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RESEARCH ARTICLE

Rotational Self-Shrinkers in Euclidean Spaces

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(Dedicated to Professor Bang-Yen CHEN on the occasion of his 80th birthday)

ABSTRACT

The rotational embedded submanifolds of $\mathbb{E}^{n+d}$ were first studied by N. Kuiper. The special examples of this type are generalized Beltrami submanifolds and toroidals submanifold. The second author and et. all recently have considered $3-$dimensional rotational embedded submanifolds in $\mathbb{E}^5$. They gave some basic curvature properties of this type of submaifolds. Self-similar flows emerge as a special solution to the mean curvature flow that preserves the shape of the evolving submanifold. In this article we consider self-similar submanifolds in Euclidean spaces. We obtained some results related with self-shrinking rotational submanifolds in Euclidean $5-$space $\mathbb{E}^5$. Moreover, we give the necessary and sufficient conditions for these type of submanifolds to be homothetic solitons for their mean curvature flows.

Keywords: *Rotational submanifold, mean curvature flow, homothetic soliton, self-shrinkers.*
AMS Subject Classification (2020): *Primary: 53C42 ; Secondary: 53C40.*

1. Introduction

Let $x : M \to \mathbb{E}^{n+d}$ be an isometric immersion of an $n-$dimensional submanifold M in the $(n+d)-$dimensional Euclidean space $\mathbb{E}^{n+d}$. The position vector x of M is determined by $x = \overrightarrow{op}$ from a point $p \in M$ to an arbitrary reference point $o \in \mathbb{E}^{n+d}$. The position vector field x of the submanifolds M has the decomposition

$$x = x^T + x^N, \tag{1.1}$$

where x^T and x^N are the tangential and normal components of x, respectively [9].

The mean curvature vector field $\overrightarrow{H}$ is one of the most important invariants of the submanifold M. The mean curvature flow is the gradient flow of the area functional on the space of the submanifold M. *Self-similar flows* arise as special solution of the mean curvature flow that preserves the shape of the evolving submanifold [26]. The most important mean curvature flow is the self-similar flow obtained from the following non-linear elliptical system:

$$\overrightarrow{H} + x^N = 0. \tag{1.2}$$

For complete self-shrinkers with constant norm of the second fundamental form see [11].

Another important mean curvature flow is inverse mean curvature flow. Homothetic solitons are the solutions of the *inverse mean curvature flow* defined by

$$-\frac{\overrightarrow{H}}{\left\|\overrightarrow{H}\right\|^2} = cx^{\perp}. \tag{1.3}$$

where, $x^{\perp}$ is the normal component of the position vector x. Here, the solutions of equation (1.3) form homothetic solitons for constant speed $c \neq 0$ inverse mean curvature flow. If $c < 0$, the inverse mean curvature flow is said to have *contracting solutions*, if $c > 0$, it is said to have *expanding solutions* [7].

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This paper is organized as follows: In section 2, some basic concepts, theorems and definitions which will be used in the other sections are given. In particular, the basic concepts of differentiable maps, submanifolds and their second fundamental form, Gauss and Weingarten equations, and mean curvature are given. In Section 3, the materials and methods used in obtaining the results are given. This section consists of four parts. In the first part, the properties of the mean curvature flow are examined and some examples are given. In the second part, the inverse mean curvature flow is discussed. In the third part, the basic properties of the mean curvature flow preserving the weighted volume are given. In the last part homothetic solitons are examined. In the final section we considered $3-$dimensional rotational submanifolds in $\mathbb{E}^5$. Further, we give some results of two types of rotational submanifolds $\mathbb{E}^5$ satisfying the self-similarity condition. Further, we obtained some results related with the 3-dimensional rotational homothetic submanifolds in $\mathbb{E}^5$. We also give some examples of these type of submanifolds.

2. Basic concepts

Let $x : M \longrightarrow \mathbb{E}^{n+d}$ be an immersed submanifold in the Euclidean space $\mathbb{E}^{n+d}$. Denote by $\chi(M)$ and $\chi^{\perp}(M)$ the space of the smooth vector fields tangent and normal to M, respectively. Given any orthonormal local vector fields $e_1, e_2, ..., e_n$ tangent to M, one considers the second fundamental map $h : \chi(M) \times \chi(M) \rightarrow \chi^{\perp}(M)$ given by

$$h(e_i, e_j) = \widetilde{\nabla}_{e_i} e_j - \nabla_{e_i} e_j; \quad 1 \leq i, j \leq n, \tag{2.1}$$

where ∇ and $\widetilde{\nabla}$ are the induced connection of M and the Riemannian connection of $\mathbb{E}^{n+d}$, respectively. This map is well-defined, symmetric and bilinear [8].

For any arbitrary orthonormal frame field $\{n_1, n_2, ..., n_d\}$ of M, recall the shape operator $A : \chi^{\perp}(M) \times \chi(M) \rightarrow \chi(M)$ given by

$$A_{n_\alpha} e_k = -\widetilde{\nabla}_{e_k} n_\alpha + D_{e_k} n_\alpha, \quad 1 \leq \alpha \leq d, 1 \leq k \leq n, \tag{2.2}$$

where D is the connection of the normal bundle of M. The shape operator is bilinear, self-adjoint and satisfies the following equation:

$$\langle A_{n_\alpha} e_j, e_i \rangle = \langle h(e_i, e_j), n_\alpha \rangle = h^\alpha_{ij}, \ 1 \leq i, j \leq n; \ 1 \leq \alpha \leq d, \tag{2.3}$$

where h^α_{ij} are the coefficients of the second fundamental form. The equations (2.1) and (2.2) are called *Gaussian formula* and *Weingarten formula* respectively. In addition,

$$h(e_i, e_j) = \sum_{\alpha=1}^{d} h^\alpha_{ij} n_\alpha, \ 1 \leq i, j \leq n \tag{2.4}$$

holds. The *mean curvature vector* $\overrightarrow{H}$ is defined by

$$\overrightarrow{H} = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i). \tag{2.5}$$

The norm $H = \left\| \overrightarrow{H} \right\|$ is called the *mean curvature* of M^n [8]. The submanifold M is called *minimal* if the the mean curvature H of M vanishes identically.

3. Material and Method

In the present section we consider mean curvature flow and homothetic solitons in $\mathbb{E}^{n+d}$. We give some well-known results and examples regarding these concepts.

3.1. Mean Curvature Flow

Let $x : M \rightarrow \mathbb{E}^{n+d}$ be an isometric immersion of an $n-$dimensional submanifold M in the Euclidean space $\mathbb{E}^{n+d}$. Define the family of smooth immersions

$$x(p, t) : M \rightarrow \mathbb{E}^{n+d}, \ x(p, 0) = x(p). \tag{3.1}$$

Then the mean curvature flow of x is a family $x_t : M \to \mathbb{E}^{n+d}$ that satisfies

$$\left(\frac{\partial}{\partial t} x_t(p)\right)^{\perp} = H(p,t),\ x_0 = x, \tag{3.2}$$

where $H(p,t)$ is the mean curvature vector of $x_t(M)$ at $x_t(p)$ and $v^{\perp}$ denotes the projection of v into the normal space of $x_t(M)$ [26]. The mean curvature flow was studied in [31] and [30]. For higher order submanifolds, see [13] and [32]. However, the mean curvature flow of all graphs was also analyzed [17]. For detailed information about the mean curvature flow see also [28].

The area functional of the isometric immersion $x : M \to \mathbb{E}^{n+1}$ for the $n-$dimensional hypersurface M is calculated by $A(x) = \int\limits_M d\mu$, where μ is the canonical measure with respect to the induced metric g of the Euclidean space $\mathbb{E}^{n+1}$. The first variation of the area functional becomes

$$\frac{d}{dt}A(x(p,t)) = \frac{d}{dt}\int\limits_M d\mu_t = -\int\limits_M H d\mu_t. \tag{3.3}$$

Therefore, the change in area is non-increasing. In other words, there will be no increase in surface area of the family M_t along the mean curvature flow, but a decrease may be possible.

Example 3.1. Let $B_r^{n+1}(x) = \left\{y \in \mathbb{E}^{n+1} : |y - x| \le r\right\}$ be an open ball in $\mathbb{E}^{n+1}$. Denote by $M_t = B_{r(t)}^{n+1}, t \in \mathbb{R}$ a family of the concantrix sphere in $\mathbb{E}^{n+1}$ with the radius of $r(t)$. As it is known, since the mean curvature remains invariant under the isometries of $\mathbb{E}^{n+1}$, equation (3.2) turns into an ordinary differential equation $\frac{d}{dt}r(t) = -\frac{n}{r(t)}$ of the radius function $r(t)$. In this case, for $r(0) = \rho$, $M_0 = \partial B_\rho$ a non-trivial solution of equation (3.4) is $r(t) = \sqrt{\rho^2 - 2nt}$, $t \in (-\infty, \frac{\rho^2}{2n})$ (see, ([16])). Consequently, for $t \mapsto \frac{\rho^2}{2n}$ the hypersphere shrinks to a point (see Figure 1).

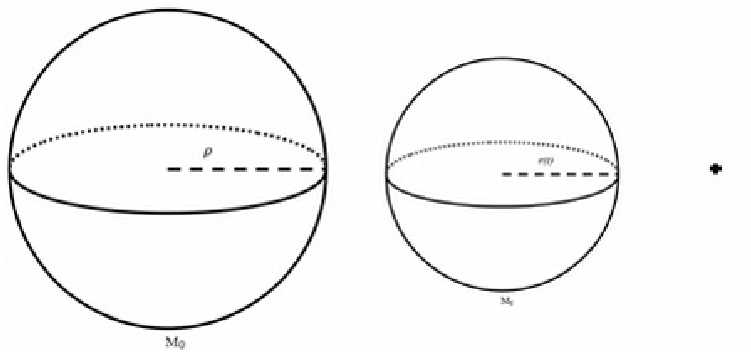

Figure 1. The hypersphere shrinks to a point

Example 3.2. Let $M_t = B_{r(t)}^{n+1-k} \times \mathbb{E}^{n+1}$, $t \in I$, $0 \le k \le n$ be a hypercylinder in $\mathbb{E}^{n+1}$. Then equation (3.2) turns into an ordinary differential equation of the radius function $r(t)$ of the form $\frac{d}{dt}r(t) = -\frac{n-k}{r(t)}$. In this case, for $r(0) = \rho$, $M_0 = \partial B_\rho$ a non-trivial solution of equation is $r(t) = \sqrt{\rho^2 - 2(n-k)t}$, $t \in (-\infty, \frac{\rho^2}{2(n-k)})$ (see, ([16])). In this case, for $t \mapsto \frac{\rho^2}{2(n-k)}$ the hypercylinder shrinks to a line (see Figure 2).

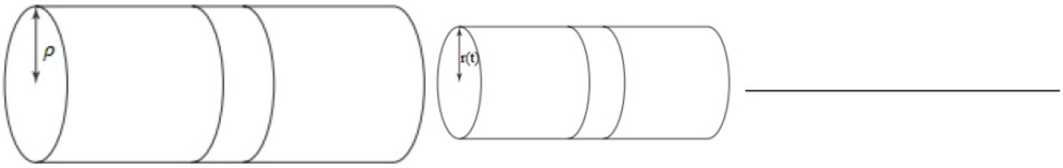

Figure 2. The hypercylinder shrinks to a line

Example 3.3. Let $M_0 \subset \mathbb{E}^3$ be torus defined as the locus of the points at a distance of ρ units from the unit circle. In this case, the mean curvature H_0 of M_0 should be positive for or $\rho < \frac{1}{2}$. Suppose the region formed by M_t is Ω_t. Thus, according to the maximum principle, as t increases, the mean curvature H_t of M_t will increase. Furthermore, the torus shrinks into a circle [16] (see Figure 3).

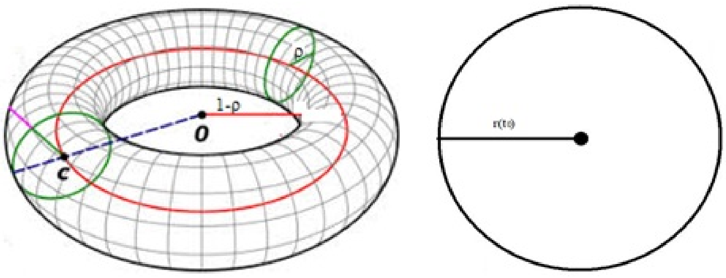

Figure 3. The torus shrinks to a circle

Definition 3.1. A submanifold M in the Euclidean space $\mathbb{R}^{n+d}$ is called self-shrinker if the curvature vector field $\overrightarrow{H}$ of M satisfies the following non-linear elliptic system:

$$\overrightarrow{H} + \lambda x^N = 0, \tag{3.4}$$

where x^N is the normal component of x and λ is a positive real valued function. The submanifold is self-expander if $\lambda < 0$.

The case of vanishing λ is the well-known case of minimal submanifold, which of course is stationary under the action of the flow [17]. Infact, self-similar submanifolds are spacial type of self-shrinker submanifolds with $\lambda = 1$ [20].

The classification and analysis of low-index mean curvature flow and automatic shrinkage are considered in [25]. For the exact surfaces that shrink themselves, see [29]. Recently, self-similar surfaces have also been studied in [18]. A review article [14] on closed hypersurfaces that shrinks and includes symmetry is noteworthy. Results on spontaneous contraction and singularity of the mean curvature flow [21] can also be seen.

In the plane $M = \Gamma \subset \mathbb{E}^2$ is a curve then the all solutions of (3.4) have been classified by Abresch and Langer [1]. They showed that if a planar curve $\Gamma \subset \mathbb{E}^2$ is self-shrinking, then there is a constant c_Γ such that $\kappa e^{-\frac{|\gamma|}{2}} = c_\Gamma$ holds on all of Γ, where κ is the curvature of Γ.

In [32] K. Smooczyk proved the following results:

Theorem 3.1. *Let $x : M \to \mathbb{E}^{n+d}$ be a closed self-shrinker then M is a minimal submanifold of the sphere $S^{n+d-1}(\sqrt{n})$ if and only if $\overrightarrow{H} \neq 0$ and $\nabla^\perp \upsilon = 0$, where $\upsilon = \frac{\overrightarrow{H}}{\|\overrightarrow{H}\|}$ is the principal normal of the submanifold M.*

Theorem 3.2. *Let $x : M \to \mathbb{E}^{n+d}$ be a compact self-shrinker. Then M is spherical submanifold if and only if $\overrightarrow{H} \neq 0$ and $\nabla^\perp \upsilon = 0$ holds identically.*

Theorem 3.3. *Let $x : M \to \mathbb{E}^{n+d}$ be a complete non compact connected self-shrinker with $\overrightarrow{H} \neq 0$ and $\nabla^\perp \upsilon = 0$. Then M must belong to one of the followings:*

$$\Gamma \times \mathbb{E}^{n-1}, N^k \times \mathbb{E}^{n-k}.$$

Here, Γ is one of the Abresch-Langer curves and N^k is a complete minimal submanifold of the sphere $S^{k+d-1} \subset \mathbb{E}^{d+k}$, $0 \leq k \leq n$.

Later, C. Arezzo and J. Sun proved the following result in [3].

Theorem 3.4. *A complete submanifold $M \subset S^{n+d-1}(r) \subset \mathbb{E}^{n+d}$ is minimal submanifold of $S^{n+d-1}(r)$ if and only if it is a self-shrinker in $\mathbb{E}^{n+d}$ for $\lambda = \frac{n}{r^2}$, where λ is a real constant.*

In [5] H-D Cao ad H. Li listed the following examples of self-shrinkers.

Example 3.4. For any positive integers $m_1, ..., m_d$ such that $m_1 + ... + m_d = n$, the submanifold $M = S^{m_1}(\sqrt{m_1}) \times ... \times S^{m_d}(\sqrt{m_d})$ is an n-dimensional compact self-shrinker in $\mathbb{E}^{n+d}$ with $\overrightarrow{H} = -x$, $\left\| \overrightarrow{H} \right\| = n$, $\|A\|^2 = d$. Here $S^{r_i}(r_i) = \left\{ x_i \in \mathbb{E}^{m_i+1} : \|x_i\|^2 = r_i^2 \right\}$, $1 \leq i \leq d$, is m_i-dimensional round sphere with radius r_i.

Example 3.5. For any positive integers $m_1, ..., m_d$, $q \leq 1$ with $m_1 + ... + m_d + q = n$, the submanifold $M = S^{m_1}(\sqrt{m_1}) \times ... \times S^{m_d}(\sqrt{m_d}) \times \mathbb{E}^q \subset \mathbb{E}^{n+d}$ is an n-dimensional complete non-compact self-shrinker in $\mathbb{R}^{n+d}$ with $\overrightarrow{H} = -x^\perp$ and

$$\left\| \overrightarrow{H} \right\|^2 = \sum_{i=1}^{n} m_i, \|A\|^2 = d.$$

3.2. Homothetic Solitons

Let $x : M \to \mathbb{E}^{n+d}$ be an isometric immersion of an $n-$dimensional submanifold M in the Euclidean space $\mathbb{E}^{n+d}$. Define the family of smooth immersion $x(p,t) : M \to \mathbb{E}^{n+d}$, $x(p,0) = x(p)$. Then the *inverse mean curvature flow* of x is a family $x_t : M \to \mathbb{E}^{n+d}$ that satisfies

$$\left(\frac{\partial}{\partial t} x_t(p) \right)^\perp = \frac{H(p,t)}{\|H(p,t)\|}, \; x_0 = x, \tag{3.5}$$

where $H(p,t)$ is the mean curvature vector of $x_t(M)$ at $x_t(p)$ and $v^\perp$ denotes the projection of v into the normal space of $x_t(M)$ [6], [15].

A lot of work has been done on the inverse mean curvature flow when M a is hypersurface, see for example [23] and [24]. Under the inverse mean curvature flow of an immersion $x : M \to \mathbb{E}^{n+d}$, the change of flow relative to homothety will be of the form $F(p,t) = e^{ct}x(p)$, $c \neq 0$, $t < 0$, where the immersion x satisfies the equation

$$-\frac{\overrightarrow{H}}{\left\| \overrightarrow{H} \right\|^2} = cx^\perp, \left\| \overrightarrow{H} \right\| \neq 0. \tag{3.6}$$

Here, the solutions of equation (3.6) form homothetic solitons for constant speed $c \neq 0$ inverse mean curvature flow. If $c < 0$, the inverse mean curvature flow is said to have contracting solutions, if $c > 0$, it is said to have expanding solutions [7]. The solution of equation (3.6) for planar curves is also given in [14]. In particular, the involutes of the classical logarithmic spiral and circle are shown to be expanding solutions of the inverse mean curvature flow.

Remark 3.1. For a homothetic soliton, $M \subset \mathbb{E}^{n+d}$ the condition (3.6) transforms into the form

$$\left\| \overrightarrow{H} \right\|^2 = \left\langle \overrightarrow{H}, \overrightarrow{H} \right\rangle = -\left\langle c \left\| \overrightarrow{H} \right\|^2 x^\perp, \overrightarrow{H} \right\rangle = -c \left\| \overrightarrow{H} \right\|^2 \left\langle x^\perp, \overrightarrow{H} \right\rangle. \tag{3.7}$$

However, since $\left\| \overrightarrow{H} \right\| \neq 0$, the relations

$$\left\langle x, \overrightarrow{H} \right\rangle = -\frac{1}{c}, \text{ or } \langle \Delta_g x, x \rangle = -\frac{1}{c}, \text{or } \Delta_g \|x\|^2 = 2\left(n - \frac{1}{c} \right), \tag{3.8}$$

can be obtained with the help of the equation (3.7). Here, g is the metric of M induced from $\mathbb{E}^{n+d}$ [15].

With the help of previous remark and using [7] one can get the following result.

Proposition 3.1. *Let $x : M \to \mathbb{E}^{n+d}$ be an isometric immersion of an $n-$dimensional submanifold M in the Euclidean space $\mathbb{E}^{n+d}$. Then M is a homothetic soliton of the inverse mean curvature flow if and only if*

$$\left\langle x, \overrightarrow{H} \right\rangle = -\frac{1}{c}$$

holds, where c is constant speed of inverse mean curvature flow.

As can be seen from the definition, the homothetic soliton is the self-similar solution of the inverse mean curvature flow.

Definition 3.2. Let $x : M \to \mathbb{E}^{n+d}$ be an isometric immersion of an $n-$dimensional submanifold M in the Euclidean space $\mathbb{E}^{n+d}$. For any smooth vector field $Z \in T(M)$ if the equation $div(Z) = 0$ holds then Z is called an incompressible vector field [10].

The following result is due to [10].

Proposition 3.2. *Let $x : M \to \mathbb{E}^{n+d}$ be an isometric immersion of an $n-$dimensional submanifold M in the Euclidean space $\mathbb{E}^{n+d}$. Then the tangent component x^T of the position vector field x is incompressible if and only if $\left\langle x, \overrightarrow{H} \right\rangle = -\frac{1}{c}$ holds, where c is constant speed of inverse mean curvature flow.*

As a consequence of the previous propositions one can get the following result.

Corollary 3.1. *Let $x : M \to \mathbb{E}^{n+d}$ be an isometric immersion of an $n-$dimensional submanifold M in the Euclidean space $\mathbb{E}^{n+d}$. If M is a homothetic soliton of the inverse mean curvature flow with $c = 1$ then the tangent component x^T of the position vector field x is incompressible.*

4. Results and Discussion

The mean curvature flow is the gradient flow of the functional field of the $n-$dimensional M submanifold. From the point of view of analysis, this flow is produced by a non-linear parabolic equation. Although the classified results of the analysis indicate the short-term existence of the mean curvature flow, understanding the long-term behavior is a difficult problem that requires controlling for possible singularities that may arise along the flow. The mean curvature vector field H is one of the most important invariants of the M submanifold. In physics, the mean curvature vector field is the torsion field applied to the submanifold originating from the destination space. Self-similar flows arise as a special solution of the mean curvature flow that preserves the shape of the submanifold [22]. The most important mean curvature flow is the self-similar flow obtained when the change becomes a homothety. The mean curvature vector H of such self-similar submanifold satisfies the following nonlinear elliptic system given in (3.4).

In [18], the present authors and E. Etemoğlu investigated self-similar surfaces in Euclidean $4-$space $\mathbb{E}^4$. Additionally, we give necessary and sufficient conditions of spherical product surfaces and surfaces with Monge patch in $\mathbb{E}^4$ to become self-similar. In the current work, as a generalization of this work, we obtain some results regarding the self-similarity of $3-$dimensional rotational submanifolds in Euclidean space $\mathbb{E}^5$.

Let $f : N^d \to \mathbb{E}^r$; $f(x) = (f_1(x), ..., f_r(x)), x \in N^d$ be an isometric immersion of $d-$dimensional Riemannian manifold N^d into r-dimensional Euclidean space $\mathbb{E}^r$. Consider the standard immersion $g : S^{q-1} \to \mathbb{E}^q$ onto unit sphere S^{q-1}. By rotating the submanifold N^d around S^{q-1} one can obtain a rotational submanifold M given with the isometric immersion

$$\varphi : M \to \mathbb{E}^{r+q-1};\ \varphi(x, y) = (f_1(x), ..., f_{r-1}(x), f_r(x)g(y)), \tag{4.1}$$

where the last component $g(y)$, being the position vector in $\mathbb{E}^q$ and $f_r(x) > 0$ for all $x \in N^d$, $y \in S^{q-1}$ [27]. If we choose N^d as the regular curve $\gamma(I)$, $I \subset \mathbb{R}$, in $r-$dimensional Euclidean space $\mathbb{E}^r$ then the resultant rotational submanifold M which lies in ambient space $\mathbb{E}^{r+q-1}$ will be represented by the isometric immersion

$$\varphi(s, y) = (f_1(s), ..., f_{r-1}(s), f_r(s)g(y)). \tag{4.2}$$

where the last component $g(y)$ represents either a unit speed spherical curve or a spherical submanifold of $\mathbb{E}^q$. Specifically, if $f_r(s) = e^{-s}$ is taken, the curve

$$\gamma(s) = \left(f_1(s), ..., f_{r-1}(s), e^{-s}\right)$$

indicates a *generalized tractrix* and the submanifold obtained by isometric immersion $\varphi = \varphi(x, y)$ is called *generalized Beltrami submanifold* [19]. In the same study, the authors showed that the generalized Beltrami submanifold is pseudo-spherical, its sectional curvature is $K = -1$.

In [4] the authors studied 3-dimensional rotational submanifolds in 5-dimensional Euclidean space $\mathbb{E}^5$. They considered the following spacial case:

For $p = 2$ and $q = 4$, the isometric immersion

$$X(s,u,v) = (f_1(s), f_2(s)g(u,v)), \tag{4.3}$$

with

$$g(u,v) = (0; a_1 \cos u, a_1 \sin u, a_2 \cos v, a_2 \sin v), \tag{4.4}$$

describes a rotational submanifold M in 5-dimensional Euclidean space $\mathbb{E}^5$ [4]. The surface given with the position vector (4.4) is a Clifford torus T^2 in E^4, such that $a_1, a_2 \in \mathbb{R}$ are real constants satisfying $a_1^2 + a_2^2 = 1$. From now on we assume that M is a rotational submanifold in 5-dimensional Euclidean space $\mathbb{E}^5$.

We choose a moving frame $\{e_1, e_2, e_3, e_4, e_5\}$ such that e_1, e_2, e_3 are tangent to M and e_4, e_5 are normal to M in the following (see [4]);

$$\begin{aligned} e_1 &= \frac{x_s}{\|x_s\|},\ e_2 = \frac{x_u}{\|x_u\|},\ e_3 = \frac{x_v}{\|x_v\|} \\ e_4 &= \frac{1}{\kappa}(f_1'', a_1 f_2'' \cos u, a_1 f_2'' \sin u, a_2 f_2'' \cos v, a_2 f_2'' \sin v) \\ e_5 &= (0, a_2 \cos u, a_2 \sin u, -a_1 \cos v, -a_1 \sin v) \end{aligned} \tag{4.5}$$

where $\kappa > 0$ is the curvature of the unit speed profile curve γ defined by

$$\kappa^2 = \|\gamma''(s)\|^2 = \frac{(f_2''(s))^2}{1 - (f_2'(s))^2}. \tag{4.6}$$

With respect to this frame we can obtain the second fundamental maps (see [4]);

$$\begin{aligned} h(e_1, e_1) &= \kappa e_4, \\ h(e_2, e_2) &= -\frac{f_2''}{\kappa f_2} e_4 - \frac{a_2}{a_1 f_2} e_5, \\ h(e_3, e_3) &= -\frac{f_2''}{\kappa f_2} e_4 + \frac{a_1}{a_2 f_2} e_5, \\ h(e_1, e_2) &= h(e_1, e_3) = h(e_2, e_3) = 0. \end{aligned} \tag{4.7}$$

Consequently, by the use of (2.5) with (4.7) the mean curvature vector $\vec{H}$ of M becomes

$$\vec{H} = \frac{1}{3}\left\{ \left(\kappa - \frac{2 f_2''}{\kappa f_2} \right) e_4 + \left(\frac{a_1^2 - a_2^2}{a_1 a_2 f_2} \right) e_5 \right\}. \tag{4.8}$$

As a consequence of (4.8) with (4.6) one can get the following result.

Corollary 4.1. *[4] Let M be a rotational submanifold in $\mathbb{E}^5$ given with the parametrization (4.3). Then M is minimal if and only if*

$$f_2(s) f_2''(s) + 2(f_2'(s))^2 - 2 = 0 \text{ and } a_1 = a_2 = \frac{1}{\sqrt{2}}. \tag{4.9}$$

We obtain the following result:

Proposition 4.1. *Let $M \subset \mathbb{E}^5$ be a the rotational submanifold given by the parametrization (4.3). Then M is a homothetic soliton of the inverse mean curvature flow if and only if*

$$(f_1 f_1'' + f_2 f_2'') \left(\frac{f_2 f_2'' - 2\left(1 - (f_2')^2\right)}{f_2 f_2''} \right) = -\frac{3}{c}. \tag{4.10}$$

Proof. With the help of (4.3) and (4.5) we get

$$\langle e_4, x\rangle = \frac{f_1 f_1'' + f_2 f_2''}{\kappa}, \ \langle e_5, x\rangle = 0. \tag{4.11}$$

Further, by the use of (4.3) with (4.8) we obtain

$$\left\langle \overrightarrow{H}, x \right\rangle = (f_1 f_1'' + f_2 f_2'')\left(\frac{f_2 f_2'' - 2\left(1-(f_2')^2\right)}{3 f_2 f_2''}\right). \tag{4.12}$$

Assume that M is a homothetic solution of the inverse mean curvature flow then equation $\left\langle x, \overrightarrow{H} \right\rangle = -\frac{1}{c}$ is satisfied. Thus, the desired result is obtained by using the equation (4.12). □

From the orthogonal decomposition (1.1) of the position vector x of M_1 we obtain

$$x^N = x - \rho'(s)e_1, \tag{4.13}$$

where $\rho(s) = \frac{1}{2}\|x\|^2$ is the square norm of the distance function of the position vector x such that

$$\rho'(s) = f_1(s)f_1'(s) + f_2(s)f_2'(s). \tag{4.14}$$

Due to [9] we obtain the following results.

Theorem 4.1. *Let M be a rotational submanifold in $\mathbb{E}^5$ given with the parametrization (4.3). Then $x = x^N$ holds identically if and only if M is a spherical submanifolds of $\mathbb{E}^5$.*

Proof. Assume that M is a rotational submanifold in $\mathbb{E}^5$ given with the parametrization (3.4). If $x = x^N$ holds identically, then we get $\rho'(s) = 0$. Further, the the meridian curve γ has arc-length parameter we have the system of differential equations

$$\begin{aligned} (f_1'(s))^2 + (f_2'(s))^2 &= 1, && (4.15)\\ f_1(s)f_1'(s) + f_2(s)f_2'(s) &= 0, && (4.16) \end{aligned}$$

which has a nontrivial solution

$$f_1(s) = \frac{c_1}{2}\sin\left(\frac{2(c_2-s)}{c_1}\right), f_2(s) = \frac{c_1}{2}\cos\left(\frac{2(c_2-s)}{c_1}\right), \ c_1, c_1 \in \mathbb{R}. \tag{4.17}$$

Consequently, it is a parametrization of a circle of radius $r = \frac{c_1}{2}$. Thus the rotational submanifold M_1 with this meridian curve is a spherical submanifolds of $\mathbb{E}^5$. The converse is clear. □

Example 4.1. If the meridian curve of the rotational submanifold $M \subset \mathbb{E}^5$ is taken as a unit circle, then the submanifold M becomes a spherical submanifold with parameterization

$$x(s,u,v) = (\cos s, a_1 \sin s \cos u, a_1 \sin s \sin u, a_2 \sin s \cos v, a_2 \sin s \sin v).$$

Since $c = -1$, M is a shrinking homothetic soliton of the inverse mean curvature flow.

In [12] Q-M Cheng and Y. Peng considered complete proper self-shrinkers of 3 dimension.

Theorem 4.2. *Let $x : M \to \mathbb{E}^5$ be an isometric immersion of an $3-$dimensional proper self-shrinker submanifold without boundary and with $H > 0$. If the principal normal $v = \frac{\overrightarrow{H}}{\|\overrightarrow{H}\|}$ is parallel in the normal bundle of M and the squared norm of the second fundamental form is constant, then M is one of the following:*
i) $S^k(\sqrt{k}) \times \mathbb{R}^{3-k}$, $1 < k < 3$, *with* $\|h\|^2 = 1$,
ii) $S^1(1) \times S^1(1) \times \mathbb{R}$, *with* $\|h\|^2 = 2$,
iii) $S^1(1) \times S^2(\sqrt{2})$, *with* $\|h\|^2 = 2$,
iv) The 3-dimensional minimal isoparametric Cartan hypersurface with $\|A\|^2 = 3$.

We have the following result.

Proposition 4.2. *Let M be a rotational submanifold in $\mathbb{E}^5$ given with the parametrization (4.3). If M is a self self-shrinker submanifold of $\mathbb{E}^5$ then*

$$\frac{1}{3}\left\{\left(\kappa-\frac{2f_2''}{\kappa f_2}\right)^2+\left(\frac{a_1^2-a_2^2}{a_1a_2f_2}\right)^2\right\}+\lambda\left(\frac{f_2f_2''-2\left(1-(f_2')^2\right)}{f_2f_2''}\right)(f_1f_1''+f_2f_2'')=0, \tag{4.18}$$

holds, where $\kappa>0$ is the curvature of the profile curve γ.

Proof. Assume that M is a rotational submanifold in $\mathbb{E}^5$ given with the parametrization (4.3). If M is a self-shrinker then by Definition 3.1

$$\left\langle\vec{H},\vec{H}\right\rangle+\lambda\left\langle\vec{H},x\right\rangle=0;\ \lambda>0, \tag{4.19}$$

holds identically. Furthermore, substituting (4.8) and (4.11) into (4.19) we obtain the desired result. □

5. Conclusion

In the 1970s, physicists and mathematicians began to seriously study the classical field equation in its purely nonlinear form and to interpret some solutions as candidates for particles of the theory. In this study, mean curvature flow and inverse mean curvature flow of isometric immersions of Euclidean submanifolds. Homothetic solitons are solutions of the inverse mean curvature flow. Especially rotational submanifolds, which have wide application areas, are discussed. Necessary and sufficient conditions are obtained for such submanifolds in $\mathbb{E}^5$ to be homothetic solitons and self-similar. Also, some examples are given to support the results obtained. We think that these results can be attributed to future studies on Ricci and Yamebe solitons of rotational submanifolds.

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