4. Then, the exponential motion is proven to be homothetic exponential motion.

This process allows us to define some theorems about velocities, pole points, and pole curves. Finally, It is found that at every t-instant an exponential motion defined by the regular curve of order n and derivations curves on the hypersurface M has only one acceleration center of order $(n - 1)$.

Keywords: *Tessarines, Homothetic exponential motion, Hypersurface, Regular curve.*

1. INTRODUCTION

In 1848, The tessarines were first time described by James Cockle as a successor to complex numbers (using more modern notation for complex numbers) and algebra similar to the quaternions. As a set, the tessarines are coincided with 4 -dimensional vector space R^4 over real numbers. Cockle used tessarines to isolate the hyperbolic cosine series and the hyperbolic sine series in the exponential series. He also showed how zero divisors arise in tessarines, inspiring him to use the term ”impossible.” The tessarines are now best known for their subalgebra of real tessarines $t = w + yj$, also called split-complex numbers, which express the parameterization of the unit hyperbola $[1 - 5]$.

In E^n, W. Clifford and James J. Mc Mahon have given a treatment of a rigid body's motion generated by the most general one parameter affine transformation [6]. Another treatment was given by H.R. Müller for the same kind of motion [7]. Subsequently, properties of the planar homothetic motions and three dimensional spherical homothetic motions are given by I. Olcaylar [8]. The exponential motions were given by A.P. Aydın [9] and the dual homothetic exponential motions were given by V. Asil [10].

To state the geometry of the motion of a point in the motion of space is significant in the study of kinematics or spatial mechanisms or in physics. The geometry of such a motion of a point or a line has a number of applications in geometric modeling and model-based manufacturing of mechanical products or in the design of robotic motions. Hacısalihoğlu [11, 12] showed some properties of 1-parameter homothetic motion in Euclidean space E^n. Subsequently, Kula and Yaylı [13] expressed Hamilton motion by means of Hamilton operators in semi-Euclidean spaces E^4_2 and showed that this motion is a homothetic motion.

In this paper we give a detailed study on homothetic exponential motions by considering tessarines. To do this, we introduce a matrix by using tessarines product and addition rules and give a variety of algebraic properties of this matrix in four dimensional Euclidean space E^4. Then, the exponential motion is proven to be homothetic exponential motion. This process allows us to define some theorems about velocities, pole points, and pole curves. Finally, It is found that at every t-instant an exponential motion defined by the regular curve of order n and derivations curves on the hypersurface M has only one acceleration center of order (n-1). We hope that these results will contribute to the study of space kinematics and physics.

2. TESSARINES

A tessarine is given as

$$X = x_0+x_1\ i_1+x_2\ i_2+x_3\ i_3$$

where the imaginary units i_1, i_2 and i_3 are governed by the rules:

$$i_1^2 = -1, i_2^2 = +1\ , i_3^2 = -1$$

and

$$i_1i_2= i_2i_1= i_3 : i_1.i_3= i_3i_1= -i_2 : i_2i_3= i_3i_2= i_1.$$

Let X and Y be tessarines. The addition, subtraction of X and Y are given by

$$X \mp Y = (x_0\mp y_0\ ) + (x_1\ \mp y_1\ )i_1+(x_2\mp y_2)i_2+(x_3\mp y^3)i_3$$

and multiplication of these numbers as follows

$$\begin{aligned} X.Y = Y.X &= (x_0+x_1\ i_1+x_2\ i_2+x_3\ i_3).(y_0+i_1y_1+i_2y_2+i_3y_3) \\ &= (x_0y_0-x_1y_1-x_2y_2+x_3y_3) + i_1(x_0y_1+x_1y_0 - [x_2y_3+x_3y_2]) \\ &+i_2(x_0y_2+x_2y_0 - [x_3y_1+x_1y_3]) + i_3(x_0y_3+x_3y_0 + [x_1y_2+x_2y_1]). \end{aligned}$$

It is easy to see that the multiplication of two tessarines is commutative. It is also convenient to write the set of tessarines as

$$T = \{X \mid X = x_0+x_1\ i_1+x_2\ i_2+x_3\ i_3 \mid (x_1, x_2\ , x_3\ , x_4\ ) \in R\}\ . (1)$$

Definition 2.1. (The conjugate of the tessarine): The conjugate of the tessarine X is shown by X_ and also there are different conjugations of tessarines according to the imaginary units i_1; i_2 and $i_3 = \{i_1$ and $i_2\}$ as follows:

$$1: X^* = (x_0-x_1i_1) + i_2(x_2-x_3i_1),$$
$$= x_0-x_1i_1+x_2i_2-x_3i_3.$$
$$2: X^* = (x_0+x_1i_1) + i_2(x_2+x_3i_1),$$
$$= x_0+x_1i_1-x_2i_2-x_3i_3.$$
$$3: X^* = (x_0-x_1i_1) - i_2(x_2-x_3i_1),$$
$$= x_0-x_1i_1-x_2i_2+x_3i_3.$$

The conjugation of X plays an important role both for algebraic and geometric properties for tessarines. Multiplication of the tessarine with conjugate is given according to the imaginary units i1; i2 and i3 as following;

$$1. XX^* = x_0^2 + x_1^2 + x_2^2 + x_3^2 + 2i_2(x_0x_2 + x_1x_3),$$
$$2. XX^* = x_0^2 - x_1^2 + x_2^2 - x_3^2 + 2i_1(x_0x_1 - x_2x_3), \qquad (2)$$
$$3. XX^* = x_0^2 + x_1^2 - x_2^2 - x_3^2 + 2i_3(x_0x_3 - x_1x_2).$$

The system T is a commutative algebra. It is referred as the tessarine algebra and shown with T, briey one of the bases of this algebra is $\{1, i_1, i_2, i_3\}$ and the dimension is 4. From equation (1), we can give the representation to show a mapping into 4x4 matrix as follows

$$\emptyset: X = x_0+x_1 i_1+x_2 i_2+x_3 i_3 \epsilon\ T \rightarrow \emptyset(X) = \begin{bmatrix} x_0 & -x_1 & x_2 & -x_3 \\ x_1 & x_0 & x_3 & x_2 \\ x_2 & -x_3 & x_0 & -x_1 \\ x_3 & x_2 & x_1 & x_0 \end{bmatrix}$$

T is algebraically isomorphic to the matrix algebra

$$A=\begin{bmatrix} x_0 & -x_1 & x_2 & -x_3 \\ x_1 & x_0 & x_3 & x_2 \\ x_2 & -x_3 & x_0 & -x_1 \\ x_3 & x_2 & x_1 & x_0 \end{bmatrix}$$

A and Ø (X) is a faithful real matrix representation of t.

Lemma 1. i. $X = Y \Leftrightarrow \emptyset(X) = \emptyset(Y)$ ii. $\emptyset(XY) \Leftrightarrow \emptyset(X)\,\emptyset(Y)$

$$\textbf{iii.}\,\emptyset(\boldsymbol{\lambda}\mathbf{X}) = \boldsymbol{\lambda}\emptyset(\mathbf{X})\,;\ \boldsymbol{\lambda} \in \mathbf{IR} \quad \textbf{iv.}\ \emptyset(\mathbf{1}) = \mathbf{I_4}.$$

3. TESSARINES AND HOMOTHETIC EXPONENTIAL MOTIONS IN 4-DIMENSIOAL EUCLIDEAN SPACE

Definition 3.1. (Matrix Exponential): The exponential transformation described as

exp : GL(n; IR) → GL(n; IR) ⊂ E^4

(t,A) → exp (tA) = e^{tA} = ξ(t) =$\sum_{k=0}^{\infty}\frac{t^k}{k!}A^k = I + tA + \frac{t^2}{2!}A^2 + \cdots$

is investigated in the view of kinematic under the condition of A. It is not difficult to show that this sum converges for all n x n complex matrices A of any finite dimension.

Definition 3.2. In equations H(t) = h(t)ξ(t) and ξ(t) = e^{tA}. The matrix A is orthogonal matrix in the sense of Euclidean space. h(t) is a non-constant scalar matrix, t a real parameter provided that

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} H & C \\ 0 & 1 \end{bmatrix}.\begin{bmatrix} X_0 \\ 1 \end{bmatrix} \qquad (3)$$

$$\begin{bmatrix} X \\ 1 \end{bmatrix} = \begin{bmatrix} h(t)\xi(t) & C \\ 0 & 1 \end{bmatrix}.\begin{bmatrix} X_0 \\ 1 \end{bmatrix}$$

which is called a homothetic exponential motion in the Euclidean space of 4-dimensions. In equation (3), X, X_0 and C are 3x1 type matrices. ξ, h and C are differentiable functions of C^∞ class of the parameter t. X and X_0 correspond to the position vectors of the same point with respect to the rectangular coordinate frames of the moving space R and the fixed space R_0, respectively. At the initial time t = t_0

we consider the coordinate systems in R and R_0 are same. We assume that h = h(t) $\neq$ constant, and to avoid the cases of pure translation and pure rotation we also assume for

$$\xi^{1}(t) = A\,\xi(t),\ C^{1} \neq 0$$

and

$$H^{1}(t) = \frac{dH}{dt} = h^{1}(t)\xi(t) + h(t)\xi^{1}(t) = (h^{1}(t) + h(t)A)\xi(t)$$

where (^{1}) indicates $\frac{d}{dt}$. On the other hand, since h = h(t) is scalar matrix, its inverse and transpose are

$h^{-1} = \frac{1}{h}I$, $h^T = h$, here T is transpose in h^T.

Since $\xi(t)$ is a orthogonal matrix, the inverse of H is

$$H^{-1} = h^{-1}\,\xi^{T},\ \xi^{-1} = \xi^{T}.$$

From the equation (3), we can also have

$$X_0 = H^{-1}X + C_0 \qquad (4)$$

where $-H^{-1}C = C_0$. Equations (3) and (4) express the coordinate transformations between the fixed and moving space.

From equation (2), let

$M = \{\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3 = ;\ \alpha_0\alpha_2 + \alpha_1\alpha_3 = 0;\ \alpha \neq 0\} \subset E^4$ be a hypersurface and

$S^3 = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ a unit hypersphere.

Let us consider the following curve;

$\alpha: I \subset R \rightarrow E^4$ defined by,

$\alpha(t) = \alpha_0(t), \alpha_1(t), \alpha_2(t), \alpha_3(t))$ for every $t \in I$.

We suppose that the curve $\alpha(t)$ is differentiable regular curve of order n, (i.e.$\left\|\alpha^{(n)}(t)\right\| \neq 0$. The operator H corresponding to _(t) is defined by the following matrix [14]:

$$A = A(\alpha\,(t)) = \begin{bmatrix} \alpha_0(t) & -\alpha_1\,(t) & \alpha_2(t) & -\alpha_3\,(t) \\ \alpha_1(t) & \alpha_0(t) & \alpha_3(t) & \alpha_2(t) \\ \alpha_2(t) & -\alpha_3\,(t) & \alpha_0(t) & -\alpha_1\,(t) \\ \alpha_3(t) & \alpha_2(t) & \alpha_1(t) & \alpha_0(t) \end{bmatrix} \quad (5)$$

Let $\|\alpha^{\iota}(t)\| = 1$; α (t) be a unit velocity curve. If α (t) does not pass through the origin, and α(t) ≠0 , the above matrix can be represent as

$$H = h\,\xi = he^{tA}$$

$$H=h\begin{bmatrix} \frac{\alpha_0(t)}{h} & \frac{-\alpha_1\,(t)}{h} & \frac{\alpha_2(t)}{h} & \frac{-\alpha_3\,(t)}{h} \\ \frac{\alpha_1(t)}{h} & \frac{\alpha_0(t)}{h} & \frac{\alpha_3(t)}{h} & \frac{\alpha_2(t)}{h} \\ \frac{\alpha_2(t)}{h} & \frac{-\alpha_3\,(t)}{h} & \frac{\alpha_0(t)}{h} & \frac{-\alpha_1\,(t)}{h} \\ \frac{\alpha_3(t)}{h} & \frac{\alpha_2(t)}{h} & \frac{\alpha_1(t)}{h} & \frac{\alpha_0(t)}{h} \end{bmatrix} \quad (6)$$

where,

h: I ⊂ R → R

t → h(t) = $\|\alpha(t)\| = \sqrt{|\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2|}$.

Theorem 2. Let α (t) ∈ S^3 ∩ M: In equation

$$H = h)\,\xi = he^{tA}$$

h is a scalar matrix then, the matrix is an orthogonal matrix i.e. the matrix ξ is SO(4).

Proof. If α (t) ∈ S^3, where $\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$; using equations (5) and (6), from equation H = h ξ, we obtain that $\xi\,\xi^T = \xi^T\,\xi = I_4$ and det ξ = 1.

Corollary 3. Let α (t) ∈M The homothetic exponential motions are regular and have only one instantaneous rotation center at all-time t in Euclidean space E^4

Proof. $H^{\iota} = (h^{\iota} + h A)\ \xi = h\ \xi\ (A + \frac{h^{\iota}(t)}{h} I_4)$, where if we define as

$\lambda = \frac{h^{\iota}}{h}$, then last equation is

$$H^{\iota}() = h\ \xi\ (A - \lambda I):$$

From above equation, we have

$$\det H^{\iota} = \det (h\ \xi\ (t)) \det(A - \lambda I).$$

Since $\det H^{\iota} = 0$, that is; H^{ι} is singular, we get

$$h = 0 \text{ or } \det (A - \lambda I) = 0:$$

Where $h \neq 0$. Otherwise, the exponential motion will be pure translation. H^{ι} is always regular.

Theorem 4. The exponential motion defined by the equation (3) in Euclidean space E^4 is a homothetic exponential motion.

Proof. The matrix determined by the equation (3); can be written

$H = h\ \xi(t) = he^{A}$,

where due to $H \in SO(4)$ this matrix determined is a motion with one parameter.

Theorem 5. Let α (t) be a unit velocity curve and $\alpha^{\iota}(t) \in M$ then the derivation operator H^{ι} of $H = h\ \xi$ is real orthogonal matrix in E^4.

Proof. Since $\alpha(t)$ is a unit velocity curve,

$$(\alpha_0^{\iota}(t))^2 + (\alpha_1^{\iota}(t))^2 + (\alpha_2^{\iota}(t))^2 + (\alpha_3^{\iota}(t))^2 = 1$$

and $\alpha^{\iota}(t) \in M$, then

$$\alpha_0^{\iota}(t)\alpha_2^{\iota}(t) + \alpha_1^{\iota}(t)\alpha_3^{\iota}(t) = 0$$

Thus, $\xi^{\iota}(\xi^{\iota})^T = (\xi^{\iota})^T\ \xi^{\iota}$ and $\det \xi^{\iota} = 1$.

Theorem 6. If $\alpha(t)$ is a unit velocity curve and $\alpha^{\iota}(t) \in M$; the exponential motion is a regular exponential motion and it is independent of h:

Proof. By using theorem 3, det $\xi^{\prime} = 1$ and thus the value of det $\xi^{\prime}$ is independent of h:

Theorem 7. If α(t) is a spherical curve on M , then the exponential motion is rotation exponential motion.

Proof. As α(t) is a spherical curve on S^3, then

$$\alpha_0^2(t) + \alpha_1^2(t) + \alpha_2^2(t) + \alpha_3^2(t) = 1$$

and $H\,H^T = H^T\,H = I_4$: H is a orthogonal matrix and det H = 1: Thus H is a rotating matrix in Euclidean space E^4.

4. VELOCITIES, POLE POINTS AND POLE CURVES OF THE MOTION

Differentiating the equation (3) with respect to t we get

$$X^{\prime} = H^{\prime}\,X_0 + H\,X_0 + C^{\prime}$$

$$= H^{\prime}\,X_0 + h\,\xi X_0 + C^{\prime}$$

where $H^{\prime}\,X_0 = h\,\xi X_0$ is the relative velocity $H\,X_0 + C^{\prime}$ is the sliding velocity, and X_0 is the absolute velocity of point X_0. In this case the following theorem can be given.

Theorem 8. In Euclidean space E^4, for homothetic exponential motion with one parameter, the absolute velocity vector of a moving system of point X_0 at that time t is the sum of the sliding velocity and relative velocity of X_0.

To find the pole point, we have to solve the equation (3); where

$$X_0 = -\,(H^{\prime}\,)^{-1}(C^{\prime}).$$

Theorem 9. If α(t) is a unit velocity curve and $\alpha^{\prime}(t) \in M$, then the pole point corresponding to each t-instant in R_0 is the rotation by $(H^{\prime})^{-1}$ of the speed vector $(C^{\prime})$ of the translation vector at that moment.

Proof. Since the matrix $H^{\prime}$ is orthogonal, then the matrix $(H^{\prime})^T$ is orthogonal, too. Thus it makes a rotation.

Corollary 10. In a homothetic exponential motion in Euclidean space E^4 the tangent vectors of curves during motion are coinciding after the rotation ξ and translation h.

5. ACCELARATIONS AND ACCELARATION CENTRES OF ORDER (*n* - *1*)

Definition 5.1. If H = h(t) ξ (t); h(t) is a scalar matrix and then ξ (t)

is an orthogonal 4x4 matrix, the n th-order derivatives of H is given by

$$H^{(n)} = \left[\sum_{k=0}^{n}\binom{n}{k}h^{(n-k)}A^k\right]\xi.$$

Definition 5.2. The set of the zeros of sliding acceleration of order n is defined the acceleration centre of order (n - 1). By the above definition, we have to solve the solution of the equation

$$H^{(n)} X_0 + C^{(n)} = 0;$$

$$\left[\sum_{k=0}^{n}\binom{n}{k}h^{(n-k)}A^k\right]\xi X_0 + C^{(n)} = 0, \tag{7}$$

where

$$H^{(n)} = \frac{d^n H}{dt^n} \text{ and } C^{(n)} = \frac{d^n C}{dt^n}$$

We know that α(t) is a regular curve of order n and $(\alpha(t))^{(n)} \in M$. Then we have

$$\alpha_0^{(n)}(t)\alpha_2^{(n)}(t) + \alpha_1^{(n)}(t)\alpha_3^{(n)}(t) = 0$$

Thus

$$\{(\alpha_1^{(n)})^2+(\alpha_2^{(n)})^2+(\alpha_3^{(n)})^2+(\alpha_4^{(n)})^2\neq 0\}$$

$\alpha_i^{(n)}=\frac{d_i^n\alpha}{dt^n}$. Also, we have

$$\det H^{(n)}=((\alpha_1^{(n)})^2+(\alpha_2^{(n)})^2+(\alpha_3^{(n)})^2+(\alpha_4^{(n)})^2)^2.$$

Then $\det H^{(n)}\neq 0$. Thus matrix $H^{(n)}$ has an inverse and by equation (7),

the acceleration centre of order (n-1) at every t-instant, is

$$X_0=\left[H^{(n)}\right]^{-1}\left[-C^{(n)}\right].$$

Example 1. Let $\alpha: I\subset R\rightarrow M\subset E^4$ be a curve given by

$$t\rightarrow\alpha(t)=\frac{1}{\sqrt{2}}(\text{cost},\text{sint},\text{sint},\text{cost}).$$

Note that $\alpha(t)\in S^3$ and since $\|\alpha^{\iota}(t)\|=1$, then α (t) is a unit velocity curve. Moreover, $\alpha^{\iota}(t)\in M$; $\alpha^{\iota\iota}(t)\in M,\dots,(\alpha(t))^n\in M$. Thus α (t) satisfies all conditions of the above theorems.

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