Yuzuncu Yil University
Journal of the Institute of Natural & Applied Sciences
https://dergipark.org.tr/en/pub/yyufbed

Research Article

Gegenbauer Parameter Effect on Gegenbauer Wavelet Solutions of Lane-Emden Equations

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Article Info

Received: 20.07.2023
Accepted: 22.11.2023
Online April 2024

DOI:10.53433/yyufbed.1330540

Keywords
Boundary value problems,
Gegenbauer wavelets,
Initial value problems,
Lane-Emden equations

Abstract: In this study, we aim to solve Lane-Emden equations numerically by the Gegenbauer wavelet method. This method is mainly based on orthonormal Gegenbauer polynomials and takes advantage of orthonormality which reduces the computational cost. As a further advantage, Gegenbauer polynomials are associated with a real parameter allowing them to be defined as Legendre polynomials or Chebyshev polynomials for some values. Although this provides an opportunity to be able to analyze the problem under consideration from a wide point of view, the effect of the Gegenbauer parameter on the solution of Lane-Emden equations has not been studied so far. This study demonstrates the robustness of the Gegenbauer wavelet method on three problems of Lane-Emden equations considering different values of this parameter.

Lane-Emden Denklemlerinin Gegenbauer Dalgacık Çözümleri Üzerinde Gegenbauer Parametresinin Etkisi

Makale Bilgileri

Geliş: 20.07.2023
Kabul: 22.11.2023
Online Nisan 2024

DOI:10.53433/yyufbed.1330540

Anahtar Kelimeler
Başlangıç değer problemleri,
Gegenbauer dalgacıkları,
Lane-Emden denklemleri,
Sınır değer problemleri

Öz: Bu çalışmada Lane-Emden denklemlerini Gegenbauer dalgacık yöntemi ile sayısal olarak çözmeyi amaçlıyoruz. Bu yöntem temel olarak ortonormal Gegenbauer polinomlarına dayanır ve hesaplama maliyetini azaltan ortonormallik avantajını kullanır. Diğer bir avantaj olarak, Gegenbauer polinomları, bazı değerleri için Legendre polinomları veya Chebyshev polinomları olarak tanımlanabilmelerini sağlayan reel bir parametre ile ilişkilendirilir. Bu durum, ele alınan problemi geniş bir bakış açısıyla analiz edebilmek için bir fırsat sağlasa da Gegenbauer parametresinin Lane-Emden denklemlerinin çözümü üzerindeki etkisi şimdiye kadar çalışılmamıştır. Bu çalışma, bu parametrenin farklı değerlerini dikkate alarak Gegenbauer dalgacık yönteminin Lane-Emden denklemlerinin üç problemi üzerindeki doğruluğunu göstermektedir.

1. Introduction

Lane-Emden equations which have been originated from the studies of the astrophysicists Jonathan Homer Lane and Robert Emden about understanding the structure of the stars, are nonlinear ordinary differential equations arising in interdisciplinary branches of the mathematical physics and the astrophysics such as thermal explosions, stellar structure, radiative cooling, isothermal gas spheres, the thermal behavior of a spherical cloud of gas and thermionic currents (Lane, 1870; Emden, 1907; Chambre, 1952; Davis, 1962; Chandrasekhar, 1967; Shawagfeh, 1993; Lima & Morgado, 2010).

These equations can be modelled as an initial value problem using the second-order differential equation

$$y''(x) + \frac{c}{x}\, y'(x) + F\big(x, y(x)\big) = g(x), \quad x \in (0,1], \tag{1}$$

with the initial conditions

$$y(0) = \alpha_1, \;\; y'(0) = \beta_1 \tag{2}$$

or as a boundary value problem if the differential equation Eq.(1) is subject to boundary conditions of different types

$$y(0) = \alpha_2, \qquad y(1) = \beta_2, \tag{3}$$

$$y'(0) = \alpha_3, \qquad y'(1) = \beta_3, \tag{4}$$

$$y'(0) = \alpha_4, \qquad \gamma_1 y(1) + \gamma_2 y'(1) = \beta_4, \tag{5}$$

where c, α_j, β_j, for $j = 1,2,3,4$; and γ_1, γ_2 are given real numbers, the nonlinear term $F(x, y(x))$ and $g(x)$ are given real-valued functions.

The initial value problem of the differential equation Eq.(1) and the initial conditions in Eq.(2) when $c = 2$, $g(x) = 0$, and $F\big(x, y(x)\big) = y^n$, for a physical parameter n in the range $0 \leq n \leq 5$, is one of the very well-known cases studied by many researchers and the analytical solutions for $n = 0,1,5$ and the numerical solutions for other values of n are obtained (Chandrasekhar, 1967; Shawagfeh, 1993; Wazwaz, 2001; Parand et al., 2010; Kumar et al., 2011; Pandey & Kumar, 2012; Singh & Kumar, 2014; Gümgüm, 2020). Another most studied problem is a similar initial value problem with a nonlinear term $F\big(x, y(x)\big) = e^y$ which is also known as isothermal gas spheres equation (Davis, 1962) and solved by Adomian decomposition method (Wazwaz, 2001), quasilinearization methods (Mandelzweig & Tabakin, 2001; Krivec & Mandelzweig, 2008), Lagrangian based analytical method (Khalique & Ntsime, 2008), Homotopy Analysis method (Liao, 2003; Van Gorder & Vajravelu, 2008), Variational Iteration method (Yildirim & Öziş, 2009), Bernstein Polynomial operational matrix of integration and differentiation methods (Kumar et al., 2011; Pandey & Kumar, 2012).

It is known that the analytical solutions exist for some types of the Lane-Emden equations. For the other cases, approximate solutions have been obtained by numerical techniques some of which are mentioned above. Especially the collocation methods based on the polynomials such as the very well-known Chebyshev polynomials and the Legendre polynomials are efficiently used to solve these problems, due to their easy implementation and accurate results (Yousefi, 2006; Adibi & Rismani, 2010; Yüzbaşı, 2011; Doha et al., 2013; Yüzbaşı & Sezer, 2013; Gürbüz & Sezer, 2014; Öztürk & Gülsu, 2014; Shiralashetti & Kumbinarasaiah, 2017; Öztürk, 2018; Ahmed, 2023; İdiz et al., 2023). In this sense, the proposed method provides a wider point of view. Since the Gegenbauer wavelet method is based on the Gegenbauer polynomials and these polynomials are associated with a parameter giving the Chebyshev polynomials and the Legendre polynomials for some values. Thus, this method enables to obtain more accurate solutions by investigation of the best value of the Gegenbauer parameter.

In this study, we deal with the implementation of the Gegenbauer wavelets and finding the best value of Gegenbauer parameter to solve the problems of the Lane-Emden equations through the residual error analysis. The paper is organized as follows. In Section 2, we define Gegenbauer polynomials and how to construct Gegenbauer wavelets. In Section 3, the method is implemented to solve the Lane-Emden equations with the given initial conditions. In Section 4, we illustrate the use of Gegenbauer wavelets for finding the solutions of several differential equations modelling different physical phenomena. In order to prove the efficiency of the method we first solved problems with analytical solutions and verify that the method produces either analytical solutions or accurate numerical solutions. Afterwards, we solved problems without analytical solutions and compared our results with the existing ones.

2. Gegenbauer Polynomials and Gegenbauer Wavelets

The Gegenbauer polynomials or ultraspherical polynomials, G_m^ν(x), are the eigensolutions of the following Sturm Liouville equation defined on the interval [-1,1] as follows (Kumar et al., 2019)

$$\frac{d}{dx}\left[(1-x^2)^{\nu+\frac{1}{2}}\frac{dy}{dx}\right]+m(m+2\nu)(1-x^2)^{\nu-\frac{1}{2}}y=0,$$

where $\nu > -\frac{1}{2}$ is the Gegenbauer parameter and $m \in \mathbb{Z}^+$ is the order of the polynomial. For $\nu = 0,\ \nu = 1$ and $\nu = \frac{1}{2}$, Gegenbauer polynomials generate very well-known Chebyshev polynomials of the first and second kind $T_m(x),\ U_m(x)$ and Legendre polynomials $P_m(x)$, respectively. That is

$$G_m^{\frac{1}{2}}(x) = P_m(x),\quad G_m^1(x) = U_m(x)\ \text{ and }\ \lim_{\nu\to 0}\frac{1}{\nu}G_m^\nu(x) = T_m(x),\ m \in \mathbb{N}.$$

The Gegenbauer polynomials can be obtained recursively by the relation

$$(m+1)G_{m+1}^\nu(x) = 2x(m+\nu)G_m^\nu(x) - (m+2\nu-1)G_{m-1}^\nu(x),$$

with the first two values $G_0^\nu(x) = 1$ and $G_1^\nu(x) = 2\nu x$. As the basic property, these polynomials are orthogonal on $[-1,1]$ with respect to the weight function $w(x) = (1-x^2)^{\nu-\frac{1}{2}}$ that is

$$\int_{-1}^{1}(1-x^2)^{\nu-\frac{1}{2}}G_m^\nu(x)G_n^\nu(x)\,dx = L_m^\nu\delta_{mn}\,,$$

where δ_{mn} is Kronecker delta and $L_m^\nu = \frac{\pi 2^{1-2\nu}\Gamma(m+2\nu)}{m!(m+\nu)[\Gamma(\nu)]^2}$ is the normalization factor and hence they constitute an orthogonal basis in Hilbert space $L^2[-1,1]$. Further information about the properties of the Gegenbauer polynomials can be found in (Reimer, 2003; Kim et al., 2012).

Gegenbauer wavelets are a family of functions, constructed from Gegenbauer polynomials $G_m^\nu(x)$, together with five arguments k, m, n, ν, x as

$$\psi_{nm}(x) = \begin{cases}\frac{1}{L_m^\nu}2^{k/2}G_m^\nu\left(2^k x - 2n + 1\right), & x \in \left[\frac{n-1}{2^k-1}, \frac{n}{2^{k-1}}\right], \\ 0, & otherwise\,,\end{cases} \tag{6}$$

where $k = 1, 2, 3, \ldots$ is the level of resolution, $n = 1, 2, \ldots, 2^{k-1}$ is the translation parameter and $x \in [0,1]$.

Any function $u(x) \in L^2[0,1]$ can be expressed in terms of Gegenbauer wavelet coefficient obtained by the inner product

$$a_{nm} = \langle u(x), \psi_{nm}(x)\rangle_{\omega_n} = \int_0^1 \omega_n(x)u(x)\psi_{nm}(x)\,dx\,,$$

for a shifted weight function $\omega_n(x) = w\left(2^k x - 2n + 1\right)$. For an approximation, we truncate the series above as

$$u(x) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} a_{nm}\,\psi_{nm}(x) = \boldsymbol{a}^T\boldsymbol{\phi}(\boldsymbol{x})\,, \tag{7}$$

where $\boldsymbol{a}$ and $\boldsymbol{\phi}(\boldsymbol{x})$ are $2^{k-1}M \times 1$ matrices in the form

$$\boldsymbol{a} = \left[a_{10}, a_{11}, a_{12}, \dots, a_{1(M-1)}, a_{20}, a_{21}, a_{22}, \dots, a_{2(M-1)}, \dots, a_{2^{k-1}0}, a_{2^{k-1}1}, \dots, a_{2^{k-1}(M-1)}\right],$$
$$\boldsymbol{\phi}(\boldsymbol{x}) = \left[\psi_{10}, \psi_{11}, \psi_{12}, \dots, \psi_{1(M-1)}, \psi_{20}, \psi_{21}, \psi_{22}, \dots, \psi_{2(M-1)}, \dots, \psi_{2^{k-1}0}, \psi_{2^{k-1}1}, \dots, \psi_{2^{k-1}(M-1)}\right].$$

Theorem 1: The Gegenbauer wavelet series expansion $\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{nm}\, \psi_{nm}(x)$ converges to $u(x)$, as 2^{k-1} and M approache to ∞.
Proof: Let $L^2[0,1]$ denote the Hilbert space. Since the Gegenbauer wavelets form an orthonormal basis in $L^2[0,1]$ any function $u(x)$ can be expanded by the series $u(x) = \sum_{j=0}^{M-1} a_{\rho j}\psi_{\rho j}(x)$ for a fixed $n = \rho$, where $1 < \rho < 2^{k-1}$ and $a_{\rho j} =< u(x), \psi_{\rho j}(x) >$.

A partial sum S_n of the sequence $\{a_{\rho j}\psi_{\rho j}\}_{j=0}^{n}$, for some $m < n < M-1$, is defined in the form $S_n = \sum_{j=1}^{n} a_{\rho j}\psi_{\rho j}(x)$. Here, the main goal for verifying the convergence is to show that S_n is a Cauchy sequence in Hilbert space. It is clear that $S_n - S_m = \sum_{j=m+1}^{n} a_{\rho j}\psi_{\rho j}(x)$ for $m < n < M-1$. With the help of orthogonality of Gegenbauer wavelets and Bessel's inequality (Arfken & Weber, 2005)

$$\|s_n - S_m\|^2 = \left\| \sum_{j=m+1}^{n} a_{\rho j}\psi_{\rho j}(x) \right\|^2 =< \sum_{j=m+1}^{n} a_{\rho j}\psi_{\rho j}(x), \sum_{i=m+1}^{n} a_{\rho i}\psi_{\rho i}(x) >$$
$$= \sum_{j=m+1}^{n} \sum_{i=m+1}^{n} a_{\rho j} a_{\rho i} < \psi_{\rho j}(x), \psi_{\rho i}(x) >,$$

for real $a_{\rho i}$, then the inner product in the last equation gives 1 when $i = j$. Hence the last summation equals to

$$\sum_{j=m+1}^{n} |a_{\rho j}|^2 \le \sum_{j=1}^{n} |a_{\rho j}|^2 \le \|u(x)\|^2 .$$

This implies that $\sum_{j=m+1}^{n} |a_{\rho j}|^2$ is bounded and therefore, $\|S_n - S_m\|^2 = \sum_{j=m+1}^{n} |a_{\rho j}|^2$ is convergent as $m, n \to \infty$. Hence, S_n is a Cauchy sequence in Hilbert space and therefore it converges to a sum $u(x)$.
Theorem 2: Let $\Omega_n = Span\{\psi_{n0}, \psi_{n1}, \psi_{n2}, \dots \psi_{n(M-1)}\}$ and $f(x)$ be a real valued function such that $f(x) \in C^M[0,1]$. Assume that $f(x) = \sum_{n=1}^{2^{k-1}} f_n(x)$ and if $a_n^T\phi_n(x)$ is the best approximation of $f_n(x)$ then $\boldsymbol{a}^{\boldsymbol{T}}\boldsymbol{\phi}(\boldsymbol{x})$ approximates $f(x)$ with the error bound:

$$\left\|f(x) - \boldsymbol{a}^{\boldsymbol{T}}\boldsymbol{\phi}(\boldsymbol{x})\right\| \le \frac{\varepsilon}{M!\, 2^{M(k-1)}\sqrt{2M+1}},$$

where $\varepsilon = \max_{x \in [0,1]} |f^M(x)|$.
Proof: The proof is based on the Taylor series expansion of $f_n(x)$ about $\frac{n-1}{2^{k-1}}$

$$\widetilde{f_n}(x) = f_n\left(\frac{n-1}{2^k - 1}\right) + \dots + \frac{1}{(M-1)!} f_n^{(M-1)}\left(\frac{n-1}{2^{k-1}}\right)\left(x - \frac{n-1}{2^{k-1}}\right)^{M-1},$$

then since $\frac{n-1}{2^{k-1}} < x < \frac{n}{2^{k-1}}$

$$\left|f_n(x)-\widetilde{f_n}(x)\right| \le \frac{1}{M!} f_n^{(M)}(x)\left(x-\frac{n-1}{2^{k-1}}\right)^M.$$

Since $a_n^T\phi_n(x)$ is the best approximation of $f_n(x)$,

$$\left\|f_n(x)-\boldsymbol{a^T\phi(x)}\right\|^2 \le \sum_{n=1}^{2^{k-1}} ||\, f_n(x)-a_n^T\phi_n(x)||^2 \le \sum_{n=1}^{2^{k-1}} ||\, f_n(x)-\widetilde{f_n}(x)||^2 \le \frac{2^{2M}\varepsilon^2}{(M!)^2 2^{2Mk}(2M+1)}$$

gives the desired result. For more details, see Usman et al. (2019).

2.1. Operational matrix of derivative

The derivatives of the vector $\boldsymbol{\phi(x)}$ defined in Eq.(7) can be obtained by means of an operational matrix D as

$$\frac{d^i}{dx^i}\boldsymbol{\phi(x)} = D^i\boldsymbol{\phi(x)}\,, \tag{8}$$

where $\boldsymbol{\phi(x)}$ is the vector of Gegenbauer wavelets, D^i is the i -th power of the $2^{k-1}M \times 2^{k-1}M$ operational matrix for differentiation, derived in Usman et al. (2019) as

$$D = \begin{pmatrix} F & 0 & 0 & \cdots 0 \\ 0 & F & 0 & \dots 0 \\ \vdots & \vdots & \vdots & \dots \vdots \\ 0 & 0 & 0 & \dots F \end{pmatrix}. \tag{9}$$

Here F is a $M \times M$ submatrix whose (r,s)-th component

$$F_{rs} = \begin{cases} \dfrac{2^{k+1}(s+\nu-1)\sqrt{(r-1+\nu)\Gamma(r)\Gamma(s-1+2\nu)}}{\sqrt{(s-1+\nu)\Gamma(s)\Gamma(r-1+2\nu)}}, & r=2,\dots,M;\ s=1,\dots,(r-1) \text{ and } r+s \text{ is odd}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\Gamma(.)$ is the Gamma function.

3. Application of Gegenbauer Wavelets to Lane-Emden Equations

We assume that the solution of the differential equation Eq. (1) has the form

$$y(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{nm}\,\psi_{nm}(x) = \boldsymbol{a^T\phi(x)},$$

where a_{nm} are the unknown coefficients for $n=1,\dots,2^{k-1}$; $m=0,\dots,M-1$; $\boldsymbol{a}$ and $\boldsymbol{\phi(x)}$ are vectors defined in Eq. (7). Our aim is to find these $2^{k-1}M$ unknown coefficients to obtain the solution. For this we need the derivatives

$$y'(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{nm}\,{\psi'}_{nm}(x) = \boldsymbol{a^T\phi'(x)} = \boldsymbol{a^T D\phi(x)},$$
$$y''(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{nm}\,{\psi''}_{nm}(x) = \boldsymbol{a^T\phi'(x)} = \boldsymbol{a^T D^2\phi(x)},$$

in order to replace into the differential equation in Eq. (1) as

$$\boldsymbol{a^T D^2\phi(x)} + \frac{\boldsymbol{c}}{\boldsymbol{x}}\,\boldsymbol{a^T D\phi(x)} + \boldsymbol{F}\big(\boldsymbol{x}, \boldsymbol{a^T\phi(x)}\big) = \boldsymbol{g(x)}. \tag{10}$$

This is an algebraic equation in terms of x involving unknown coefficients. But in order to obtain all of these coefficients explicitly, we need $2^{k-1}M$ algebraic equations. Two of these equations are provided by the initial or boundary conditions in Eq. (3) as

$$y(0) = \boldsymbol{a^T \phi(0)} = \alpha_1, \qquad y'(0) = \boldsymbol{a^T D \phi(0)} = \beta_1,$$

or

$$\begin{gathered} y(0) = \boldsymbol{a^T \phi(0)} = \alpha_2, \qquad y(1) = \boldsymbol{a^T \phi(1)} = \beta_2, \\ y'(0) = \boldsymbol{a^T D \phi(0)} = \alpha_3, \qquad y'(1) = \boldsymbol{a^T D \phi(1)} = \beta_3, \\ y'(0) = \boldsymbol{a^T D \phi(0)} = \alpha_4,\ \gamma_1 y(1) + \gamma_2 y'(1) = \gamma_1 \boldsymbol{a^T \phi(1)} + \gamma_2 \boldsymbol{a^T D \phi(1)} = \beta_4 \ . \end{gathered}$$

The remaining $2^{k-1}M - 2$ equations are provided by replacing the first $2^{k-1}M - 2$ roots x_i of very well-known Chebyshev polynomials as collocation points in Eq. (10)

$$\boldsymbol{a^T D^2 \phi(x_i)} + \frac{\boldsymbol{c}}{\boldsymbol{x_i}} \boldsymbol{a^T D \phi(x_i)} + \boldsymbol{F}\left(\boldsymbol{x_i}, \boldsymbol{a^T \phi(x_i)}\right) = \boldsymbol{g(x_i)},$$

for $i = 1,2, \dots, 2^{k-1}M - 2$. The obtained equations form a system of $2^{k-1}M$ algebraic equations to be solved for a_{nm} by Matlab tools.

4. Numerical Examples

This section illustrates the use of the current method to solve three problems. All of these problems are solved by a code, including *fsolve* function, written in Matlab R2021b and the computations are performed in Lenovo AMD Ryzen 7 3700U 8 GB RAM computer.

In the first example, we consider a problem with analytical solution to examine the effectiveness of the current method. We show that when the exact solution is polynomial, the method produces the solution itself. In the second and third examples, we continue with problems without exact solutions. Compared the results with the ones in the literature and via the residual error analysis, we check the robustness of the method and present the effect of Gegenbauer parameter on the solution.

4.1. Example 1

The first example is a second order linear and nonhomogeneous problem (Mall & Chakraverty, 2015)

$$\begin{gathered} y''(x) + \frac{8}{x} y'(x) + xy(x) = x^5 - x^4 + 44x^2 - 30x, \quad x \in (0,1], \\ y(0) = y'(0) = 0. \end{gathered} \tag{11}$$

This problem has an analytical solution of fourth degree polynomial $y(x) = x^4 - x^3$, so we used a fourth order polynomial with randomly chosen $\nu = 2.5$ and approximated the function and its derivatives as follows

$$\begin{gathered} y(x) = \frac{\sqrt{30}}{4} a_{10} + \frac{3\sqrt{5}}{2} a_{11}(2x-1) + \sqrt{1155} a_{12}\left(x^2 - x + \frac{3}{14}\right) + \frac{\sqrt{2730}}{4} a_{13}(12x^3 - \\ 18x^2 + 8x - 1) + \sqrt{70} a_{14}\left(\frac{165}{2}x^4 - 165x^3 + \frac{225}{2}x^2 - 30x + \frac{5}{2}\right) , \\ y'(x) = 3\sqrt{5} a_{11} + \sqrt{1155} a_{12}(2x-1) + \sqrt{2730} a_{13}(9x^2 - 9x + 2) + \\ \sqrt{70} a_{14}(330x^3 - 495x^2 + 225x - 30) , \\ y''(x) = 2\sqrt{1155} a_{12} + \sqrt{2730} a_{13}(18x - 9) + \sqrt{70} a_{14}(990x^2 - 990x + 225) , \end{gathered} \tag{12}$$

where $a_{10}, a_{11}, a_{12}, a_{13}, a_{14}$ are the unknown coefficients. To determine these coefficients, we firstly substitute Eq. (12) into Eq. (11) and insert the first three roots of the Chebyshev polynomial into the resulting equation. The obtained equations and the initial conditions form a system of five equations. This system is solved by *fsolve* function of Matlab to obtain

$$a_{10} = -0.0435,\ a_{11} = -0.0248,\ a_{12} = 0.0040,\ a_{13} = 0.0064,\ a_{14} = 0.0014,$$

which results in the analytic solution.

4.2. Example 2

The second problem is used in modelling the oxygen diffusion in a spherical cell (Wazwaz, 2011) as well as heat conduction through a solid (Lima & Morgado, 2010)

$$y''(x) + \frac{2}{x}\, y'(x) - \frac{ny(x)}{y(x)+k} = 0, \quad x \in (0,1],$$
$$y'(0) = 0, \qquad 5y(1) + y'(1) = 5,$$

where $n = 0.76129,\ k = 0.03119$ and $f(x,y) = \frac{ny(x)}{y(x)+k}$ is the heat generating function.

This problem does not have an exact solution. In order to check the robustness of the proposed method, we define residual error function $ER(x_i)$ in the form

$$ER(x_i) = y_M''(x_i) + \frac{2}{x_i}\, y_M'(x_i) - \frac{ny_M(x_i)}{y_M(x_i)+k} = 0,$$

where $y_M(x_i)$ is the approximate solution at the collocation points. First, the problem is solved for several values of Gegenbauer parameter ν then the best value of this parameter is investigated through the residual errors $ER(x_i)$ at $x_i \in (0,1]$. The parameter giving the smallest value of $ER(x_i)$ for $x_i \in (0,1]$ is determined as the best parameter for the problem of the Lane-Emden equation.

Figure 1 and Table 1 show the values of the error $ER(x_i)$ of the method for different values of ν. One can see from this figure and this table that the errors are increasing when the parameter ν is increasing and the smallest errors are obtained for ν =-0.49.

Table 1. Error $ER(x_i)$ obtained from different values of ν at the points x_i

x_i	$ER(x_i)$ with $\nu = -0.49$	$ER(x_i)$ with $\nu = 2.5$	$ER(x_i)$ with $\nu = 5$	$ER(x_i)$ with $\nu = 10$	$ER(x_i)$ with $\nu = 30$
0	1.103257e-06	2.398957e-05	8.779339e-05	4.480180e-04	9.597005e-03
0.1	5.843838e-07	2.156675e-06	1.304410e-06	1.878467e-05	8.856819e-04
0.2	3.446693e-07	1.256834e-06	5.725870e-06	2.927864e-05	5.641982e-04
0.3	7.557322e-07	1.953095e-06	2.625605e-06	2.222495e-07	1.831005e-04
0.4	8.237166e-09	5.505205e-07	1.483441e-06	4.563612e-06	2.464657e-05
0.5	1.019222e-06	2.249278e-06	3.376932e-06	5.843993e-06	1.757727e-05
0.6	1.002441e-06	8.700669e-07	1.599169e-07	4.661690e-06	4.216164e-05
0.7	6.384774e-07	4.731577e-06	8.729478e-06	8.988542e-06	5.150524e-04
0.8	2.424914e-06	1.049723e-06	1.845242e-05	1.708220e-04	4.978212e-03
0.9	6.172315e-07	4.710728e-05	1.816439e-04	9.517936e-04	2.070513e-02
1	1.850748e-05	2.048626e-04	6.627298e-04	3.078282e-03	6.062056e-02

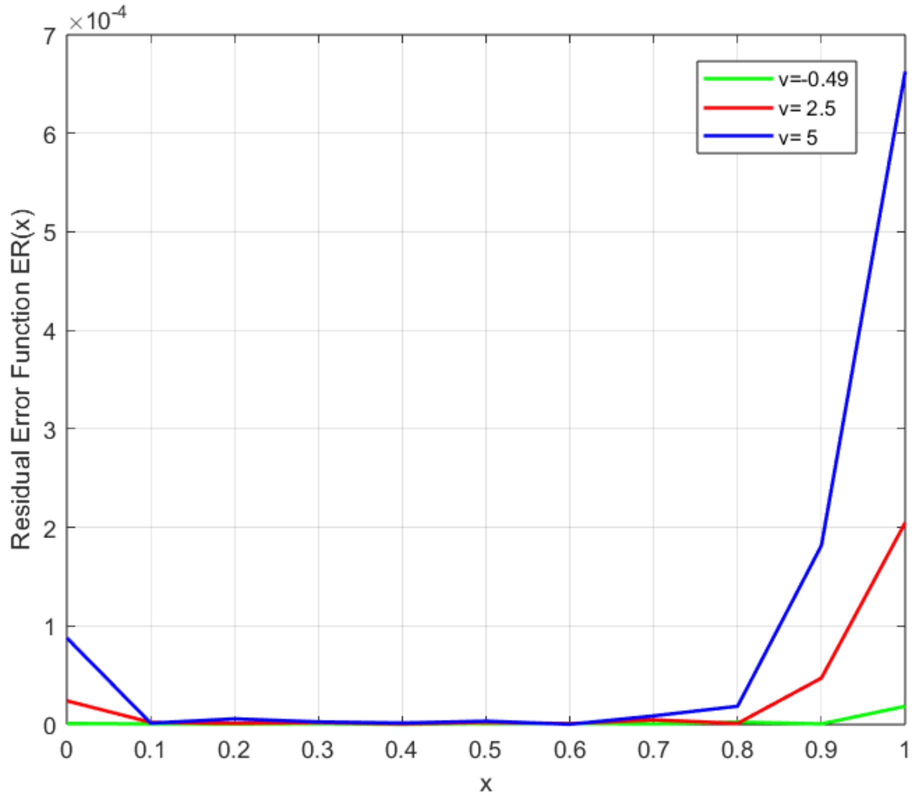

Figure 1. Error $ER(x_i)$ obtained from different values of ν are shown.

Hence, we solved this problem by taking $\nu = -0.49$ and M = 7 which corresponds to a 6th order polynomial. Table 2 presents the results of the current method and other numerical methods. Here TWM (Gümgüm, 2020) and VIM (Wazwaz, 2011) are respectively Taylor wavelet method and the variational iteration method. Gümgüm (2020) and Wazwaz (2011) used 6th degree polynomials. Çağlar et al. (2009) used B-Spline method (BSM) with n = 40 collocation points and Mohsenyzadeh et al. (2015) used Bernoulli polynomials of 14th degree (BP).

These results show that the first six decimal places in all numerical solutions are compatible. Taking into account of both the polynomial degrees and the amount of the utilized collocation points used, one can observe that the current method is as efficient as TWM and VIM and more powerful than BSM and BP.

Table 2. Comparison of the current method with other numerical methods

x_i	GWM, $y_6(x_i)$	TWM, $y_6(x_i)$	VIM, $y_6(x_i)$	BSM, n = 40	BP, $y_{14}(x_i)$
0	0.8284832928	0.8284835573	0.8284832761	0.8284832512	0.8284832818
0.1	0.8297060995	0.8297063594	0.8297060781	0.8297060537	0.8297060924
0.2	0.8333747216	0.8333750006	0.8333747193	0.8333746961	0.8333747335
0.3	0.8394899053	0.8394901811	0.8394898996	0.8394898784	0.8394899139
0.4	0.8480527944	0.8480530523	0.8480527701	0.8480527521	0.8480527849
0.5	0.8590649470	0.8590651943	0.8590649108	0.8590648975	0.8590649271
0.6	0.8725283313	0.8725285872	0.8725282997	0.8725282940	0.8725283199
0.7	0.8884452986	0.8884455738	0.8884452781	0.8884452838	0.8884453056
0.8	0.9068185356	0.9068188164	0.9068185095	0.9068185305	0.9068185480
0.9	0.9276509944	0.9276512446	0.9276509392	0.9276509752	0.9276509883
1	0.9509458010	0.9509459977	0.9509457539	0.9509457898	0.9509457984

These results can be analyzed better in Figure 2. One can see that they all coincide.

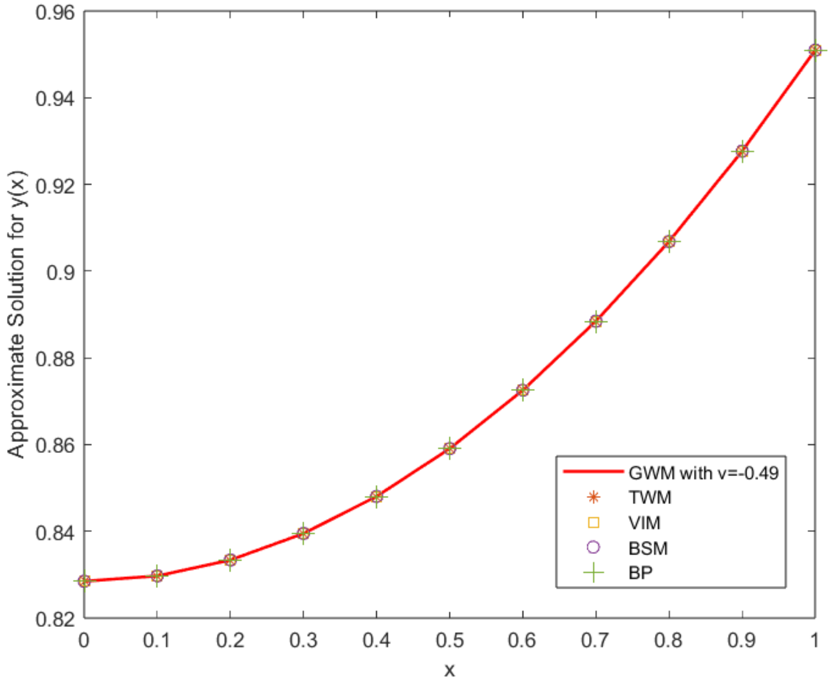

Figure 2. Numerical results of the mentioned methods.

4.3. Example 3 The final example is defined as in (Wazwaz, 2001)

$$y''(x) + \frac{2}{x}\, y'(x) + \sinh(y) = 0,$$
$$y(0) = 1, \;\; y'(0) = 0, \qquad x \geq \; 0.$$

This problem does not have an exact solution. In order to check the robustness of the current method, we define residual error function in the form

$$ER(x_i) = y_M''(x_i) + \frac{2}{x_i}\, y_M'(x_i) + \sinh\,(y_M(x_i)) = 0,$$

where $y_M(x_i)$ is the approximate solution at the collocation points. We solved the problem for different values of the Gegenbauer parameter ν then the best value of this parameter is investigated through the residual errors $ER(x_i)$ at $x_i \in (0,1]$. The parameter giving the smallest value of $ER(x_i)$ for $x_i \in (0,1]$ is determined as the best parameter for the problem of the Lane-Emden equation.

Table 3. Error $ER(x_i)$ obtained from different values of ν

x_i	$ER(x_i)$ with $\nu = -0.49$	$ER(x_i)$ with $\nu = 2.5$	$ER(x_i)$ with $\nu = 6$
0	2.635574e-17	1.481564e-17	2.616733e-17
0.1	5.963773e-11	5.963773e-11	5.963773e-11
0.2	5.585695e-11	5.585731e-11	5.585726e-11
0.3	6.826894e-11	6.826894e-11	6.826889e-11
0.4	7.205352e-11	7.205341e-11	7.205330e-11
0.5	6.091321e-10	6.091326e-10	6.091327e-10
0.6	2.464924e-10	2.464879e-10	2.464878e-10
0.7	1.377397e-09	1.377369e-09	1.377369e-09
0.8	2.255828e-09	2.255956e-09	2.255957e-09
0.9	2.100228e-09	2.099742e-09	2.099742e-09
1	3.012828e-08	3.012983e-08	3.012983e-08

Table 3 shows the values of the error $ER(x_i)$ of the current method for different values of ν. As one can see in Table 3 the errors do not much differ for several values of ν in this example. For this reason, we solved the problem with $\nu = -0.49$, as we did in the previous example. As earlier studies, Parand et al. (2010) solved this problem by Hermite collocation method (HCM) and Wazwaz (2001) used Adomian Decomposition method (ADM) to obtain a series solution of the form

$$y(x) = 1 - \frac{e^2 - 1}{12e} x^2 + \frac{e^4 - 1}{480e^2} x^4 - \frac{2e^6 + 3e^2 - 3e^4 - 2}{30240e^3} x^6 + \frac{61e^8 - 104e^6 + 104e^2 - 61}{26127360e^4} x^8.$$

The results of the current method for $M = 9$ which correspond to 8th degree polynomial, ADM in Wazwaz (2001) and HCM with $M = 10$ in Parand et al. (2010) are given in Table 4. We present the comparison of the values at some points that we can find in (Parand et al., 2010)

Table 4. Comparison of the current method with other numerical methods

x_i	GWM	ADM	HCM
0	1.0000000000	1.0000000000	1.0000000000
0.1	0.9980428414	0.9980428414	0.9981138095
0.2	0.9921894348	0.9921894348	0.9922758837
0.5	0.9519610927	0.9519611019	0.9520376245
1	0.8182429284	0.8182516669	0.8183047481

By Table 4, we observe that at least the first two decimal places in all numerical solutions are compatible while the results of the proposed method and Wazwaz (Wazwaz, 2001) match up to at least 4 decimal places. For better comparison, the series solution of Wazwaz (Wazwaz, 2001) can be embedded into the residual error function in the place of $y_M(x)$. Figure 3 presents the comparison of the errors obtained by the current method with $\nu = -0.49$ and by ADM (Wazwaz, 2001).

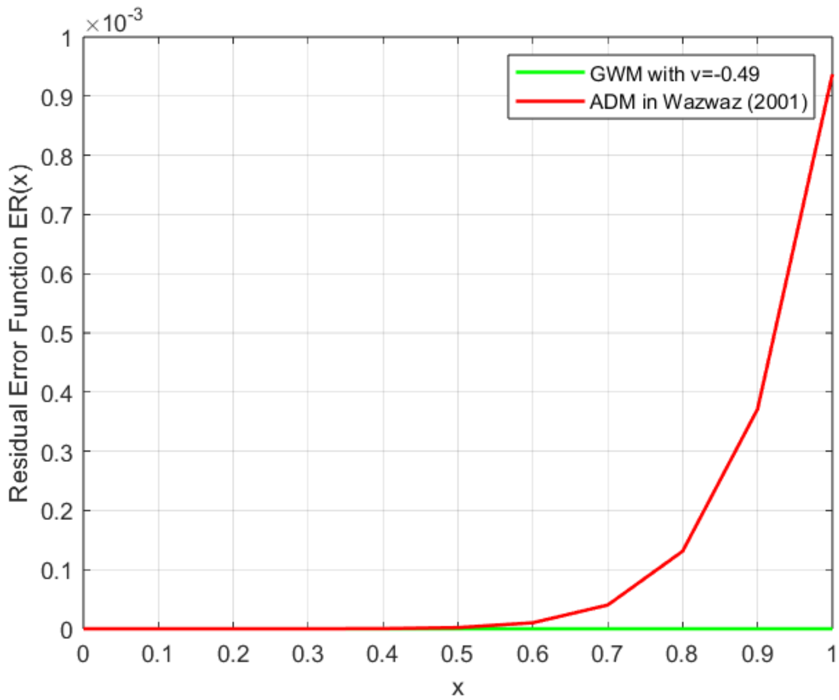

Figure 3. The errors obtained by the proposed method and ADM in Wazwaz (2001) are shown.

As one can observe in the tables and figures the proposed method is more efficient than the other two methods.

5. Conclusion

In this study, several singular linear and nonlinear problems are treated by the proposed method. Error and convergence analysis of the method is given. Two main advantages of this method are that, unlike some numerical techniques, there is no need to linearize the nonlinear terms and to discretize the domain. Therefore, the computation cost is less and the method is quite easy to implement. We simply replace the unknown function and its derivatives with the approximating function and its derivatives and reduce the equation to a system of nonlinear algebraic equations. We observe that an analytical solution is possible to attain once the solution is a polynomial. Comparing the results to variants shows that the method is efficient, reliable and highly accurate.

Acknowledgements

I would like to thank Nilgün Dungan Jr. for proofreading the article. No funding is available for this research.

References

Adibi, H., & Rismani, A. M. (2010). On using a modified Legendre-spectral method for solving singular IVPs of Lane–Emden type. *Computers and Mathematics with Applications*, 60, 2126-2130. doi:10.1016/j.camwa.2010.07.056

Ahmed, H. M. (2023). Numerical solutions for singular Lane-Emden equations using shifted Chebyshev polynomials of first kind. *Contemporary Mathematics*, 4(1), 132-149. doi:10.37256/cm.4120232254

Arfken, G. B., & Weber, H. J. (2005). *Mathematical Methods for Physicists* (6th Ed.). London: Elsevier Academic Press.

Chambre, P. L. (1952). On the solution of the Poisson-Boltzmann equation with application to the theory of thermal explosions. *The Journal of Chemical Physics*, 20, 1795-1797. doi:10.1063/1.1700291

Chandrasekhar, S. (1967). *Introduction to the Study of Stellar Structure.* New York: Dover.

Çağlar, H., Çağlar, N., & Özer, M. (2009). B-spline solution of non-linear singular boundary value problems arising in physiology. *Chaos, Soliton & Fractals*, 39, 1232-1237. doi:10.1016/j.chaos.2007.06.007

Davis, H.T. (1962). *Introduction to Nonlinear Differential and Integral Equations*. New York: Dover,

Doha, E. H, Abd-Elhameed, W. M., & Youssri, Y. H. (2013). Second kind Chebyshev operational matrix algorithm for solving differential equations of Lane-Emden type. *New Astronomy*, 23-24, 113-117. doi:10.1016/j.newast.2013.03.002

Emden, R. (1907). *Gaskugeln: Anwendungen der Mechanischen Warmetheorie auf Kosmologische und Meteorologische Probleme*. Berlin: Teubner.

Gümgüm, S. (2020). Taylor wavelet solution of linear and nonlinear Lane-Emden equations. *Applied Numerical Mathematics*, 158, 44-53. doi:10.1016/j.apnum.2020.07.019

Gürbüz, B., & Sezer, M. (2014). Laguerre polynomial approach for solving Lane-Emden type functional differential equations. *Applied Mathematics and Computation*, 242, 255-264. doi:10.1016/j.amc.2014.05.058

İdiz, F., Tanoğlu, G., & Aghazadeh, N. (2023). A numerical method based on Legendre wavelet and quasilinearization technique for fractional Lane-Emden type equations. *Numerical Algorithms*, 95, 181-206. doi:10.1007/s11075-023-01568-z

Khalique, C. M., & Ntsime, P. (2008). Exact solutions of the Lane-Emden-type equations. *New Astronomy*, 13(7), 476-480. doi:10.1016/j.newast.2008.01.002

Kim, D. S., Kim, T., & Rim, S. H. (2012). Some identities involving Gegenbauer polynomials. *Advances in Difference Equations*, 2012, 219. doi:10.1186/1687-1847-2012-219

Krivec, R., & Mandelzweig, V. B. (2008). Quasilinearization approach to computations with singular potentials. *Computer Physics Communications*, 179(12), 865-867. doi:10.1016/j.cpc.2008.07.006

Kumar, N., Pandey, R. K., & Cattani, C. (2011). Solution of the Lane-Emden Equation Using the Bernstein Operational Matrix of Integration. *ISRN Astronomy and Astrophysics*, 2011, 351747. doi:10.5402/2011/351747

Kumar, S., Pandey, P., & Das, S. (2019). Gegenbauer wavelet operational matrix method for solving variable-order non-linear reaction–diffusion and Galilei invariant advection–diffusion equations. *Computational and Applied Mathematics*, 162, 1-22. doi:10.1007/s40314-019-0952-z

Lane, J. H. (1870). On the theoretical temperature of the sun, under the hypothesis of a gaseous mass maintaining its internal heat and depending on the laws of gases known to terrestrial experiment. *The American Journal of Science and Arts*, 50, 57-74. doi:10.2475/ajs.s2-50.148.57

Liao, S. J. (2003). A new analytic algorithm of Lane-Emden type equations. *Applied Mathematics and Computation*, 142(1), 1-16. doi:10.1016/S0096-3003(02)00943-8

Lima, P. M., & Morgado, L. (2010). Numerical modeling of oxygen diffusion in cells with Michaelis–Menten uptake kinetics. *Journal of Mathematical Chemistry*, 48, 145-158. doi:10.1007/s10910-009-9646-x.

Mandelzweig, V. B., & Tabakin, F. (2001). Quasilinearization approach to nonlinear problems in physics with application to nonlinear ODEs. *Computer Physics Communications*, 141(2), 268-281. doi:10.1016/S0010-4655(01)00415-5

Mall, S., & Chakraverty, S. (2015). Numerical solution of nonlinear singular initial value problems of Emden-Fowler type using Chebyshev Neural Network method. *Neurocomputing*, 149, 975-982. doi:10.1016/j.neucom.2014.07.036

Mohsenyzadeh, M., Maleknejad, K., & Ezzati, R. (2015). A numerical approach for the solution of a class of singular boundary value problems arising in physiology. *Advances in Difference Equations*, 231.

Öztürk, Y. (2018). Solution for the system of Lane–Emden type equations using Chebyshev Polynomials. *Mathematics*, 6, 181. doi:10.3390/math6100181

Öztürk, Y., & Gülsu, M. (2014). An operational matrix method for solving Lane-Emden equations arising in astrophysics. *Mathematical Methods in the Applied Sciences*, 37(15), 2227-2235. doi:10.1002/mma.2969

Pandey, R. K., & Kumar, N. (2012). Solution of Lane–Emden type equations using Bernstein operational matrix of differentiation. *New Astronomy*, 17(3), 303-308. doi: 10.1016/j.newast.2011.09.005.

Parand, K., Dehghan, M., Rezaei, A. R., & Ghaderi, S. M. (2010). An approximation algorithm for the solution of the nonlinear Lane-Emden type equations arising in astrophysics using Hermite functions collocation method. *Computer Physics Communications*, 181, 1096-1108. doi: 10.1016/j.cpc.2010.02.018

Reimer, M. (2003). Gegenbauer Polynomials. In *Multivariate Polynomial Approximation*, (pp. 19-38). Birkhäuser Verlag: Springer Basel.

Shawagfeh, N.T. (1993). Nonperturbative approximate solution for Lane-Emden equation. *Journal of Mathematical Physics*, 34, 4364-4369. doi:10.1063/1.530005

Shiralashetti, S. C., & Kumbinarasaiah, S. (2017). Theoretical study on continuous polynomial wavelet bases through wavelet series collocation method for nonlinear Lane–Emden type equations. *Applied Mathematics and Computation*, 315, 591-602. doi:10.1016/j.amc.2017.07.071

Singh, R., & Kumar, J. (2014). An efficient numerical technique for the solution of nonlinear singular boundary value problems. *Computer Physics Communications*, 185, 1282-1289. doi:10.1016/j.cpc.2014.01.002

Wazwaz, A. M. (2001). A new algorithm for solving differential equations of Lane-Emden type. *Applied Mathematics and Computation*, 118, 287-310. doi:10.1016/S0096-3003(99)00223-4

Wazwaz, A. M. (2011). The variational iteration method for solving nonlinear singular boundary value problems arising in various physical models. *Communications in Nonlinear Science and Numerical Simulation*, 16, 3881-3886. doi:10.1016/j.cnsns.2011.02.026

Usman, M., Hamid, M., Zubair, T., Haq, R., & Wang, W. (2019). Operational-matrix-based algorithm for differential equations of fractional order with Dirichlet boundary conditions. *The European Physical Journal Plus*, 134, 279-294. doi: 10.1140/epjp/i2019-12653-7

Van Gorder, R. A., & Vajravelu, K. (2008). Analytic and numerical solutions to the Lane-Emden Equations. *Physics Letters*, 372(39), 6060-6065. doi:10.1016/j.physleta.2008.08.002

Yildirim, A., & Öziş, T. (2009). Solutions of singular IVPs of Lane-Emden type by the variational iteration method. *Nonlinear Analysis, Theory, Methods & Applications*, 70(6), 2480-2484. doi:10.1016/j.na.2008.03.012

Yousefi, A.S. (2006). Legendre wavelets method for solving differential equations of Lane–Emden type. *Applied Mathematics and Computation*, 181, 1417-1422. doi:10.1016/j.amc.2006.02.031

Yüzbaşı, Ş. (2011). A numerical approach for solving a class of the nonlinear Lane-Emden type equations arising in astrophysics. *Mathematical Methods in the Applied Sciences*, 34(18), 2218-2230. doi:10.1002/mma.1519

Yüzbaşı, Ş., & Sezer, M. (2013). An improved Bessel collocation method with a residual error function to solve a class of Lane-Emden differential equations, *Mathematical Computer Modelling*, 57, 1298-1311. doi:10.1016/j.mcm.2012.10.032