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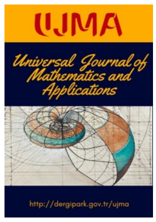

Conchoidal Surfaces in Euclidean 3-space Satisfying $\Delta x_i = \lambda_i x_i$

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Abstract

In this paper, we study the conchodial surfaces in 3-dimensional Euclidean space with the condition $\Delta x_i = \lambda_i x_i$ where Δ denotes the Laplace operator with respect to the first fundamental form. We obtain the classification theorem for these surfaces satisfying under this condition. Furthermore, we have give some special cases for the classification theorem by given the radius function $r(u,v)$ with respect to the parameter u and v.

1. Introduction

The invention of the conchoid was attributed to Greek mathematician Nicomedes by Pappus and other classical authors in the second century BC. Based on the oldest data, the conchoid curve was designed by Nicomedes as a result of the problem of dividing an angle into three equal parts, which has been a problem for many mathematicians for many years. The word conchoid is derived from the Greek word ”conch”, which means crustacean, and is also referred to as mussel shell shape in the literature.

This curve became a favourite of many mathematicians in the 17th century as an example of new methods in analytical geometry and calculus. For this reason, Newton suggested that it should be treated as a ‘standard’ curve [1]. In 1837, Pierre Wantzel showed that an arbitrary angle is not divisible by three in the classical way, and therefore conchoid curves were obtained, which can be examples of many curves. The best known of these curves are Hippias’ quadratrix curve, Nicomedes’ conchoid, Pascal’s limachon and cycloid curves.

The conchoid structure is usually best applied to curves in the Euclidean plane $\mathbb{E}^2$ [2]. A conchoid curve is obtained by using a planar curve, a fixed point and a fixed distance. The set of points on the line at a fixed distance from a moving point on a planar curve gives the conchoid of this planar curve [3]. In [2], the concept of a conchoidal curve is generalized to the concept of a conchoidal transformation of two curves, and when one of the two curves is a circle, the conchoidal transformation becomes a classical conchoidal curve. It is known that conchoid curves have many applications. In particular, they have been used in the construction of buildings and structures and are also used in physics, astronomy, optics, electromagnetic research, biology and medical engineering applications(see, [3]- [5]).

The conchoid transformation has been applied to surfaces in Euclidean 3-space in ([6]- [11]) in order to construct new classes of surfaces and making them accessible to the algorithms implemented in CAGD systems. The concept of conchoid surface is also based on the concept of curve and studies on conchoid surfaces of quadrics, conchoid surfaces of sphere, conchoid surfaces of ruled surfaces have been carried out. In addition, in [12] conchoid curves and surfaces in 3-dimensional Euclidean space are considered and the curvatures that determine the geometric properties of these curves and surfaces are calculated. Also in ([13]) the authors computed the types of spacelike conchoid curves in the Minkowski plane and in ([14]) the authors examined the condition which is the conchoidal surface and the surface of revolution given with a conchoid curve to be a Bonnet surface in Euclidean 3-space. The latest studies in Euclidean 3-space is conchoidal twisted surface which isformed by the synchronized anti-symmetric rotation matrix of a planar conchoidal curve ([15]).

This paper is organised as follows: In section 2 we give some basic concepts of the surfaces in $\mathbb{E}^3$ and also surfaces satisfying the condition $\Delta x_i = \lambda_i x_i$. In section 3 we consider conchoidal surfaces in $\mathbb{E}^3$and we gave the results of Gaussian and mean curvature of these surfaces with respect to the given paper in [12]. In the final section we consider conchoidal surfaces in $\mathbb{E}^3$ satisfying the condition $\Delta x_i = \lambda_i x_i$. We obtain the classification theorem for these surfaces satisfying under this condition. Furthermore, we have give some special cases for the classification theorem by given the radius function $r(u,v)$ with respect to the parameter u and v.

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2. Basic Concepts

2.1. Surfaces in $\mathbb{E}^3$

Let M be a smooth surface in $\mathbb{E}^3$ given with the patch $X(u,v):(u,v)\in D\subset\mathbb{E}^2$. The tangent space to M at an arbitrary point $p=X(u,v)$ of M span $\{X_u,X_v\}$. The unit normal vector field or surface normal N is defined by

$$N(u,v)=\frac{X_u\times X_v}{\|X_u\times X_v\|}(u,v)$$

at those points $(u,v)\in D$ at which $X_u\times X_v$ does not vanish, i.e., X is regular.
Let $X:D\subset\mathbb{E}^2\to\mathbb{E}^3$ be a regular patch. Then the Gaussian curvature and mean curvature of the surface are given by the formulas

$$K=\frac{eg-f^2}{EG-F^2}$$

and

$$H=\frac{eG+gE-2fF}{2(EG-F^2)}$$

where

$$E=\langle X_u,X_u\rangle,\quad F=\langle X_u,X_v\rangle,\quad G=\langle X_v,X_v\rangle$$

and

$$e=\langle X_{uu},N\rangle,\quad f=\langle X_{uv},N\rangle,\quad g=\langle X_{vv},N\rangle$$

are the coefficients of first and second fundamental form of the surface respectively. Recall that a surface M is said to be flat and minimal if its Gaussian curvature and mean curvature vanishes respectively [22,23].

2.2. Surfaces satisfying $\Delta x_i=\lambda_i x_i$

The definition of submanifolds of finite type was introduced by B.Y. Chen in the late 1970s in order to understand the total mean mean curvature for general Euclidean submanifolds. So, the author introduced the notions of order and type for Euclidean submanifolds. By applying such notions, he introduced the notions of finite type submanifolds an finite type maps. The family of finite-type submanifolds is quite large. The most important and widely known; minimal submanifolds in Euclidean space, minimal submanifolds on hyperspheres and all parallel submanifolds [24].
Let u_i,u_j be a local coordinate system of M. For the array g_{ij} $(i,j=1,2)$ consisting of components of the induced metric on M, we denote by $g^{ij}=(g_{ij})^{-1}$ the inverse matrix of the array g_{ij}. Then the Laplacian operator Δ of the induced metric on M is given

$$\Delta=-\frac{1}{\sqrt{\det(g_{ij})}}\sum_{i,j}\frac{\partial}{\partial u_i}\left(\sqrt{\det(g_{ij})}g^{ij}\frac{\partial}{\partial u_i}\right).$$

An isometric immersion $x:M\to\mathbb{E}^m$ of a submanifold M in Euclidean m-space $\mathbb{E}^m$ is said to be of finite type if x identified with the position vector field of M in $\mathbb{E}^m$ can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M, that is;

$$x=x_0+\sum_{i=1}^{k}x_i$$

where x_0 is a constant map, x_1, x_2, ..., x_k non-constant maps such that $\Delta x_i=\lambda_i x_i$, $\lambda_i\in\mathbb{R}$, $1\le i\le k$. If λ_1, λ_2, ..., λ_k are different, then M is said to be of k-type.
Similarly, a smooth map φ of an n-dimensional Riemannian manifold M of $\mathbb{E}^m$ is said to be of finite type if φ is a finite sum of $\mathbb{E}^m$-valued eigenfunctions of Δ (see, [24], [25]).
It is well known the Beltrami formula [24] ;

$$\Delta\overrightarrow{x}=-2\overrightarrow{H}$$

which shows, in particular, that M is minimal surface in $\mathbb{R}^3$ if and only if its coordinate functions are harmonic. Moreover, T. Takahashi [26] states that minimal surfaces and spheres are the only surfaces in $\mathbb{R}^3$ satisfying the condition

$$\Delta\overrightarrow{x}=\lambda\overrightarrow{x},\quad \lambda\in\mathbb{R}.$$

On the other hand Garay [16] determined the complete surfaces of revolution in $\mathbb{R}^3$ whose component functions are eigenfunctions of their Laplace operator i.e.

$$\Delta x_i=\lambda_i x_i\ \ \lambda_i\in\mathbb{R} \tag{2.1}$$

Later Lopez [17] studied the hypersurfaces in $\mathbb{R}^{n+1}$, Bekkar and Zoubir [18] classified the surfaces of revolution with non zero Gaussian curvature in the 3-dimensional Euclidean space $\mathbb{E}^3$ and Lorentzian-Minkowski spaces and Bekkar and Senoussi [19] studied the factorable surfaces in the 3-dimensional Euclidean and Minkowski space under the condition (2.1). Also Difi et al. [20] studied the translation-factorable surfaces in 3-dimensional Euclidean and Lorentzian spaces satisfying the condition $\Delta x_i = \lambda_i x_i$. Zoubi et al. [21] gave a classification of surfaces of coordinate finite type in the Lorentz–Minkowski 3-Space.
In this paper we classify the conchoidal surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$ satisfying the condition $\Delta x_i = \lambda_i x_i$ where $\lambda_i \in \mathbb{R}$.

3. Conchoidal Surfaces in $\mathbb{E}^3$

In this section some results on conchoid surfaces are given. Gaussian and mean curvatures of conchoid surfaces given in 3-dimensional Euclidean space have been investigated in the paper Bulca et al. [12].
The conchoid surface M_d of a given surface M with respect to a point O is roughly speaking the surface obtained by increasing the radius function of M with respect to O by a constant d. Consider $M \subset \mathbb{R}^3$ be a regular surface, distance $d \in \mathbb{R}$, with respect to a given fixed point $O = (0,0,0) \in \mathbb{R}^3$. Let M be represented by polar representation

$$x(u,v) = r(u,v)\rho(u,v) \tag{3.1}$$

with $\|\rho(u,v)\| = 1$. Taking into account parametrization $\rho(u,v) = (\cos u \cos v, \sin u \cos v, \sin v)$ of the unit sphere S^2, so $\rho(u,v)$ is called spherical part of $x(u,v)$ and $r(u,v)$ its radius function. The conchoidal surface M_d of M at distance d parametrized by

$$x_d(u,v) = (r(u,v) \pm d)\rho(u,v) \tag{3.2}$$

(see, [9]).

Theorem 3.1. *([12])Let M be a regular surface given with the parametrization (3.1). Then the Gaussian and mean curvature of M becomes*

$$K = -\frac{\delta^2(u,v) - \psi(u,v)\xi(u,v)\cos^2 v}{r^2\left((r^2+r_v^2)\cos^2 v + r_u^2\right)^2} \tag{3.3}$$

and

$$H = -\frac{\cos v(r^2+r_v^2)\psi(u,v) + \cos v(r^2\cos^2 v + r_u^2)\xi(u,v) + 2r_u r_v \delta(u,v)}{2r^2\left((r^2+r_v^2)\cos^2 v + r_u^2\right)^{3/2}} \tag{3.4}$$

respectively,where

$$\begin{aligned}
\delta(u,v) &= r r_{uv}\cos v - 2r_u r_v \cos v + r r_u \sin v,\\
\psi(u,v) &= 2r_u^2 + r r_v \sin v \cos v + r^2\cos^2 v - r r_{uu},\\
\xi(u,v) &= 2r_v^2 + r^2 - r r_{vv}
\end{aligned}$$

are the differentiable functions.

Corollary 3.2. *([12])Let M be a regular surface given with the parametrization (3.1).*
i) If the radius function $r(u,v)$ be an u-parameter function then the Gaussian and mean curvature of M

$$\begin{aligned}
K &= \frac{\cos^2 v\left(2r_u^2 + r^2\cos^2 v - r r_{uu}\right) - r_u^2\sin^2 v}{(r^2\cos^2 v + r_u^2)^2},\\
H &= -\frac{\cos v\left(3r_u^2 + 2r^2\cos^2 v - r r_{uu}\right)}{2\left(r^2\cos^2 v + r_u^2\right)^{3/2}}
\end{aligned}$$

ii) If the radius function $r(u,v)$ be a v-parameter function then the Gaussian and mean curvature of M

$$\begin{aligned}
K &= \frac{(r_v\sin v + r\cos v)\left(2r_v^2 + r^2 - r r_{vv}\right)}{r\cos v(r^2+r_v^2)^2},\\
H &= -\frac{(r_v\sin v + r\cos v)\left(r^2+r_v^2\right) + r\cos v\left(2r_v^2 + r^2 - r r_{vv}\right)}{2r\cos v\left(r^2+r_v^2\right)^{3/2}}
\end{aligned}$$

Theorem 3.3. *([12])Let M_d be a conchoidal surface of M given with the parametrization (3.2). Then the Gaussian and mean curvature of M_d becomes*

$$\widetilde{K} = -\frac{\widetilde{\delta}^2(u,v) - \widetilde{\psi}(u,v)\widetilde{\xi}(u,v)\cos^2 v}{(r\pm d)^2\left(((r\pm d)^2 + r_v^2)\cos^2 v + r_u^2\right)^2}$$

and

$$\widetilde{H} = -\frac{\widetilde{\xi}(u,v)\cos v\left((r\pm d)^2\cos^2 v + r_u^2\right) + \widetilde{\psi}(u,v)\cos v\left((r\pm d)^2 + r_v^2\right) + 2r_u r_v\widetilde{\delta}(u,v)}{2(r\pm d)^2\left(((r\pm d)^2 + r_v^2)\cos^2 v + r_u^2\right)^{3/2}}$$

respectively, where

$$\widetilde{\delta}(u,v) = rr_{uv}\cos v - 2r_u r_v \cos v + rr_u \sin v,$$
$$\widetilde{\psi}(u,v) = 2r_u^2 + rr_v \sin v \cos v + r^2\cos^2 v - rr_{uu},$$
$$\widetilde{\xi}(u,v) = 2r_v^2 + r^2 - rr_{vv}$$

are the differentiable functions.

Corollary 3.4. *([12])Let M_d be a regular surface given with the parametrization (3.2).*
i) If the radius function $r(u,v)$ be an u-parameter function then the Gaussian and mean curvature of M_d

$$\widetilde{K} = \frac{\cos^2 v\left(2r_u^2 + (r\pm d)^2\cos^2 v - (r\pm d)r_{uu}\right) - r_u^2\sin^2 v}{((r\pm d)^2\cos^2 v + r_u^2)^2},$$
$$\widetilde{H} = -\frac{\cos v\left(3r_u^2 + 2(r\pm d)^2\cos^2 v - (r\pm d)^2 r_{uu}\right)}{2\left((r\pm d)^2\cos^2 v + r_u^2\right)^{3/2}}$$

ii) If the radius function $r(u,v)$ be a v-parameter function then the Gaussian and mean curvature of M_d

$$\widetilde{K} = \frac{(r_v\sin v + (r\pm d)\cos v)\left(2r_v^2 + (r\pm d)^2 - (r\pm d)r_{vv}\right)}{(r\pm d)\cos v((r\pm d)^2 + r_v^2)^2},$$
$$\widetilde{H} = -\frac{(r_v\sin v + (r\pm d)\cos v)\left((r\pm d)^2 + r_v^2\right) + (r\pm d)\cos v\left(2r_v^2 + (r\pm d)^2 - (r\pm d)r_{vv}\right)}{2(r\pm d)\cos v\left((r\pm d)^2 + r_v^2\right)^{3/2}}.$$

4. Conchoidal Surfaces in Euclidean 3-space Satisfying $\Delta x_i = \lambda_i x_i$

In this section we consider a conchoidal surfaces given with the parametrization (3.2) which is satisfying the condition $\Delta x_i = \lambda_i x_i$. Firstly we consider the polar representation of the surfaces M given with the parametrization (3.1). The coefficients of the first fundamental form and the unit normal vector field of the surface M are:

$$E = r^2\cos^2 v + r_u^2,$$
$$F = r_u r_v,$$
$$G = r^2 + r_v^2,$$

and

$$N = \frac{(r_v\cos u\cos v\sin v + r\cos u\cos^2 v + r_u\sin u, r_v\sin u\cos v\sin v + r\sin u\cos^2 v - r_u\cos u, -r_v\cos^2 v + r\cos v\sin v)}{\sqrt{(r^2 + r_v^2)\cos^2 v + r_u^2}}. \tag{4.1}$$

Further, the coefficients of the second fundamental form as follows;

$$e = -\frac{\cos v\left(2r_u^2 + rr_v\sin v\cos v + r^2\cos^2 v - rr_{uu}\right)}{\sqrt{(r^2 + r_v^2)\cos^2 v + r_u^2}},$$
$$f = \frac{rr_{uv}\cos v - 2r_u r_v\cos v + rr_u\sin v}{\sqrt{(r^2 + r_v^2)\cos^2 v + r_u^2}},$$
$$g = -\frac{\cos v\left(2r_v^2 + r^2 - rr_{vv}\right)}{\sqrt{(r^2 + r_v^2)\cos^2 v + r_u^2}}.$$

The Laplacian Δ of M is given by with respect to the Beltrami formula is $\Delta x = -2\overrightarrow{H}$. So if we use this formula we can obtain,

$$\begin{aligned} \Delta x_1 &= -2Hn_1 \\ \Delta x_2 &= -2Hn_2 \\ \Delta x_3 &= -2Hn_3 \end{aligned} \tag{4.2}$$

where H and n_i are defined in (3.4) and (4.1) respectively. If the the polar representation of the surfaces M given with the parametrization (3.1) is constructed with component functions which are eigenfunctions of its Laplacian, we shall have that

$$\begin{aligned} \Delta\left(r(u,v)\cos u\cos v\right) &= \lambda_1 r(u,v)\cos u\cos v \\ \Delta\left(r(u,v)\sin u\cos v\right) &= \lambda_2 r(u,v)\sin u\cos v \\ \Delta r(u,v)\sin v &= \lambda_3 r(u,v)\sin v \end{aligned} \tag{4.3}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. Using the equations (4.1),(4.2) and (4.3) we obtain

$$\begin{aligned} -2H(r_v\cos u\cos v\sin v + r\cos u\cos^2 v + r_u\sin u) &= \lambda_1 W\cos u\cos v, \\ -2H(r_v\sin u\cos v\sin v + r\sin u\cos^2 v - r_u\cos u) &= \lambda_2 W\sin u\cos v, \\ -2H(-r_v\cos^2 v + r\sin v\cos v) &= \lambda_3 W\sin v, \end{aligned} \tag{4.4}$$

where

$$W=r\sqrt{(r^2+r_v^2)\cos^2 v+r_u^2}$$

We distinguish two special cases according to whether this surface satisfying the condition given by (4.4)
Case 1. For the first case we suppose that the radius function $r(u,v)$ given with the parameter u. So, if the function $r=r(u)$ then the mean curvature of the surface M and the conditions of $\Delta x_i=\lambda_i x_i$ are

$$H=-\frac{\cos v\left(3r_u^2+2r^2\cos^2 v-rr_{uu}\right)}{2\left(r^2\cos^2 v+r_u^2\right)^{3/2}},$$

and

$$-2H(r\cos u\cos^2 v+r_u\sin u)=\lambda_1 W\cos u\cos v, \tag{4.5}$$
$$-2H(r\sin u\cos^2 v-r_u\cos u)=\lambda_2 W\sin u\cos v, \tag{4.6}$$
$$-2H(r\cos v)=\lambda_3 W. \tag{4.7}$$

Furthermore, we explore the classification of the surface M given with the parametrization (3.1) satisfying the equation (2.1);

1) Let $\lambda_3=0$ then the equation (4.5) gives rise to $H=0$ which means that the surface is minimal. We get also by the equations (4.5) and (4.6) $\lambda_1=\lambda_2=0$.
2) Let $\lambda_3\neq 0$, so $H\neq 0$. We get four cases for these condition.

i) If $\lambda_1=0$ and $\lambda_2\neq 0$ then $H\neq 0$. So, from (4.5) we have

$$r\cos u\cos^2 v+r_u\sin u=0. \tag{4.8}$$

The solution of the differential equation (4.8) we obtain the radius function

$$r(u)=\frac{C_1}{\sqrt{(\sin u)^{\cos 2v+1}}},$$

where C_1 is a real constant.
ii) If $\lambda_1\neq 0$ and $\lambda_2=0$ then $H\neq 0$. So, from (4.6) we have

$$r\sin u\cos^2 v-r_u\cos u=0. \tag{4.9}$$

The solution of the differential equation (4.9) we obtain the radius function

$$r(u)=\frac{C_2}{\sqrt{\cos(u)^{\cos 2v+1}}}$$

where C_2 is a real constant.
iii) If $\lambda_1\neq 0$ and $\lambda_2\neq 0$. Equations (4.5) and (4.6) imply that:

$$r\cos u\cos^2 v+r_u\sin u\neq 0,$$
$$r\sin u\cos^2 v-r_u\cos u\neq 0.$$

Also, the Equations (4.5) and (4.7) imply that,

$$(r\cos u\cos^2 v+r_u\sin u)\lambda_3=\lambda_1 r\cos u\cos^2 v, \tag{4.10}$$

and the Equations (4.6) and (4.7) imply that;

$$(r\sin u\cos^2 v-r_u\cos u)\lambda_3=\lambda_2 r\sin u\cos^2 v. \tag{4.11}$$

So, the solution of the differential equations (4.10) and (4.11) we obtain the radius function

$$r(u)=C_3\sqrt{\cos(u)^{\dfrac{(\cos(2v)+1)(\lambda_2-\lambda_3)}{\lambda_3}}}$$

or

$$r(u)=C_4\sqrt{\sin(u)^{\dfrac{(\cos(2v)+1)(\lambda_1-\lambda_3)}{\lambda_3}}}$$

where C_3, C_4 are real constants.
iv) If $\lambda_1=0$ and $\lambda_2=0$ then from the equations (4.5) and (4.6) we get,

$$r\cos u\cos^2 v+r_u\sin u=0,$$
$$r\sin u\cos^2 v-r_u\cos u=0$$

The solution of these differential equations we obtain $H=0$. So this is a contradiction.

Case 2. For the second case we suppose that the radius function $r(u,v)$ given with the parameter v. So, if the function $r=r(v)$ then the mean curvature of the surface M and the conditions of $\Delta x_i=\lambda_i x_i$ are

$$H=-\frac{(r_v\sin v+r\cos v)\left(r^2+r_v^2\right)+r\cos v\left(2r_v^2+r^2-rr_{vv}\right)}{2r\cos v\left(r^2+r_v^2\right)^{3/2}}$$

and

$$-2H(r_v\sin v+r\cos v)=\lambda_1 W \tag{4.12}$$
$$-2H(r_v\sin v+r\cos v)=\lambda_2 W \tag{4.13}$$
$$-2H(-r_v\cos^2 v+r\sin v\cos v)=\lambda_3 W\sin v \tag{4.14}$$

Furthermore, we explore the classification of the surface M given with the parametrization (3.1) satisfying (2.1);

1) Let $\lambda_3=0$. We get two cases for these condition. i) If $\lambda_3=0$ then the equation (4.12) gives rise to $H=0$ which means that the surface is minimal. We get also by the equations (4.12) and (4.13) $\lambda_1=\lambda_2=0$.
ii) If $-r_v\cos^2 v+r\sin v\cos v=0$ then the solution of this differential equation we obtain the radius function

$$r(v)=\frac{C_5}{\cos v} \tag{4.15}$$

where C_5 is a real constant. For the radius function given with (4.15) one can get $H\neq 0$, so we obtain $\lambda_1=\lambda_2=\frac{1}{C_5^2}$.

2) Let $\lambda_3\neq 0$, so $H\neq 0$ and $(-r_v\cos^2 v+r\sin v\cos v)\neq 0$. For the equations (4.12) and (4.13) we get $\lambda_1=\lambda_2$ three cases for these condition.
i) If $\lambda_1=0$ and $\lambda_2\neq 0$ (or $\lambda_1\neq 0$ and $\lambda_2=0$). So, from (4.12) and (4.13) we have

$$r_v\sin v+r\cos v=0. \tag{4.16}$$

The solution of the differential equation (4.16) we obtain the radius function

$$r(v)=\frac{C_6}{\sin v},$$

where C_6 is a real constant. For this radius function we get $H=0$, so this is a contradiction.
ii)If $\lambda_1=\lambda_2\neq 0$ Then the Equations (4.12) and (4.14) imply that,

$$\lambda_3\sin v(r_v\sin v+r\cos v)=\lambda_1\cos v(-r_v\cos v+r\sin v), \tag{4.17}$$

So, the solution of the differential equation (4.17) we obtain the radius function

$$r(v)=\frac{\sqrt{2}C_7}{\sqrt{\lambda_3(1-\cos(2v))+\lambda_1(1+\cos(2v))}}.$$

iii) If $\lambda_1=\lambda_2=0$ then from the equations (4.12) we get $H=0$ or $r_v\sin v+r\cos v=0$.So this is a contradiction.

Theorem 4.1. *Let M be surface given with the parametrization (3.1) in $\mathbb{E}^3$. If the radius function $r(u,v)$ given with the parameter u, then M satisfies $\Delta r_i=\lambda_i r_i$, $(i=1,2,3)$ if and only if the following statements hold:*
i) M has zero mean curvature,
ii) The radius function $r=r(u)$ is

$$r(u)=\frac{C_1}{\sqrt{(\sin u)^{\cos 2v+1}}}\ \textit{or}\ r(u)=\frac{C_2}{\sqrt{\cos(u)^{\cos 2v+1}}},$$

iii) The radius function $r=r(u)$ is

$$r(u)=C_3\sqrt{\cos(u)^{\frac{(\cos(2v)+1)(\lambda_2-\lambda_3)}{\lambda_3}}}\ \textit{or}\ r(u)=C_4\sqrt{\sin(u)^{\frac{(\cos(2v)+1)(\lambda_1-\lambda_3)}{\lambda_3}}}.$$

Theorem 4.2. *Let M be surface given with the parametrization (3.1) in $\mathbb{E}^3$. If the radius function $r(u,v)$ given with the parameter v, then M satisfies $\Delta r_i=\lambda_i r_i$, $(i=1,2,3)$ if and only if the following statements hold:*
i) M has zero mean curvature,
ii) The radius function $r=r(v)$ is

$$r(v)=\frac{C_5}{\cos v}\ \textit{or}\ r(v)=\frac{C_6}{\sin v},$$

iii) The radius function $r=r(v)$ is

$$r(v)=\frac{\sqrt{2}C_7}{\sqrt{\lambda_3(1-\cos(2v))+\lambda_1(1+\cos(2v))}}.$$

Using the similar way we obtain the conchoidal surface M_d of M at distance d given with the parametrization (3.2) satisfying the condition $\Delta x_i = \lambda_i x_i$.

Theorem 4.3. *Let M_d be conchodial surface given with the parametrization (3.2) in $\mathbb{E}^3$. If the radius function $r(u,v)$ given with the parameter u, then M_d satisfies $\Delta r_i = \lambda_i r_i$, $(i = 1,2,3)$ if and only if the following statements hold:*
i) M_d has zero mean curvature,
ii) The radius function $r = r(u)$ is

$$r(u) = \pm d + \frac{C_1}{\sqrt{(\sin u)^{\cos 2v+1}}} \text{ or } r(u) = \pm d + \frac{C_2}{\sqrt{\cos(u)^{\cos 2v+1}}},$$

iii) The radius function $r = r(u)$ is

$$r(u) = \pm d + C_3 \sqrt{\cos(u)^{\frac{(\cos(2v)+1)(\lambda_2-\lambda_3)}{\lambda_3}}} \text{ or } r(u) = \pm d + C_4 \sqrt{\sin(u)^{\frac{(\cos(2v)+1)(\lambda_1-\lambda_3)}{\lambda_3}}}.$$

Theorem 4.4. *Let M_d be conchodial surface given with the parametrization (3.2) in $\mathbb{E}^3$. If the radius function $r(u,v)$ given with the parameter v, then M_d satisfies $\Delta r_i = \lambda_i r_i$, $(i = 1,2,3)$ if and only if the following statements hold:*
i) M_d has zero mean curvature,
ii) The radius function $r = r(v)$ is

$$r(v) = \pm d + \frac{C_5}{\cos v} \text{ or } r(v) = \pm d + \frac{C_6}{\sin v},$$

iii) The radius function $r = r(v)$ is

$$r(v) = \pm d + \frac{\sqrt{2}C_7}{\sqrt{\lambda_3(1-\cos(2v)) + \lambda_1(1+\cos(2v))}}.$$

5. Conclusion

In this study, we study the conchodial surfaces in 3-dimensional Euclidean space with the condition $\Delta x_i = \lambda_i x_i$ where Δ denotes the Laplace operator with respect to the first fundamental form. We give a result for this condition for the special cases of radius function $r(u,v)$. In future studies, this problem can be done for the general solution for radius function. It is possible to consider these kind of surfaces in the other spaces or higher dimensional Euclidean spaces.

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