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DEĞİŞMEZLİK GRUBU İÇİN ALT YÖRÜNGESEL ÇİZGELER

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ÖZET

Bu çalışmada, değişmezlik grubu için alt yörüngesel çizgeleri araştırdık. $F_{u,N}$ çizgesinde devre olabilmesi için gerekli ve yeterli şartları verdik. Ayrıca $\Gamma[N]$ grubunun üretici eliptik elemanlarıyla, $F_{u,N}$ çizgesindeki devrelerin uzunlukları arasındaki bazı bağıntıları gösterdik.
Anahtar Kelimeler: *Değişmezlik grubu, Yörünge, Blok, Alt yörüngesel çizge, Devre.*

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SUBORBITAL GRAPHS FOR THE INVARIANCE GROUP

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ABSTRACT

In this paper, we investigate some suborbital graphs for the invariance group. We gave a necessary and sufficient condition for the graph $F_{u,N}$ to be circuit. And also we went further to show some relations between the lengths of circuits in F $F_{u,N}$ and the generate elliptic elements of the group $\Gamma[N]$.

Keywords: *Invariance group, Orbit, Block, Suborbital graph, Circuit*

1. INTRODUCTION

Towards the end of the 19th century, some significant results which could serve as a basis for discrete groups theory were first displayed by Henry Poincare and these were used in the generalization of elliptic functions theory. Many scientists carried out studies on functions left invariant by these discrete groups which were named Fuchsian groups and systematic work of which were developed by Henry Poincare. With the discovery of Non-Euclidean geometry and invariant theory in the 19th century, linear fractional transformation groups gained a particular importance and were deeply studied by analysis and algebraic methods, due to them being suitable to topologic group structure. Due to their importance in elliptic curves, integral quadratic forms and elliptic modular functions, congruence subgroups of Γ modular group $\Gamma(N), \Gamma_0(N), \Gamma_1(N)$ etc. groups were mostly studied. It has become evident in recent years that congruence subgroups of Γ modular group played an important role in proving Pierre de Fermat's Last Theorem in 1637.

The main purpose on this issue in previous studies and in this study, is to set the foundations of a new method which would help to identify the congruence subgroups in modular group much better, which have been subject to many studies and gaining particular importance since 1970s and to reveal how the producing elements of the congruence subgroups can be gained by this method (in fact, one of the most important discrete group the normalizer of $\Gamma_0(N)$ in $PSL(2,\mathbb{R})$).With this corresponded which we name as the graph method, the relations between the length of some closed circuits and the orders of the elliptic elements in the subgroups are examined. It is by this way that the signature problem was transferred to the suborbital graphs and a new approach was tried to be obtained.

2. PRELIMINARIES

Let $\mathbb{C}$ the field of complex number and $\hat{\mathbb{C}}$ the Riemann sphere. The modular fractional linear transformation is a map $T\colon \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$, $T(z) := \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{Z}$ are parameters which satisfy $ad - bc = 1$. It is isomorphic to the integral uimodular matrix group

$$\Gamma \coloneqq \mathrm{PSL}(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Γ is clearly discrete group and hence a Fuchsian group. It is known that modular group acts freely properly discontinuously on complex upper half plane $\mathrm{H} \coloneqq \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$. For a natural number N let z_N denote the residue class of $z \in \mathbb{Z}$ modulo N and let $\mathbb{Z}_N$ denote the ring of all residue classes modulo N. As before Γ denotes the group of homogeneous modular transformations which is isomorphic to the special linear group $\mathrm{SL}(2,\mathbb{Z})$. Correspondingly, we use the sembol Γ_N for the group $\mathrm{SL}(2,\mathbb{Z}_N)$.

Definition 2.1. The ring homomorphism $z \to z_N$ of $\mathbb{Z}$ onto $\mathbb{Z}_N$ induces the group homomorphism σ of Γ into Γ_N with $\sigma \colon \Gamma \to \Gamma_{\mathrm{N}}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a_N & b_N \\ c_N & d_N \end{pmatrix}$.

The kernel $\Gamma(N) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} modN \right\}$

of σ is a normal subgroup of Γ and is called the homogeneous principal congruence group of level N. And also the image $\sigma(\Gamma)$ is isomorphic to the full group Γ_N. The homogeneous group

$$\Gamma[N] \coloneqq \Gamma(N) \cup (-I)\Gamma(N)$$

will likewise be called the principal congruence group.

For example $\Gamma_1 \coloneqq \Gamma[2] \cup \Gamma[2]T, T^2 = -I$ is also called the theta group.

Let $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be a matrix of order N. We assume that the entries of M are relatively prime, $(\alpha, \beta, \gamma, \delta) = 1$. The linear transformation corresponding to M will be called a transformation of order N. For fixed N we will let μ_N denote both the set of all such matrices and their corresponding transformations. By a transformation group Γ_M of order N we mean the group $\Gamma_M \coloneqq \Gamma \cap M^{-1}\Gamma M$, which may be considered either as a group of matrices or as a group of linear transformations.

Let j be a non-constant modular function for the full modular group. Extended complex upper half plane we will be denoted by $\mathrm{H}^* \coloneqq \mathrm{H} \cup \mathbb{Q} \cup \{\infty\}$ and then we define the function j_M by $j_M(\tau) \coloneqq j\big(M(\tau)\big)$, $\tau \in \mathrm{H}^*$.

Definition 2.2. The invariance group for all transforms j_M of j of order N is the group $\Gamma^*(N) \coloneqq \left\{T \in \Gamma : T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} modN\right\}$.

From this we see that the index of the principal congruence group $\Gamma(N)$ in $\Gamma^*(N)$ is equal to the number η of incongruent solutions modulo N of the congruence $x^2 \equiv 1\ (modN)$. If N has the prime factorization

$$N = 2^a p_1^{r_1} \ldots p_s^{r_s} \quad \text{with} \quad 2 < p_1 < \cdots < p_s \quad \text{and} \quad s > 0$$

then η known to be

$$\eta = \begin{cases} 2^s & , \quad a = 0 \quad \text{or} \quad a = 1 \\ 2^{s+1} & , \quad a = 2 \\ 2^{s+2} & , \quad a > 2 \end{cases}$$

thus the index of $\Gamma^*(N)$ in Γ is equal to $|\Gamma : \Gamma^*(N)| = \frac{|\Gamma:\Gamma(N)|}{\eta}$. We note especially that $\eta = \begin{cases} 1 & , \quad N = 1 \quad \text{or} \quad N = 2 \\ 2 & , \quad N = p_1^{r_1} \quad \text{or} \quad N = 2p_1^{r_1} \end{cases}$ and in this cases $\Gamma^*(N) = \Gamma[N]$.

In addition to that a complex form can be put on the quotient group H/Γ to get a noncompact Riemann surface. A general compactification of Riemann surface is achieved by adding finitely many points named the cusps of Γ. In particularly , this is done by considering the action of Γ on the H^*. The group Γ acts on the subset $\mathbb{Q} \cup \{\infty\}$. If Γ acts transitively on $\mathbb{Q} \cup \{\infty\}$, the space H^*/Γ becomes the special compactification of H/Γ.

Remark 2.3. Actually quotient groups of complex upper half plane H that are compact do form for Fuchsian groups Γ other than subgroups of the modular group which is known; a class of them constructed from quaternion algebras is also of significance in number theory and combinatoric theory and it also has the merit of being fairly interesting to many number theorists.

3. PERMUTATION GROUPS AND IMPRIMITIVE ACT

Every element of the extended set of rational $\widehat{\mathbb{Q}} \coloneqq \mathbb{Q} \cup \{\infty\}$ can be represented as a reduced fraction $\frac{x}{y}$ with $x, y \in \mathbb{Z}$ and $(x, y) = 1$. Since $\frac{x}{y} = \frac{-x}{-y}$, this representation is not unique. We represent ∞ as $\frac{1}{0} = \frac{-1}{0}$. The action of the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma^*(N)$ on $\frac{x}{y}$ is $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \frac{x}{y} \longrightarrow \frac{\alpha x + \beta y}{\gamma x + \delta y}$. The action of a matrix on $\frac{x}{y}$ and on $\frac{-x}{-y}$ is identical.

Definition 3.1. Let (G, Ω) be transitive permutation group, consisting of a group G acting on a set Ω transitively. An equivalence relation $\approx$ on Ω is called $G - invariant$ if whenever $v_1, v_2 \in \Omega$ satisfy $v_1 \approx v_2$ then $g(v_1) \approx g(v_2)$ for all $g \in G$. The equivalence classes are called blocks.

We call (G, Ω) imprimitive, if Ω admits some G −invariant equivalence relation different from

(i) the identity relation, $v_1 \approx v_2$ if and only if $v_1 = v_2$;

(ii) the universal relation, $\nu_1 \approx \nu_2$ for all $\nu_1, \nu_2 \in \Omega$.

Otherwise (G, Ω) is called primitive. These two relations are supposed to be trivial relations. In conclusion we have,

Lemma 3.2. Let (G, Ω) be a transitive permutation group. (G, Ω) is primitive if and only if G_ε, the stabilizer of $\varepsilon \in \Omega$ is a maximal subgroup of G for each ε.

From the lemma above we see that whenever, for some ε, $G_\varepsilon < H < G$, then Ω admits some G –invariant equivalence relation other than the trivial cases. Because of the transitivity, every element of Ω has the form $g(\varepsilon)$ for some $g \in G$. Thus one of the non-trivial G -invariant equivalence relation on Ω is given as follows:

$$g_1(\varepsilon) \approx g_2(\varepsilon) \text{ if and only if } g_2 \in g_1 H.$$

The number of blocks is the index $|G{:}H|$.

We have the following statements:

4. MAIN CALCULATION

Lemma 4.1. (i) $\Gamma^*(N)$ acts transitively on $\widehat{\mathbb{Q}}$.

(ii) Let $\Gamma[N]$ which is the principal congruence subgroups of Γ. Then $\Gamma^*_\infty(N) < \Gamma[N] < \Gamma^*(N)$ for each N, where $\Gamma^*_\infty(N)$ is the stabilizer of ∞ in $\widehat{\mathbb{Q}}$ is the set of $\left\{\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{Z}\right\}$.

Proof.

(i) It is enough to show that the orbit containing ∞ is $\widehat{\mathbb{Q}}$. If $\frac{x}{y} \in \widehat{\mathbb{Q}}$, then as $(x, y) = 1$, there exits $\upsilon_1, \upsilon_1 \in \mathbb{Z}$ with $x\upsilon_1 - y\upsilon_1 = 1$. Then the element $\begin{pmatrix} x & \upsilon_2 \\ y & \upsilon_1 \end{pmatrix}$ of $\Gamma^*(N)$ sends ∞ to $\frac{x}{y}$.

(ii) Since the action is transitive, stabilizers of any two points in $\widehat{\mathbb{Q}}$ are conjugate. So it is sufficient to consider the stabilizer $\Gamma^*_\infty(N)$ of

∞. The consists of the elements of the form $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ with $\lambda \in \mathbb{Z}$. Hence

$$T(\infty) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Then $\alpha = 1,\ \gamma = 0, \delta = 1$ and $\beta = \lambda \in \mathbb{Z}$. Therefore $T = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$. So $\Gamma_{\infty}^{*}(N)$ is the infinite cyclic group by the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. That is, $\Gamma_{\infty}^{*}(N) = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle$.

And also, it is clearly that $\Gamma_{\infty}^{*}(N) < \Gamma[N] < \Gamma^{*}(N)$. Hence, this completes the proof.

We now consider imprimitivity of the action on $\Gamma^{*}(N)$ on $\widehat{\mathbb{Q}}$. This will be a special case of the following:

We will define an equivalence relation $\approx$ induced on $\widehat{\mathbb{Q}}$ by $\Gamma^{*}(N)$. Then $\Gamma^{*}(N)$ acts imprimitively on $\widehat{\mathbb{Q}}$. Let $\approx$ denote the $\Gamma^{*}(N)$ invariant equivalence relation on $\widehat{\mathbb{Q}}$ by $\Gamma[N]$, and let $\rho_1 = \frac{\alpha_1}{\gamma_1}$ and $\rho_2 = \frac{\alpha_2}{\gamma_2}$ be elements of $\widehat{\mathbb{Q}}$. Then there are the elements $T_1 \coloneqq \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}$ and $T_2 \coloneqq \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}$ in $\Gamma^{*}(N)$ such that $\rho_1 = T_1(\infty)$, $\rho_2 = T_2(\infty)$. So we have

$\rho_1 \approx \rho_2$ if and only if $T_1^{-1}T_2 \in \Gamma[N]$. And so from the above we can easily calculate that

$T_1^{-1}T_2 = \begin{pmatrix} \alpha_2\delta_1 - \gamma_2\beta_1 & \delta_1\beta_2 - \delta_2\beta_1 \\ \alpha_1\gamma_2 - \alpha_2\gamma_1 & \alpha_1\delta_2 - \gamma_1\beta_2 \end{pmatrix} \in \Gamma[N]$. Hence $\alpha_1\gamma_2 - \alpha_2\gamma_1 \equiv 0\ (modN)$ is obtained. Similarly if $\xi_1 = \frac{\beta_1}{\delta_1}$ and $\xi_2 = \frac{\beta_2}{\delta_2}$ then $\xi_1 = T_1(0)$, $\xi_2 = T_2(0)$. Thus $\xi_1 \approx \xi_2$ if and only if $T_1^{-1}T_2 \in \Gamma[N]$. It has clearly here $\delta_1\beta_2 - \delta_2\beta_1 \equiv 0\ (modN)$ is achieved.

By our general discussion of imprimitivity, the number of the blocks under $\approx$ is given by $|\Gamma^*(N):\Gamma[N]|$. ∞ and 0 are blocks respectively

$$[\infty] := \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \left\{ \frac{a}{b} \mid a,b \in \mathbb{Z},\ (a,b) = 1 \text{ and } b \equiv 0\ (modN) \right\} \text{ and}$$

$$[0] := \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left\{ \frac{a}{b} \mid a,b \in \mathbb{Z},\ (a,b) = 1 \text{ and } a \equiv 0\ (modN) \right\}.$$

Now we have two definition,

Definition 4.2. $X \neq \emptyset$ is a set and $\Delta \subset X \times X$ is a relation. $G=(X,\Delta)$ pair is called a graph. Elements of X are vertices of graph and elements of Δ are edges of the graph. If $(a,b) \in \Delta$ this is indicated as $a \longrightarrow b$. If $(a,b) \in \Delta$ or $(b,a) \in \Delta$, a and b are connected by an edge. In this case, a and b are called neighboring vertices.

Definition 4.3. Let $a = a_0, a_1, a_2, \dots, a_n = b$ be a sequence of G graph vertices. If for $1 \leq i \leq n$, a_{i-1} and a_i are connected with an edge, then this is indicated with the expression from a to b there is a path with the length of n. If $a = b$ and $a_0, a_2, \dots, a_{n-1}$ vertices are all different, then this is called a n edged circuit. Furthermore, if for the pairs of a_i, a_{i+1}, $a_i \rightarrow a_{i+1}$ then this is a circuit directed at a circuit. A three edged circuit is called a triangle, four edged circuit is quadrilateral and six edged circuit is called a hexagon.

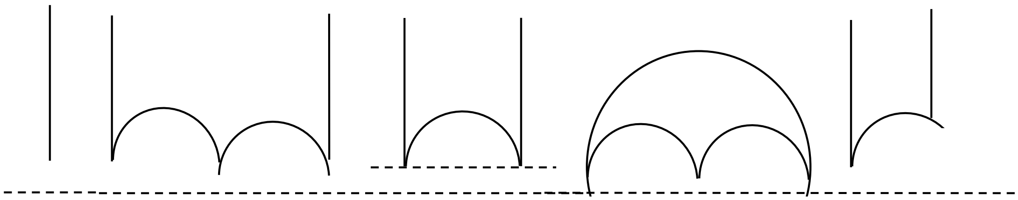
Sims introduced the idea of suborbital graphs for a finite permutation groups G acting on a set Ω. These are graphs with a vertex set Ω, on which G induces automorphism. We summarize Sims' theory as follows.

Let (G,Ω) denote a transitive permutation group. Then G acts on a Cartesian product set $\Omega \times \Omega$ by $\Pi: G \times (\Omega \times \Omega) \longrightarrow (\Omega \times \Omega)$. For $(\nu_1, \nu_2) \in \Omega \times \Omega$ and $g \in G$, we define

$\Pi\big(g,(v_1,v_2)\big)=\big(g(v_1),g(v_2)\big)$. Therefore $(G,\Omega\times\Omega\ )$ becomes a permutation group. The orbit of this action are called suborbitals of G, that containing (v_1,v_2) being denoted by $O(v_1,v_2)$. From $O(v_1,v_2)$ we can form a suborbital graph $G(v_1,v_2)$: its vertices are the elements of Ω, and there is a directed edge from a to b if $(a,b)\in O(v_1,v_2)$. A directed edge from a to b is denoted by $a\longrightarrow b$. In this case we will say that there exists an edge $a\longrightarrow b$ in $G(v_1,v_2)$. We can also say that the reader is refereed to [1], [2], [5], [13], [14] and [15] for some relevant previous work on suborbital graphs.

In this paper our calculation concern $\Gamma^*(N)$, so we can draw this edge as a hyperbolic geodesic in the complex upper half plane H. Here graph is a combination of hyperbolic lines. In this study, we investigate that G and Ω are $\Gamma^*(N)$ and $\widehat{\mathbb{Q}}$, respectively. Since $\Gamma^*(N)$ acts transitively on $\widehat{\mathbb{Q}}$, each suborbital contains a pair (∞,ρ) for some $\rho\in\widehat{\mathbb{Q}}$; writing $\rho=\frac{u}{N}$, $(u,N)=1$, we denote this suborbital by $O_{u,N}:=O(\frac{1}{0},\frac{u}{N})$ and the corresponding suborbital graph by $G_{u,N}:=G(\frac{1}{0},\frac{u}{N})$. $G_{u,N}$ is a disjoint union of all subgraphs forming blocks with respect to $\approx$ $\Gamma^*(N)$ invariant equivalence relation. $\Gamma^*(N)$ permutes these blocks transitively and so all of the subgraphs are isomorphic. Therefore it is sufficient to do the calculations only for the block $[\infty]$.

Let $F_{u,N}:=F(\frac{1}{0},\frac{u}{N})$ denote the subgraph of $G_{u,N}$ whose vertices form the block $[\infty]$. we represent the edges of $F_{u,N}$ as hyperbolic geodesics in the upper half-plane H. And also we can show that the subgraph $F_{u,N}$ of $G_{u,N}$ does not cross in the upper half-plane.



2-gon H - Quadrilateral H - Triangles

Figure 1. Circuits

Theorem 4.4. Let $\frac{\alpha_1}{\gamma_1}$ and $\frac{\alpha_2}{\gamma_2}$ be in the block $[\infty]$. Then there is an edge

$\frac{\alpha_1}{\gamma_1} \longrightarrow \frac{\alpha_2}{\gamma_2}$ in $F_{u,N}$ if and only if either

(a) $\alpha_2 \equiv u\alpha_1\ (modN),\ \gamma_2 \equiv u\gamma_1\ (modN)$ and $\alpha_1\gamma_2 - \gamma_1\alpha_2 = N$

(b) $\alpha_2 \equiv -u\alpha_1\ (modN),\quad \gamma_2 \equiv -u\gamma_1\ (modN)$ and $\alpha_1\gamma_2 - \gamma_1\alpha_2 = -N.$

Proof. Let $\frac{\alpha_1}{\gamma_1} \longrightarrow \frac{\alpha_2}{\gamma_2} \in F_{u,N}$, then there exists some $T := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma^*(N)$ such that $T\left(\frac{1}{0}\right) = \frac{\alpha}{\gamma} = \frac{\alpha_1}{\gamma_1}$ and $T\left(\frac{u}{N}\right) = \frac{\alpha u+\beta N}{\gamma u+\delta N} = \frac{\alpha_2}{\gamma_2}$. Hence $\alpha = \alpha_1$, $\gamma = \gamma_1$. Then these equations $\alpha_2 \equiv u\alpha_1\ (modN)$ and $\gamma_2 \equiv u\gamma_1\ (modN)$ are satisfied. So we have the matrix equation

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & N \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \gamma_1 & \gamma_2 \end{pmatrix}.$$

If we take determinant, it is easily seen that $\alpha_1\gamma_2 - \gamma_1\alpha_2 = N$.

Conversely, we suppose that $\alpha_2 \equiv u\alpha_1 \,(modN)$, $\gamma_2 \equiv u\gamma_1 \,(modN)$ and $\alpha_1\gamma_2 - \gamma_1\alpha_2 = N$. Then there exist integers θ_1 and θ_2 such that $\alpha_2 = u\alpha_1 + \theta_1 N$ and $\gamma_2 = u\gamma_1 + \theta_2 N$. In this case

$$\begin{pmatrix} \alpha_1 & \theta_1 \\ \gamma_1 & \theta_2 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & N \end{pmatrix} = \begin{pmatrix} \alpha_1 & u\alpha_1 + \theta_1 N \\ \gamma_1 & u\gamma_1 + \theta_2 N \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \gamma_1 & \gamma_2 \end{pmatrix}$$

is obtained. Since $\alpha_1\gamma_2 - \gamma_1\alpha_2 = N$ from determinants we get $\alpha_1\theta_2 - \gamma_1\theta_1 = 1$. Consequently, $\begin{pmatrix} \alpha_1 & \theta_1 \\ \gamma_1 & \theta_2 \end{pmatrix} \in \Gamma^*(N)$ and $\frac{\alpha_1}{\gamma_1} \longrightarrow \frac{\alpha_2}{\gamma_2} \in F_{u,N}$.

(b) The proof for minus sign is similar. We get above matrix equation with α_2 and γ_2 replaced by $-\alpha_2$ and $-\gamma_2$, so that $\frac{\alpha_1}{\gamma_1} \longrightarrow \frac{-\alpha_2}{-\gamma_2} = \frac{\alpha_2}{\gamma_2} \in F_{u,N}$.

Theorem 4.5. The graph $F_{u,N}$ contains directed triangles if and only if $u^2 \pm u + 1 \equiv 0 \,(modN)$.

Proof. Assume first that $F_{u,N}$ has a triangle $\frac{k_0}{l_0} \longrightarrow \frac{m_0}{n_0} \longrightarrow \frac{x_0}{y_0} \longrightarrow \frac{k_0}{l_0}$. It can be easily shown that $\Gamma[N]$ permutes the vertices and edges of $F_{u,N}$ transitively. Therefore we suppose that the above triangle is transformed under $\Gamma[N]$ to the $\frac{1}{0} \longrightarrow \frac{u}{N} \longrightarrow \frac{x_0}{y_0} \longrightarrow \frac{1}{0}$.

Without loss of generality, from the edge of $\frac{u}{N} \overset{<}{\rightarrow} \frac{x_0}{y_0 N}$ the equation of $x_0 \equiv -u^{\,2}(modN)$ and from the $uy_0N - Nx_0 = -N$ equation, $x_0 = uy_0 + 1$ is achieved.

For $y_0 = 1$ situation, $\frac{u}{N} \longrightarrow \frac{x_0}{N}$ and $x_0 = u + 1$ and eventually $\frac{u}{N} \longrightarrow \frac{u+1}{N}$ is found. And also $u + 1 \equiv -u^{\,2}(modN)$ then $u^{\,2} + u + 1 \equiv 0(modN)$. Again $y_0 = 2$ can not be true because for

$\frac{x_0}{2N} \longrightarrow \frac{1}{0}$ there is not an edge condition. Similarly if $\frac{u}{N} \overset{>}{\rightarrow} \frac{x_0}{y_0 N}$ holds then we conclude that $u^2 - u + 1 \equiv 0(modN)$. Consequently we have $u^2 \pm u + 1 \equiv 0\ (modN)$.

On the other hand suppose that $u^2 \pm u + 1 \equiv 0\ (modN)$. Then, using Theorem 4.4., we see that $\frac{1}{0} \longrightarrow \frac{u}{N} \longrightarrow \frac{u \pm 1}{N} \longrightarrow \frac{1}{0}$ is a triangle in $F_{u,N}$. Hence, as the hyperbolic triangle the following shape:

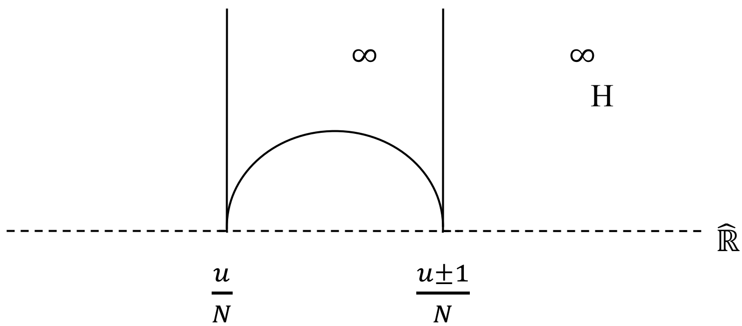


Figure 2. H -Triangle in $F_{u,N}$

Corollary 4.6. For some $u \in \mathbb{N}$, $F_{u,N}$ contains a triangle if and only if the group $\Gamma[N]$ has a period 3.

Proof. Firstly suppose $F_{u,N}$ contains a triangle. Then, Theorem 4.5. shows that $u^2 \pm u + 1 \equiv 0\ (modN)$. Therefore we have the elliptic element

$\varphi := \begin{pmatrix} -u & \frac{u^2 \pm u + 1}{N} \\ -N & u \pm 1 \end{pmatrix}$ of order 3 in $\Gamma^*(N)$. That is, $\varphi^3 = \mp I$. The elements of this form must be in $\Gamma[N]$. It is clear that, $\varphi \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$

$\binom{u}{N}$, $\varphi\binom{u}{N}=\binom{u\pm 1}{N}$, $\varphi\binom{u\pm 1}{N}=\binom{1}{0}$. That is, $\frac{1}{0}\longrightarrow\frac{u}{N}\longrightarrow\frac{u\pm1}{N}\longrightarrow\frac{1}{0}$.

Conversely, suppose that $\Gamma[N]$ has a period for order 3, so $\Gamma[N]$ contains an elliptic element of order 3. Let this element be $\varphi_1=\begin{pmatrix} a & b \\ N & -a\pm 1\end{pmatrix}$, $\det\varphi_1=1$. From this we get $N|(a^2\pm a+1)$. Therefore $F_{u,N}$ contains a triangle.

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