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Strongly Lacunary $\mathcal{I}^*$-Convergence and Strongly Lacunary $\mathcal{I}^*$-Cauchy Sequence

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Abstract
In this paper, we defined the concepts of lacunary $\mathcal{I}^*$-convergence and strongly lacunary $\mathcal{I}^*$-convergence. We investigated the relations between strongly lacunary $\mathcal{I}$-convergence and strongly lacunary $\mathcal{I}^*$-convergence. Also, we defined the concept of strongly lacunary $\mathcal{I}^*$-Cauchy sequence and investigated the relations between strongly lacunary $\mathcal{I}$-Cauchy sequence and strongly lacunary $\mathcal{I}^*$-Cauchy sequence.

Keywords: *Ideal, Lacunary sequence, $\mathcal{I}$-Convergence, $\mathcal{I}$-Cauchy Sequence*

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1. Introduction and definitions

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [1] and Schoenberg [2]. The concept of $\mathcal{I}$-convergence in a metric space, which is a generalized from of statistical convergence, was introduced by Kostyrko et al. [3]. Later it was further studied many others. Nabiev et al. [4] studied on $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence with some properties. Recently, Das et al. [5] introduced new notions, namely $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence by using ideal. Also, Yamancı and Gürdal [6] introduced the notions lacunary $\mathcal{I}$-convergence and lacunary $\mathcal{I}$-Cauchy in the topology induced by random n-normed spaces and prove some important results. Debnath [7] studied the notion of lacunary ideal convergence in intuitionistic fuzzy normed linear spaces as a variant of the notion of ideal convergence. Tripathy et al. [8] introduced the concept of lacunary $\mathcal{I}$-convergent sequences. A lot of development have been made about the statistical convergence and ideal convergence defined in different setups [9–11].

In this paper, we defined the concepts of lacunary $\mathcal{I}^*$-convergence and strongly lacunary $\mathcal{I}^*$-convergence. We investigated the relations between strongly lacunary $\mathcal{I}$-convergence and strongly lacunary $\mathcal{I}^*$-convergence. Also, we defined the concept of strongly lacunary $\mathcal{I}^*$-Cauchy sequence and investigated the relations between strongly

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lacunary $\mathcal{I}$-Cauchy sequence and strongly lacunary $\mathcal{I}^*$-Cauchy sequence.

Now, we recall some basic concepts and definitions (see [3, 4, 6–8, 12–21]).
A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if
(i) $\emptyset \in \mathcal{I}$,
(ii) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$,
(iii) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.
A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is a filter if and only if
(i) $\emptyset \notin \mathcal{F}$,
(ii) If $A, B \in F$, then $A \cap B \in \mathcal{F}$,
(iii) If $A \in \mathcal{F}$ and $B \supseteq A$, then $B \in \mathcal{F}$.
$\mathcal{I}$ is a non-trivial ideal in $\mathbb{N}$, then the set

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter in $\mathbb{N}$, called the filter associated with $\mathcal{I}$.

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the property (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \cdots\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, \cdots\}$ such that $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence (x_n) of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathcal{I}.$$

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence (x_n) of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-Cauchy sequence if for each $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - x_N| \geq \varepsilon\} \in \mathcal{I}.$$

A sequence (x_n) is said to be $\mathcal{I}^*$-convergent to L if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ such that

$$\lim_{k \to \infty} x_{m_k} = L.$$

A sequence (x_n) is said to be $\mathcal{I}^*$-Cauchy sequence if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ such that the subsequence $x_M = (x_{m_k})$ is an ordinary Cauchy sequence, that is,

$$\lim_{k,p \to \infty} |x_{m_k} - x_{m_p}| = 0.$$

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that

$$k_0 = 0 \text{ and } h_r = k_r - k_{r-1} \to \infty$$

as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by

$$I_r = (k_{r-1}, k_r]$$

and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Throughout the paper, we take $\theta = \{k_r\}$ be a lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal.
A sequence (x_n) of elements of $\mathbb{R}$ is said to be strongly lacunary convergent to $L \in \mathbb{R}$ if

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{n \in I_r} |x_n - L| = 0.$$

A sequence (x_n) is said to be a strongly lacunary $\mathcal{I}$-convergent to L, if for every $\varepsilon > 0$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n \in I_r} |x_n - L| \geq \varepsilon \right\} \in \mathcal{I}.$$

In this case, we write $x_n \to L[\mathcal{I}_\theta]$.

A sequence (x_n) is said to be a strongly lacunary $\mathcal{I}$-Cauchy if for every $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n \in I_r} |x_n - x_N| \geq \varepsilon \right\} \in \mathcal{I}.$$

Lemma 1.1. *[4] Let $\{P_i\}_1^\infty$ be a countable collection of subsets of $\mathbb{N}$ such that $P_i \in F(\mathcal{I})$ for each i, where $F(\mathcal{I})$ is a filter associate with an admissible ideal $\mathcal{I}$ with property (AP). Then there exists a set $P \subset \mathbb{N}$ such that $P \in F(\mathcal{I})$ and the set $P \backslash P_i$ is finite for all i.*

2. Main results

In this section, firstly, we gave the concepts of lacunary $\mathcal{I}^*$-convergence and strongly lacunary $\mathcal{I}^*$-convergence. We investigated the relations between strongly lacunary $\mathcal{I}$-convergence and strongly lacunary $\mathcal{I}^*$-convergence. Then after, we gave the concept of strongly lacunary $\mathcal{I}^*$-Cauchy sequence and investigated the relations between strongly lacunary $\mathcal{I}$-Cauchy sequence and strongly lacunary $\mathcal{I}^*$-Cauchy sequence.

Definition 2.1. [12]. A sequence (x_n) is said to be lacunary $\mathcal{I}^*$-convergent to L if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ we have

$$\lim_{\substack{r \to \infty \\ (r \in M')}} \frac{1}{h_r} \sum_{k \in I_r} x_{m_k} = L.$$

In this case, we write $x_n \to L(\mathcal{I}_\theta^*)$.

Definition 2.2. A sequence (x_n) is said to be strongly lacunary $\mathcal{I}^*$-convergent to L if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ we have

$$\lim_{\substack{r \to \infty \\ (r \in M')}} \frac{1}{h_r} \sum_{k \in I_r} |x_{m_k} - L| = 0.$$

In this case, we write $x_n \to L[\mathcal{I}_\theta^*]$.

Theorem 2.1. *If a sequence (x_n) is strongly lacunary $\mathcal{I}^*$-convergent to L, then it is lacunary $\mathcal{I}^*$-convergent to L.*

Proof. Let $x_n \to L[\mathcal{I}_\theta^*]$. Then, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ $(i.e. H = \mathbb{N} \backslash M' \in \mathcal{I})$ and for every $\varepsilon > 0$ there is a $r_0 = r_0(\varepsilon) \in \mathbb{N}$ such that for all $r > r_0$ we have

$$\frac{1}{h_r} \sum_{k \in I_r} |x_{m_k} - L| < \varepsilon, \ (r \in M').$$

Then, we have

$$\begin{aligned} \left| \frac{1}{h_r} \sum_{k \in I_r} x_{m_k} - L \right| &\leq \frac{1}{h_r} \sum_{k \in I_r} |x_{m_k} - L| \\ &< \varepsilon, \ (r \in M') \end{aligned}$$

for every $\varepsilon > 0$ and all $r > r_0 = r_0(\varepsilon)$ and so $x_n \to L(\mathcal{I}_\theta^*)$. □

Theorem 2.2. *If a sequence (x_n) is strongly lacunary $\mathcal{I}^*$-convergent to L, then it is strongly lacunary $\mathcal{I}$-convergent to L.*

Proof. Let $x_n \to L[\mathcal{I}_\theta^*]$. Then, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ $(i.e. H = \mathbb{N} \backslash M' \in \mathcal{I})$ and for every $\varepsilon > 0$ there is a $r_0 = r_0(\varepsilon) \in \mathbb{N}$ such that for all $r > r_0$ we have

$$\frac{1}{h_r} \sum_{k \in I_r} |x_{m_k} - L| < \varepsilon, \ (r \in M').$$

Then,

$$A(\varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |x_{m_k} - L| \geq \varepsilon \right\} \subset H \cup \{1, 2, \cdots, r_0\}.$$

Since $\mathcal{I}$ is an admissible ideal, we have

$$H \cup \{1, 2, \cdots, r_0\} \in \mathcal{I}$$

and so $A(\varepsilon) \in \mathcal{I}$. Hence, $x_n \to L[\mathcal{I}_\theta]$. □

Theorem 2.3. *Let $\mathcal{I}$ be a admissible ideal with property (AP). If (x_n) is strongly lacunary $\mathcal{I}$-convergent to L, then it is strongly lacunary $\mathcal{I}^*$-convergent to L.*

Proof. Assume that $x_n \to L[\mathcal{I}_\theta]$. Then, for every $\varepsilon > 0$,

$$T(\varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n \in I_r} |x_n - L| \geq \varepsilon \right\} \in \mathcal{I}.$$

Put

$$T_1 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n \in I_r} |x_n - L| \geq 1 \right\} \text{ and } T_p = \left\{ r \in \mathbb{N} : \frac{1}{p} \leq \frac{1}{h_r} \sum_{n \in I_r} |x_n - L| < \frac{1}{p-1} \right\},$$

for $p \geq 2$ and $p \in \mathbb{N}$. It is clear that $T_i \cap T_j = \emptyset$ for $i \neq j$ and $T_i \in \mathcal{I}$ for each $i \in \mathbb{N}$. By property (AP) there is a sequence $\{V_p\}_{p \in \mathbb{N}}$ such that $T_j \Delta V_j$ is a finite set for each $j \in \mathbb{N}$ and

$$V = \bigcup_{j=1}^{\infty} V_j \in \mathcal{I}.$$

We prove that,

$$\lim_{\substack{r \to \infty \\ (r \in M')}} \frac{1}{h_r} \sum_{n \in I_r} |x_n - L| = 0,$$

for $M' = \mathbb{N} \backslash V \in \mathcal{F}(\mathcal{I})$. Let $\delta > 0$ be given. Choose $q \in \mathbb{N}$ such that $\frac{1}{q} < \delta$. Then,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n \in I_r} |x_n - L| \geq \delta \right\} \subset \bigcup_{j=1}^{q-1} T_j.$$

Since $T_j \Delta V_j$ is a finite set for $j \in \{1, 2, \cdots, q-1\}$, there exists $r_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{q-1} T_j \right) \cap \{ r \in \mathbb{N} : r \geq r_0 \} = \left(\bigcup_{j=1}^{q-1} V_j \right) \cap \{ r \in \mathbb{N} : r \geq r_0 \}.$$

If $r \geq r_0$ and $r \notin V$, then

$$r \notin \bigcup_{j=1}^{q-1} V_j \text{ and so } r \notin \bigcup_{j=1}^{q-1} T_j.$$

We have

$$\frac{1}{h_r} \sum_{n \in I_r} |x_n - L| < \frac{1}{q} < \delta.$$

This implies that

$$\lim_{\substack{r \to \infty \\ (r \in M')}} \frac{1}{h_r} \sum_{n \in I_r} |x_n - L| = 0.$$

Hence, we have $x_n \to L[\mathcal{I}_\theta^*]$. This completes the proof. □

Definition 2.3. [12]. A sequence (x_n) is said to be lacunary $\mathcal{I}^*$-Cauchy sequence if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ we have

$$\lim_{\substack{r \to \infty \\ (r \in M')}} \frac{1}{h_r} \sum_{k,p \in I_r} (x_{m_k} - x_{m_p}) = 0.$$

Definition 2.4. A sequence (x_n) is said to be strongly lacunary $\mathcal{I}^*$-Cauchy sequence if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ we have

$$\lim_{\substack{r\to\infty \\ (r\in M')}} \sum_{k,p\in I_r} |x_{m_k} - x_{m_p}| = 0.$$

Theorem 2.4. *If the sequence (x_n) is strongly lacunary $\mathcal{I}^*$-Cauchy sequence, then (x_n) is lacunary $\mathcal{I}^*$-Cauchy sequence.*

Proof. Suppose that (x_n) is strongly lacunary $\mathcal{I}^*$-Cauchy sequence. Then, for every $\varepsilon > 0$, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$

$$\frac{1}{h_r} \sum_{k,p\in I_r} |x_{m_k} - x_{m_p}| < \varepsilon, \ \ (r \in M')$$

for every $\varepsilon > 0$ and all $r > r_0 = r_0(\varepsilon)$. Then, we have

$$\begin{aligned} \left| \frac{1}{h_r} \sum_{k,p\in I_r} (x_{m_k} - x_{m_p}) \right| &\leq \frac{1}{h_r} \sum_{k,p\in I_r} |x_{m_k} - x_{m_p}| \\ &< \varepsilon, \ \ (r \in M') \end{aligned}$$

for every $\varepsilon > 0$ and all $r > r_0 = r_0(\varepsilon)$ and so (x_n) is lacunary $\mathcal{I}^*$-Cauchy sequence. □

Theorem 2.5. *If the sequence (x_n) is strongly lacunary $\mathcal{I}^*$-Cauchy sequence, then (x_n) is strongly lacunary $\mathcal{I}$-Cauchy sequence.*

Proof. Suppose that (x_n) is strongly lacunary $\mathcal{I}^*$-Cauchy sequence. Then, for every $\varepsilon > 0$, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$

$$\frac{1}{h_r} \sum_{k,p\in I_r} |x_{m_k} - x_{m_p}| < \varepsilon, \ \ (r \in M')$$

for every $\varepsilon > 0$ and all $r > r_0 = r_0(\varepsilon)$. Let $N = N(\varepsilon) \in I_{r_0+1}$. Then, for every $\varepsilon > 0$ and all $r > r_0 = r_0(\varepsilon)$

$$\frac{1}{h_r} \sum_{k\in I_r} |x_{m_k} - x_N| < \varepsilon, \ \ (r \in M').$$

Now, let $H = \mathbb{N} \setminus M'$. It is clear that $H \in \mathcal{I}$. Then,

$$A(\varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n\in I_r} |x_n - x_N| \geq \varepsilon \right\} \subset H \cup \{1, 2, \cdots, r_0\}.$$

Since $\mathcal{I}$ is an admissible ideal, we have

$$H \cup \{1, 2, \cdots, r_0\} \in \mathcal{I}$$

and so $A(\varepsilon) \in \mathcal{I}$. Hence, (x_n) is strongly lacunary $\mathcal{I}$-Cauchy sequence. □

Theorem 2.6. *If $\mathcal{I}$ admissible ideal with property (AP). The sequence (x_n) is strongly lacunary $\mathcal{I}$-Cauchy sequence, then (x_n) is strongly lacunary $\mathcal{I}^*$-Cauchy sequence.*

Proof. Assume that (x_n) is strongly lacunary $\mathcal{I}$-Cauchy sequence. Then, for every $\varepsilon > 0$ there exists an $N = N(\varepsilon)$ such that

$$A(\varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n\in I_r} |x_n - x_N| \geq \varepsilon \right\} \in \mathcal{I}.$$

Let

$$P_i = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{n\in I_r} |x_n - x_{m_i}| \geq \frac{1}{i} \right\}, \ i = 1, 2, \ldots,$$

where $m_i = N\left(\frac{1}{i}\right)$. It is clear that $P_i \in \mathcal{F}(\mathcal{I})$ for $i = 1, 2, \cdots$. Since $\mathcal{I}$ has the (AP) property, then by Lemma 1.1 there exists a set $P \subset \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I})$ and $P \setminus P_i$ is finite for all i. Now, we show that

$$\lim_{\substack{r\to\infty \\ (r\in P)}} \frac{1}{h_r} \sum_{n,m\in I_r} |x_n - x_m| = 0.$$

To prove this let $\varepsilon > 0$, $j \in \mathbb{N}$ such that $j > \frac{2}{\varepsilon}$. If $r \in P$ then $P \setminus P_j$ is a finite set, so there exists $r_0 = r_0(j)$ such that $r \in P_j$ for all $r > r_0(j)$. Therefore, for all $r > r_0(j)$

$$\frac{1}{h_r} \sum_{n\in I_r} |x_n - x_{m_j}| < \frac{1}{j} \text{ and } \frac{1}{h_r} \sum_{m\in I_r} |x_m - x_{m_j}| < \frac{1}{j}.$$

Hence, for all $r > r_0(j)$ it follows that

$$\begin{aligned} \frac{1}{h_r} \sum_{n,m\in I_r} |x_n - x_m| &\leq \frac{1}{h_r} \sum_{n\in I_r} |x_n - x_{m_j}| + \frac{1}{h_r} \sum_{m\in I_r} |x_m - x_{m_j}| \\ &< \frac{1}{j} + \frac{1}{j} < \varepsilon. \end{aligned}$$

Thus, for any $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon)$ such that for all $r > r_0(\varepsilon)$ and $r \in P \in \mathcal{F}(\mathcal{I})$

$$\frac{1}{h_r} \sum_{n,m\in I_r} |x_n - x_m| < \varepsilon.$$

This shows that the sequence (x_n) is strongly lacunary $\mathcal{I}^*$-Cauchy sequence. □

Theorem 2.7. *If a sequence (x_n) is strongly lacunary $\mathcal{I}^*$-convergent to L, then (x_n) is strongly lacunary $\mathcal{I}$-Cauchy sequence.*

Proof. Let $x_n \to L[\mathcal{I}_\theta^*]$. Then, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ such that for the set $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ we have

$$\lim_{\substack{r\to\infty \\ (r\in M')}} \frac{1}{h_r} \sum_{k\in I_r} |x_{m_k} - L| = 0.$$

It shows that there exists $r_0 = r_0(\varepsilon)$ such that

$$\frac{1}{h_r} \sum_{k\in I_r} |x_{m_k} - L| < \frac{\varepsilon}{2},\ (r \in M')$$

for every $\varepsilon > 0$ and all $r > r_0$. Since

$$\begin{aligned} \frac{1}{h_r} \sum_{k,p\in I_r} |x_{m_k} - x_{m_p}| &\leq \frac{1}{h_r} \sum_{k\in I_r} |x_{m_k} - L| + \frac{1}{h_r} \sum_{p\in I_r} |x_{m_p} - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\ (r \in M') \end{aligned}$$

for all $r > r_0$, so we have

$$\lim_{\substack{r\to\infty \\ (r\in M')}} \frac{1}{h_r} \sum_{k,p\in I_r} |x_{m_k} - x_{m_p}| = 0$$

i.e., (x_n) is a strongly lacunary $\mathcal{I}^*$-Cauchy sequence. Then, by Theorem 2.5 (x_n) is a strongly lacunary $\mathcal{I}$-Cauchy sequence. □

Conclusions and future work

We investigated the concepts of strongly lacunary $\mathcal{I}^*$-convergence and strongly lacunary $\mathcal{I}^*$-Cauchy sequence. These concepts can also be studied for the double sequence in the future.

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