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Some Results On Nonsmooth Systems including Max-type Functions

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Abstract. In this paper, we consider optimal control systems with continuous-time which is governed by systems of ordinary differential equation including max-type functions. We derive some properties concerning nonsmooth concepts, and a special form of adjoint condition involved in the maximum principle giving necessary conditions of optimality for optimal control problems of these systems.

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1. Introduction and Preliminary

Nonsmooth (dynamical) systems governed by differential equations including a discontinuous function or a nondifferentiable function or a deviating argument, have been usually used for mathematical modeling of dynamic behavior in many different disciplines. Nonsmooth electrical circuits with diode elements, mechanical systems with Coulomb friction and impact, automatic control systems, switching systems and networked control systems are major examples of these systems [3, 5, 8, 15, 17]. The class of nonsmooth systems is a more comprehensive class because it also includes smooth differential equations. Optimal control of these systems has an important place in optimal control theory [8, 9, 11, 21].

Since functions called max-type functions are frequently encountered in nonsmooth optimization, there are many studies and results about the properties of these functions related to various notions defined in nonsmooth analysis and convex analysis [6, 8, 9, 11, 18]. These functions can be in the form as the maximum of the functions with finite number, or more generally, they can be depending on a parameter on a compact set. A classic example is an simple electrical circuit with diode element consisting a capacitor and impressed voltage as expressed by McClamroch [8, 15]. This system is a nonsmooth control system governed by following differential equation including the max-type function on its right-hand side

$$\dot{y}(t) = \max\left\{r_1\left(u\left(t\right) - y\left(t\right)\right), r_2\left(u\left(t\right) - y\left(t\right)\right)\right\},$$

where the control u denotes the impressed voltage and r_1, r_2 are positive constant such that $r_1 > r_2$, and y is the voltage across the capacitor. Another classic example is a regulator model constructed so that it responds to the maximum deviation of voltage on the certain time interval. The system describing the action of the regulator is governed by following differential equation

$$\dot{y}(t) = -\delta y(t) + p \max_{s\in[t-h,t]} y(s) + f_0(t),$$

where δ and p are constants, y is the voltage, and f_0 is the perturbed effect [17].

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In this paper, we consider the following general optimal control system

$$\dot{x}(t) = \psi\,(t, x(t), u(t)) \text{ a.e., } u(t) \in U(t) \text{ a.e.,} \tag{1.1}$$

where "a.e." means "for almost all $t \in [0, 1]$" (Lebesgue measure), $U : [0, 1] \rightrightarrows \mathbb{R}^m$ is a set-valued function. In [13], we worked a vector-valued max-type function in the system (1.1) such that its component functions have the form as the maximum of the functions with finite number. Differently, we deal with the component functions as the max-type (or min-type) functions depending on parameters on compact sets in this paper.

Let Q_i and R_i be compact sets in $\mathbb{R}^k$ and let $\varphi_i : Q_i \times [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $\phi_i : R_i \times [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be upper semicontinuous (u.s.c.) and lower semicontinuous (l.s.c.) in $(t, u) \in [0, 1] \times \mathbb{R}^m$, respectively for each $i = 1, ..., n$. As usual, vector-valued max-type and min-type functions $\psi_{\max}$,$\psi_{\min} : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$\psi_{\max} = \left((\psi_{\max})_1\,, \dots (\psi_{\max})_n\right), \ (\psi_{\max})_i\,(t, s, u) = \max_{q \in Q_i} \varphi_i(q, t, s, u),$$

$$\psi_{\min} = \left((\psi_{\min})_1\,, \dots (\psi_{\min})_n\right), \ (\psi_{\min})_i\,(t, s, u) = \min_{r \in R_i} \phi_i(r, t, s, u).$$

Now, we deal with the system (1.1) such that $\psi : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is defined by

$$\psi = (\psi_1, ..., \psi_n)\,, \psi_i = \alpha_i\,(\psi_{\max})_i + \beta_i\,(\psi_{\min})_i\,. \tag{1.2}$$

Dynamic systems and equations similar to above forms are frequently studied in automatic control theory and boundary value problems [1, 3, 4, 10, 12, 14, 17, 20].

The problem we are mainly concerned with is that of minimizing the objective functional of Lagrange type

$$\int_0^1 L\,(t, x(t), u(t))\,dt \tag{1.3}$$

over (1.1), where $L : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. We call this problem problem (1.1)-(1.3). In this paper, we derive some nonsmooth properties of ψ and a special form of adjoint condition involved in the maximum principle giving necessary conditions of optimality under weak hypotheses.

We organize this paper as follows. First, we obtain some results about the generalized Jacobian of ψ in s. After that, combining together these with the result in [7], we obtain the special form of adjoint condition for optimal control problems of these systems. We now state some nonsmooth concepts, the basic elements of optimal control theory, and some auxiliary results used throughout this paper. Details in the same notation may be found in [7, 8, 13, 18].

Let us consider any scalar-valued function $f : \mathbb{R}^n \to \mathbb{R}$ and let $s_0 \in \mathbb{R}^n$; let d be a direction of $\mathbb{R}^n$.

Definition 1.1 ([8, 13]). The *one-sided directional derivative* of f at s_0 in the direction d, denoted $f'(s_0; d)$, is defined as:

$$f'(s_0; d) = \lim_{\lambda \downarrow 0, \lambda \neq 0} \frac{f(s_0 + \lambda d) - f(s_0)}{\lambda}. \tag{1.4}$$

In addition, it is considered that $f'(s_0; 0) = 0$. f is called *one-sided directionally differentiable* at s_0 if limit (1.4) exists for all d. In particular, if the condition $\lambda \downarrow 0$ in (1.4) is replaced with $\lambda \to 0$ and the limit exists and is finite for all d, then f is called *two-sided directionally differentiable* at s_0.

Definition 1.2 ([8]). It is assumed that f is Lipschitz near s_0. If the one-sided directional derivative $f'(s_0; d)$ exist and is equal to $f^0(s_0; d)$ for each d, then f is called *Clarke regular at* s_0, where by $f^0(s_0; d)$ it is denoted the generalized directional derivative of f at s_0 in the direction d.

Lemma 1.3. *If f is two-sided directionally differentiable at $s_0 \in \mathbb{R}^n$, then $(-f)$ is too and the property that $f'(s_0; kd) = kf'(s_0; d)$ holds, where k is any real number and d is any direction in $\mathbb{R}^n$; if f is two-sided directionally differentiable at s_0, Lipschitz near s_0 and regular at s_0, then $(-f)$ is too.*

Proof. Under the directionally differentiability hypothesis stated in the lemma, it is easy to see from Definition 1.1 that properties that $(-f)'\,(s_0; d) = -f'(s_0; d)$ and $f'(s_0; kd) = kf'(s_0; d)$ hold. It is clear that, if f is Lipschitz near s_0, then $(-f)$ is too. Moreover, under the directionally differentiability and the Lipschitz hypotheses stated in the lemma, it follows from Proposition 2.1.1 in [8] that $(-f)^0\,(s_0; d) = f^0(s_0; -d) = f'(s_0; -d) = -\,f'(s_0; d) = -f^0(s_0; d)$. Consequently, under the corresponding hypotheses in the lemma, combining the last relations we derive the desired results. $\square$

Let $\mathbb{R}^{n\times n}$ denotes the set of all $n\times n$ matrices with real elements. For a given A in $\mathbb{R}^{n\times n}$ and ζ in $\mathbb{R}^n$; by $A^*\zeta$ we denote the matrix product of the transpose of A and ζ where it is viewed ζ as $n\times 1$. In particular, we write $A\in\mathbb{R}^{n\times n}$ in the form (a_i), where a_i is the ith row of A. If x and y are elements in $\mathbb{R}^n$, then by $x\cdot y$ we mean the standard inner-product between x and y. By $\|\cdot\|$ we denote the euclidian norm on $\mathbb{R}^n$. By ϑ_i for each i, we denote $\max\limits_{r\in R_i}(-\phi_i)$ (i.e., $\vartheta_i := -(\psi_{\min})_i$. For a given nonempty set C in $\mathbb{R}^n$; by σ_C we denote the support function of C. By "co" we mean "convex hull". By Ω_f we denote all the points at which f fails to be differentiable in s where f is any of the functions in (1.1) and (1.2).

Lemma 1.4 ([19]). *Let C and D be nonempty, convex and compact sets in $\mathbb{R}^n$. Then, $C\subseteq D$ if and only if $\sigma_C(r)\leq\sigma_D(r)$ for all $r\in\mathbb{R}^n$.*

Lemma 1.5. *Let $\{A_\alpha:\alpha\in I)\}$ be a family of compact sets in $\mathbb{R}^n$ such that $A:=\bigcup\limits_{\alpha\in I}A_\alpha$ is compact. Then,* $\sigma_{\mathrm{co}A}(\cdot)=\max\limits_{\alpha\in I}(\sigma_{A_\alpha}(\cdot))$.

Proof. Note that, it is easily obtain the result in the lemma from the following relation [9]:

$$\sigma_{\mathrm{co}A}(r)=\max_{a\in\mathrm{co}A}(a\cdot r)=\max_{a\in A}(a\cdot r),\ \forall r\in\mathbb{R}^n.$$

□

The following basic result is easily obtained as a consequence of Carathéodory's Theorem [18].

Lemma 1.6 ([9]). *Let A and B be nonempty sets in $\mathbb{R}^n$. Then, the following properties hold:* $\mathrm{co}\,[A+B]=\mathrm{co}A+\mathrm{co}B$, $\mathrm{co}\,[kA]=k\mathrm{co}A$, *where k is any real number.*

Let $\mathcal{L}$ and $\mathcal{B}^m$ be the collection of Lebesgue measurable subsets of $[0,1]$ and Borel subsets of $\mathbb{R}^m$, respectively. The smallest $\sigma-$algebra of subsets of $[0,1]\times\mathbb{R}^m$ generated by Cartesian products of sets in $\mathcal{L}$ and $\mathcal{B}^m$ is denoted by $\mathcal{L}\times\mathcal{B}^m$. A Lebesgue measurable function $u:[0,1]\to\mathbb{R}^m$ satisfying "$u(t)\in U(t)$ a.e." is called a control for problem (1.1)-(1.3). $x\in AC([0,1],\mathbb{R}^n)$ satisfying differential equation in (1.1) such that $x(0)\in C_0$ and $x(1)\in C_1$, is called a trajectory corresponding to the control u, where $AC([0,1],\mathbb{R}^n)$ denotes the space of absolutely continuous functions from $[0,1]$ into $\mathbb{R}^n$, and C_0, C_1 are given closed sets in $\mathbb{R}^n$. Any control-trajectory pair (u,x) is called an admissible pair. We denote the admissible pair (v,z) as solution to the problem; that is, (v,z) minimizes the functional over all admissible pair (u,x). By $B(z(t),\delta)$ we denote the δ-neighborhood of $z(t)$ for each t.

We consider that $\varphi_i(q,t,\cdot,u)$ is Lipschitz and regular, and that $(\psi_{\max})_i(t,\cdot,u)$ is directionally differentiable, in the following senses, respectively: there exists a function $(K_{\max})_i$ in $L^1[0,1]$ such that

$$|\varphi_i(q,t,s_1,u)-\varphi_i(q,t,s_2,u)|\leq(K_{\max})_i(t)\,|s_1-s_2|,\ \forall s_1,s_2\in B(z(t),\varepsilon),\tag{1.5}$$

where $t\in[0,1]$, u and q are any elements of $U(t)$ and Q_i, respectively; for each t,q, $\varphi_i(q,t,\cdot,v(t))$ is Clarke regular at $z(t)$; $(\psi_{\max})_i(t,\cdot,v(t))$ is two-sided directionally differentiable at $z(t)$. In addition, we consider that $L(t,\cdot,u)$ is Lipschitz, and that $(-\phi_i)(r,t,\cdot,u)$ is Lipschitz (with rank $(K_{\min})_i$) and regular, and that $(\psi_{\min})_i(t,\cdot,u)$ is directionally differentiable, analogously as above.

Definition 1.7. Let f be defined similar to φ_i such that $f(q,t,\cdot,u)$ be Lipschitz near a given $s_0\in\mathbb{R}^n$ for fixed (q,t,u). The *generalized gradient set* of $f(q,t,\cdot,u)$ *at* s_0, denoted $\partial_s f(q,t,s_0,u)$, is defined as:

$$\partial_s f(q,t,s_0,u)=\mathrm{co}\left\{p:p=\lim_{k\to\infty}\nabla_s f(q,t,s_k,u),\ s_k\to s_0,\ s_k\notin\Omega_f\right\},$$

under the Lipschitz condition, the *generalized Jacobian set* of $\psi_{\max}(t,\cdot,v(t))$ *at* $z(t)$, denoted $\mathcal{J}_s\psi_{\max}(t,z(t),v(t))$, is defined as:

$$\mathcal{J}_s\psi_{\max}(t,z(t),v(t))=\mathrm{co}\left\{A:A=\lim_{k\to\infty}J_s\psi_{\max}(t,s_k,v(t)),\ s_k\to z(t),\ s_k\notin\Omega_{\psi_{\max}}\right\},$$

where by ∇_s and J_s we mean classical gradient and classical Jacobian in s, respectively [8]. Similary, $\mathcal{J}_s\psi_{\min}$ and $\mathcal{J}_s\psi$ are obtained by replacing $\psi_{\max}$ with $\psi_{\min}$ and ψ, respectively. For fixed t,r,u; $\partial_s\phi_i(r,t,s_0,u))$ is defined analogously.

Lemma 1.8. *Let $f(t,\cdot,u)$ be Lipschitz near $s_0\in\mathbb{R}^n$, Clarke regular and two-sided directionally differentiable at s_0 for fixed (t,u). Then, we have $\sigma_{\lambda\partial_s f(t,s_0,u)}(\cdot)=\lambda\sigma_{\partial_s f(q,t,s_0,u)}(\cdot)=\lambda f^0(t,s_0,u;\cdot)$, where the directional derivative is taken with respect to s-variable, λ is any real number.*

Proof. It follows from Proposition 1.4 in [6] that $\sigma_{\partial_s f(t,s_0,u)}(\cdot) = f^0(t, s_0, u; \cdot)$. From this property we have $\sigma_{\lambda\partial_s f(t,s_0,u)}(\cdot) = \sigma_{\partial_s(\lambda f(t,s_0,u))}(\cdot) = (\lambda f)^0(t, s_0, u; \cdot)$. We derive from all this, in view of Lemma 1.3, the desired results. □

We put

$$M_{i,u}(s) := \{q \in Q_i : (\psi_{\max})_i (t, s, u) = \varphi_i(q, t, s, u)\},$$
$$m_{i,u}(s) := \{r \in R_i : (\psi_{\min})_i (t, s, u) = \phi_i(r, t, s, u)\},$$

(where (t, u) is fixed and $i = 1, ..., n$) and use notations for the following sets for $t \in [0, 1]$ and $i = 1, ..., n$:

$$W_{i,1}(t) = \{ p : p \in \partial_s\varphi_i (q, t, z(t), v(t)), q \in M_{i,v(t)}(z(t))\},$$
$$W_{i,2}(t) = \{ p : p \in \partial_s\phi_i (r, t, z(t), v(t)), r \in m_{i,v(t)}(z(t))\}.$$

After, by $W_i(t)$ we denote the following set

$$\{ p : p \in \alpha_i\partial_s\varphi_i (q, t, z(t), v(t)) + \beta_i\partial_s\phi_i (r, t, z(t), v(t)), q \in M_{i,v(t)}(z(t)), r \in m_{i,v(t)}(z(t))\}.$$

2. Main Results

Lemma 2.1. (**a**) *If $\varphi_i (\cdot, t, \cdot, u)$ and $(q, s) \to \partial_s\varphi_i(q, t, s, u)$ are u.s.c., and that $\varphi_i(q, t, \cdot, u)$ is Lipschitz and regular, then $(\psi_{\max})_i (t, \cdot, u)$ is Lipschitz and regular in the same senses, and then $W_{i,1}(t)$ is compact, and then $\partial_s (\psi_{\max})_i (t, z(t), v(t)) = \mathrm{co}W_{i,1}(t)$.*

(**b**) *If $\phi_i (\cdot, t, \cdot, u)$ and $(r, s) \to \partial_s\phi_i(r, t, s, u)$ are l.s.c., and that $(-\phi_i)(r, t, \cdot, u)$ is Lipschitz and regular, and that $(\psi_{\min})_i (t, \cdot, u)$ is directionally differentiable, then $(\psi_{\min})_i (t, \cdot, u)$ is Lipschitz and regular, and then $W_{i,2}(t)$ is compact, and then $\partial_s (\psi_{\min})_i (t, z(t), v(t)) = \mathrm{co}W_{i,2}(t)$.*

Proof. For $t \in [0, 1]$, $u \in U(t)$, $s_1, s_2 \in B(z(t), \varepsilon)$, $q' \in M_{i,u}(s_1)$ (here, it used the property that $M_{i,u}(s_1)$ is nonempty due to the hypotheses of (a)) by using (1.5), we have

$$\begin{aligned}(\psi_{\max})_i (t, s_1, u) = \varphi_i(q', t, s_1, u) &\le \varphi_i(q', t, s_2, u) + (K_{\max})_i (t) |s_1 - s_2| \\ &\le (\psi_{\max})_i (t, s_2, u) + (K_{\max})_i (t) |s_1 - s_2|.\end{aligned}$$

We may similarly obtain the same inequality with s_1 and s_2 switched. So, we have $\left|(\psi_{\max})_i (t, s_1, u) - (\psi_{\max})_i (t, s_2, u)\right| \le (K_{\max})_i |s_1 - s_2|$. Besides, for (a), by writing $(\psi_{\min})_i = -\vartheta_i$ from the definition of ϑ_i, we easily have

$$\left|(\psi_{\min})_i (t, s_1, u) - (\psi_{\min})_i (t, s_2, u)\right| \le (K_{\min})_i |s_1 - s_2|.$$

Under the hypotheses of concerning $(\psi_{\max})_i$ in (a), by applying Theorem 2.1 in [6], we obtain that $(\psi_{\max})_i(t, \cdot, u)$ is Lipschitz and regular, and that $\partial_s (\psi_{\max})_i (t, z(t), v(t)) = \mathrm{co}W_{i,1}(t)$. Moreover, it follows from Theorem 2.1 in [6] that $W_{i,1}(t)$ is contained in $\partial_s (\psi_{\max})_i (t, z(t), v(t))$. Thus, $W_{i,1}(t)$ is bounded. Now, suppose that we have a convergent sequence of points p_j in $\partial_s\varphi_i \left(q_j, t, z(t), v(t)\right)$ where $q_j \in M_{i,v(t)}(z(t))$ and $p_j \to p_0$ as $j \to \infty$. We may suppose that q_j converges to some q_0 in $M_{i,v(t)}(z(t))$, since $M_{i,v(t)}(z(t))$ is compact subset of Q_i (see, Theorem 2.43 in [2]). p_0 must be in the set $\partial_s\varphi_i (q_0, t, z(t), v(t))$, since $(q, s) \to \partial_s\varphi_i(q, t, s, u)$ is u.s.c.. Thus, $W_{i,2}(t)$ is bounded and closed, so it is compact. Taking into account Lemma 1.3 for $-\phi_i$ and the hypotheses of concerning ϑ_i (i.e., $\vartheta_i = \max\limits_{r\in R_i} (-\phi_i) = -(\psi_{\min})_i$) in (b), it is easily seen that the hypotheses in Theorem 2.1 in [6] are satisfied. Thus, it follows from the theorem that $\vartheta_i(t, \cdot, u)$ is Lipschitz and regular, and that

$$\partial_s\vartheta_i(t, z(t), v(t)) = \{ p : p \in \partial_s (-\phi_i) (r, t, z(t), v(t)), r \in m_{i,v(t)}(z(t))\}.$$

From the well-known property of the generalized gradient set regarding the scalar multiplier (see, Proposition 2.3.1 in [8]), $\partial_s\vartheta_i = \partial_s(-(\psi_{\min})_i) = -\partial_s (\psi_{\min})_i = -\mathrm{co}W_{i,2}$. Consequently, it is obtained that $(\psi_{\min})_i (t, \cdot, u)$ is Lipschitz and regular, and that $\partial_s (\psi_{\min})_i (t, z(t), v(t)) = \mathrm{co}W_{i,2}(t)$. □

Let $K_i := |\alpha_i| (K_{\max})_i + |\beta_i| (K_{\min})_i$. Observe that $K_i \in L^1 [0, 1]$. In this case, It easily follows from the proof of Lemma 2.1 that the following.

Corollary 2.2. *If for each i, $\varphi_i (\cdot, t, \cdot, u)$ and $\phi_i (\cdot, t, \cdot, u)$ are u.s.c. and l.s.c., respectively, and that $\varphi_i(q, t, \cdot, u)$ and $\phi_i(r, t, \cdot, u)$ are Lipschitz then for each i, $\psi_i(t, \cdot, u)$ is Lipschitz (so is $\psi(t, \cdot, u)$) (with rank K_i for ψ_i) in the same senses.*

Lemma 2.3. *Assume that for each i, the hypotheses of both statements of Lemma 2.1 are satisfied, and that $(\psi_{\max})_i(t,\cdot,u)$ is directionally differentiable. Then, for arbitrary $\zeta, r \neq 0_n$ in $\mathbb{R}^n$, for each t, we have*

$$\overline{\lim_{\substack{s\to z(t)\\ s\notin\Omega_\psi}}}\left\{\sum_{i=1}^{n} r_i\psi_i'(t,s,v(t);\zeta)\right\} = \sum_{i=1}^{n}[\alpha_i r_i \max_{q\in M_{i,v(t)}(z(t))}(\varphi_i'(q,t,z(t),v(t);\zeta)) + \beta_i r_i \min_{\gamma\in m_{i,v(t)}(z(t))}(\phi_i'(\gamma,t,z(t),v(t);\zeta))], \tag{2.1}$$

where the directional derivatives are taken with respect to s-variable.

Proof. Under the hypotheses of the theorem; combining the properties of scalar multiples and finite sums about the generalized gradients in [8] with results in Lemma 2.1, $W_i(t)$ can be written in the following form: $\alpha_i W_{i,1}(t)+\beta_i W_{i,2}(t)$. It follows from Lemma 1.6 that $\mathrm{co}W_i(t) = \alpha_i\mathrm{co}W_{i,1}(t)+\beta_i\mathrm{co}W_{i,2}(t)$. It follows from Lemma 1.3, Lemma 1.8 and Lemma 2.1 that $\sigma_{\mathrm{co}W_{i,1}(t)}(r_i\zeta) = (\psi_{\max})_i'(t,z(t),v(t);r_i\zeta) = r_i(\psi_{\max})_i'(t,z(t),v(t);\zeta) = r_i\sigma_{\mathrm{co}W_{i,1}(t)}(\zeta)$. Consequently, by using all this, Lemma 1.3, Lemma 1.5, Lemma 1.8, Lemma 2.1, and the facts that both the convex hull and any scalar multiple of a compact set in $\mathbb{R}^n$ are compact, we have

$$\begin{aligned}
&\sum_{i=1}^{n}[\alpha_i r_i \max_{q\in M_{i,v(t)}(z(t))}(\varphi_i'(q,t,z(t),v(t);\zeta)) + \beta_i r_i \min_{\gamma\in m_{i,v(t)}(z(t))}(\phi_i'(\gamma,t,z(t),v(t);\zeta))]\\
&=\sum_{i=1}^{n}[\alpha_i r_i \max_{q\in M_{i,v(t)}(z(t))}\left(\sigma_{\partial_s\varphi_i(q,t,z(t),v(t))}(\zeta)\right) + \beta_i r_i \max_{\gamma\in m_{i,v(t)}(z(t))}\left(\sigma_{\partial_s(-\phi_i)(\gamma,t,z(t),v(t))}(\zeta)\right)]\\
&=\sum_{i=1}^{n}[\alpha_i\sigma_{\mathrm{co}W_{i,1}(t)}(r_i\zeta)+\beta_i\sigma_{\mathrm{co}W_{i,2}(t)}(r_i\zeta)]\\
&=\sum_{i=1}^{n}\sigma_{\mathrm{co}W_i(t)}(r_i\zeta).
\end{aligned}$$

Moreover, note that it follows from Lemma 2.1 and Corollary 2.2 that $\partial_s(\psi)_i(t,z(t),v(t)) = \mathrm{co}W_i(t)$. Thus, in view of Lemma 1.3 and Lemma 2.1, it follows from (2.2) that

$$\sum_{i=1}^{n}\sigma_{\mathrm{co}W_i(t)}(r_i\zeta) = \sum_{i=1i}^{n} r_i\psi_i'(t,z(t),v(t);\zeta). \tag{2.2}$$

The following is a special form of inequality (6) in [13]. It was showed in [13] that the inequality holds. Thus, the following holds for $\zeta, r\neq 0_n$,

$$\overline{\lim}\left\{\sum_{i=1}^{n} r_i\left[\nabla_s\psi_i(t,s,v(t))\cdot\zeta\right] : \ s\to z(t), s\notin\Omega_\psi\right\} \geq \sum_{i=1}^{n} r_i\psi_i'(t,z(t),v(t);\zeta). \tag{2.3}$$

In addition, it is clear from the definition of limit superior that reverse of (2.3) always holds for $\zeta, r\neq 0_n$. Combining the above results completes the proof. □

Theorem 2.4. *Assume that the hypotheses of of Lemma 2.3 are satisfied. Then, for an arbitrary $\zeta\in\mathbb{R}^n$, for each t, we have*

$$(\mathcal{J}_s\psi(t,z(t),v(t)))^*\zeta = \{A^*\zeta : a_i\in\mathrm{co}W_i(t), i=1,...,n\}. \tag{2.4}$$

Proof. For convenience, we denote by $(\mathcal{J}\psi)\zeta$ and W, respectively, the left and right sides of the equality (2.4). First we note that these sets are nonempty, convex and compact, as easily seen from the basic properties of the generalized gradient and Jacobian in [8] and [11] (see [13]). We now try to determine the support functions of these sets. After we apply Lemma 1.4.

For any $r=(r_1,...,r_n)$ in $\mathbb{R}^n$, the support function of $(\mathcal{J}\psi)\zeta$ is determined in the following form [8]:

$$\begin{aligned}
\sigma_{(\mathcal{J}\psi)\zeta}(r) &= \overline{\lim}\left\{r\cdot((J_s\psi(t,s,v(t)))^*\zeta) \ : \ s\to z(t), s\notin\Omega_\psi\right\}\\
&=\overline{\lim}\left\{\sum_{i=1}^{n} r_i\left[\nabla_s\psi_i(t,s,v(t))\cdot\zeta\right] : \ s\to z(t), s\notin\Omega_\psi\right\},
\end{aligned}$$

$$\begin{aligned}\sigma_W(r) &= \max_{A^*\zeta\in W} r\cdot(A^*\zeta) = \max\left\{\sum_{i=1}^{n}\left[r_i\left(a_i\cdot\zeta\right)\right]:\ a_i\in \operatorname{co}W_i(t)\right\}\\ &= \sum_{i=1}^{n}\max_{a_i\in\operatorname{co}W_i(t)}\left(r_i\left(a_i\cdot\zeta\right)\right) = \sum_{i=1}^{n}\max_{a_i\in\operatorname{co}W_i(t)}\left(a_i\cdot\left(r_i\zeta\right)\right)\\ &= \sum_{i=1}^{n}\sigma_{\operatorname{co}W_i(t)}\left(r_i\zeta\right).\end{aligned}$$

It follows from (2.1) and (2.2) that $\sigma_{(\partial\psi)\zeta}(r) = \sigma_W(r)$. Thus, by applying Lemma 1.4 we obtain (2.4). □

Theorem 2.5. *Let the admissible pair (v,z) solve problem (1.1)-(1.3). Assume that the hypotheses of Theorem 2.4 and the followings are satisfied for each $i = 1,...,n$, for each s in $B(z(t),\varepsilon)$, $(\psi_{\max})_i(\cdot,s,\cdot)$, $(\psi_{\min})_i(\cdot,s,\cdot)$ and $L(\cdot,s,\cdot)$ are $\mathcal{L}\times\mathcal{B}^m$-measurable; U has $\mathcal{L}\times\mathcal{B}^m$-measurable graph; L is Lipschitz. Then, there exist a function $\bar{p}\in AC([0,1],\mathbb{R}^n)$ together with a scalar λ equal to 0 or -1 satisfying condition that $|\lambda| + \|\bar{p}(t)\| \neq 0$, for $t\in[0,1]$,and the following condition called the adjoint condition:*

$$-\dot{\bar{p}}(t) \in \left\{\bar{p}(t)A + \lambda\xi : a_i\in\operatorname{co}W_i(t), \xi\in\partial_s L\left(t,z(t),v(t)\right), i = 1,...,n\right\}\ \ a.e.. \tag{2.5}$$

Proof. Under the hypotheses of the theorem; $\psi(\cdot,s,\cdot)$) is $\mathcal{L}\times\mathcal{B}^m$-measurable for each s in $B(z(t),\varepsilon)$. Moreover, by observing Corollary 2.2, Theorem 2.4, and that it is viewed $\bar{p} = (\bar{p}_1,...,\bar{p}_n)$ as $1\times n$, where $\bar{p}_i$ is ith component function of $\bar{p}$, we see that the hypotheses of Corollary 2 in [7] are satisfied. Thus, combining results in Theorem 2.4 and Corollory 2 in [7], we obtain the results in this theorem. □

Example 2.6. Let us consider the system of the form (1.1) such that $n = 2, \psi_1 = \max_{q\in[0,1]} q(s_2 - s_1^2), \psi_2 = u$. Let $U(t)\subseteq [0,1]$, where it is considered that the graph of U is equal to $[0,1]\times[0,1]$. And also, let $L = \frac{1}{2}u^2$, $x(0) = x(1) = (0,0)$. We check that for all $t\in[0,1]$, $(z,v) = (0,0)$ is optimal.

Observe that ψ_1 is not differentiable at $(t,(0,0),u)$ in s. Hence, this system is nonsmooth. ψ_2 can be written the following form in (1.2): $\psi_2 = \min_{r\in R_2} u$, where R_2 is any compact set in $\mathbb{R}$. $\varphi_1 = q(s_2 - s_1^2)$ is continuously differentiable in s. Hence, it is Lipschitz, regular and $\partial_s\varphi_1(q,t,z,v) = \{\nabla_s\varphi_1(q,t,z,v)\} = \{(0,q)\}$. It can be easily verified that the rest of the hypotheses of both statements of Lemma 2.1 are satisfied. It follows from Lemma 2.1 that $\partial_s\psi_1(t,z,v) = \{(0,a) : a\in[0,1]\}$ and $(\psi_1)^0(t,z,v;d) = \max_{a\in[0,1]}(d\cdot a)$, $\forall d\in\mathbb{R}^2$. In addition, it can be easily verified that $\psi_1(t,\cdot,v)$ is two-sided directionally differentiable at z, and that $(\psi_1)'(t,z,v;d) = \max_{a\in[0,1]}(d\cdot a)$, where $d\in\mathbb{R}^2$ and the directional derivatives are taken with respect to s-variable, and hence that hypotheses in Theorem 2.4 and Theorem 2.5 are satisfied. From these theorems, we derive the followings:

$$\begin{aligned}\left(\partial_s\psi(t,z,v)\right)^*\zeta &= \{(0,a\zeta_1) : a\in[0,1]\}\ \ \text{for } \zeta = (\zeta_1,\zeta_2)\in\mathbb{R}^2\,,\\ (-\dot{\bar{p}}_1(t), -\dot{\bar{p}}_2(t)) &\in\ \{(0,a\bar{p}_1(t)) : a\in[0,1]\}\ \ \text{a.e..}\end{aligned}$$

From here, it is easy to see that $\lambda = -1$ and $\bar{p}(t) = (\bar{p}_1(t),\bar{p}_2(t)) = (c_1, -ac_1t + c_2)$ which satisfy the condition in Theorem 2.5, where $a\in[0,1]$ and $c_1,c_2\in\mathbb{R}$.

3. Concluding Remarks

In this paper, nonsmooth systems with continuous-time which is governed by systems of ordinary differential equation including max-type (or min-type) functions depending on parameters on compact sets are studied. An optimality condition regarding the adjoint condition involved in the maximum principle giving necessary conditions of optimality for optimal control problems of these systems, has been obtained. In general, the optimality condition (6) in Corollory 2 in [7] is also valid for these problem. The optimality condition (2.5) in Theorem 2.5 in this paper, unlike the condition associated with the generalized Jacobian set of the vector-valued function in [7], can be directly expressed with the functions φ_i and ϕ_i in max-type and min-type component functions in (1.2). In this sense, it can provide an easier and more practically calculable criterion for determining the optimal solution to the problem. In addition, Example 2.6, which is a simple application of the results presented here, proves that Theorem 2.5 can be applied to nonsmooth cases.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

Authors Contribution Statement

All authors contributed to the study conception and design. The first draft of the manuscript was written by Serkan Ilter and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

References

[1] Agarwal, R., Hristova, S., *Strict stability in terms of two measures for impulsive differential equations with 'supremum'*, Appl. Anal., **91**(7)(2012), 1379–1392.
[2] Aliprantis, C.D., Border, K.C., Infinite Dimensional Analysis A Hitchhiker's Guide, 3nd ed., Springer-Verlag, Berlin, 2007.
[3] Bainov, D., Hristova, S., Differential Equations with Maxima, Taylor and Francis, 2011.
[4] Bantsur, N.R., Trofimchuk, O.P., *On the existence and stability of periodic and almost periodic solutions of quasilinear equations with maxima*, Ukrainian Mathematical Journal, **50**(6)(1998), 747–754.
[5] Brogliato, B., Nonsmooth Mechanics: Models, Dynamics and Control, Springer-Verlag, London, 1999.
[6] Clarke, F.H., *Generalized gradients and applications*, Trans. Amer. Math. Soc., **205**(1975), 247–262.
[7] Clarke, F.H., *The Maximum Principle under minimal hypotheses*, SIAM J. Control and Optimization, **14**(6)(1976), 1078–1091.
[8] Clarke, F.H., Optimization and Nonsmooth Analysis, SIAM Classics in Applied Mathematics, Philadelphia, 1990.
[9] Demyanov, V.F., Rubinov, A.M., Quasidifferential Calculus, Optimization Software, Inc. Publications Division, New York, 1986.
[10] Geiger, T., Wachsmuth, D., Wachsmuth, G. *Optimal control of ODEs with state suprema*, Mathematical Control and Related Fields, **11**(3)(2021), 555–578.
[11] Giorgi, G., Guerraggio, A., Thierfelder, J., Mathematics of Optimization: Smooth and Nonsmooth Case, Elsevier Science, 2004.
[12] Hristova, S., Golev, A., Stefanova, K., *Approximate method for boundary value problems of anti-periodic type for differential equations with maxima*, Boundary Value Problems, **1**(2013), 12.
[13] Ilter, S., *Weak maximum principle for optimal control problems of nonsmooth systems*, Applied Mathematics and Computation, **218**(3)(2011), 805–808.
[14] Jankovski, T., *Systems of differential equations with maxima*, Doklady Akademii Nauk, **8**(1997), 57–60.
[15] McClamroch, N.H., State Models of Dynamic Systems, Springer-Verlag, New York-Heidelberg-Berlin, 1980.
[16] Otrocol, D., *Systems of functional differential equations with maxima, of mixed type*, Electron. J. Qual. Theory Differ. Equ., **5**(2014), 1–9.
[17] Popov, E., Automatic Regulation and Control, Nauka, Moscow, 1966.
[18] Roberts, A.W., Varberg, D.E., Convex Functions, Academic Press, New York-London, 1973.
[19] Rockafellar, R.T., Convex Analysis, Princeton University, New Jersey, 1970.
[20] Shpakovich, V.P., Muntyan, V.I., *Method of averaging for differential equations with maxima*, Ukrainian Mathematical Journal, **39**(5)(1987), 543–545.
[21] Vinter, R.B., Optimal Control, Birkhäuser, Boston, 2000.