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F_pR – Linear Skew Constacyclic Codes

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F_p R – Lineer Çarpık Sabit Devirli Kodlar

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Öz

Bu makalede, F_pR halkası üzerinde skew constacyclic kodlar olarak adlandırılan özel bir doğrusal kod sınıfını olan çalışıyoruz, burada $R = F_p + vF_p$, p tek asal sayıdır ve $v^2 = v$. Bu kodlar $F_p^m R^n$ halkasının bir alt kümesi olarak tanımlanır. R nin bir θ otomorfizması için, $R[x, \theta]$ skew polinom halkasının yapısal özelliklerini araştırıyoruz. Ayrıca, F_pR halkası üzerinde skew constacyclic kodların üreteç polinomlarını ve Gray görüntülerini belirliyoruz.

Anahtar Kelimeler: *Lineer kodlar; Skew polinom halkaları; Skew constacyclic kodlar; Skew devirli kodlar.*

Abstract

In this paper, we study a special class of linear codes, called skew constacyclic codes, over the ring F_pR, where $R = F_p + vF_p$, p is an odd prime number and $v^2 = v$. These codes are defined as a subset of the ring $F_p^m R^n$. For an automorphism θ of R, we investigate the structural properties of skew polynomial ring $R[x, \theta]$. We also determine the generator polynomials and the Gray images of the skew constacyclic codes over the ring F_pR.

Keywords: *Linear codes; Skew polynomial rings; Skew constacyclic codes; Skew cyclic codes.*

1. Introduction

Codes over finite rings have attracted considerable interest for several decades. One of the significant class of linear codes is known as cyclic codes. Since cyclic codes have very rich algebraic structures, these codes have been examined by many researchers (Zhu et al. 2010, Siap *et al.* 2011, Dinh et al 2020).

Recently, Bouncher *et al.* investigated skew cyclic codes over finite fields (Bouncher *et. al* 2007). These codes were obtained through non-commutative polynomial rings. They showed that skew cyclic codes have many advantages over well-known linear codes of the same dimension and length. Inspired by this study, there are numerous papers on skew cyclic codes over finite fields. For instance; Gursoy *et al.* considered skew cyclic codes over $F_q + vF_q$ (Gursoy *et al.* 2014). Siap *et al.* studied skew cyclic codes for arbtrary length and obtained optimal linear codes over finite fields (Siap *et al.* 2011).

Mixed alphabets were first introduced by Delsarte (Delsarte 1973). Later, many papers over mixed alphabet codes were studied (Aksoy and Caliskan 2021, Li *et al.* 2021, Dinh *et al*. 2020, Caliskan et al. 2023). The most striking among these studies is the skew cyclic codes over the mixed alphabets. Benbelkacem *et al.* considered skew cyclic codes over F_4R (Benhelkacem *et al.* 2022). Li, Gao and Fu presented linear skew cyclic codes on F_qR (Li *et al.* 2021). Besides, Abualrub and Aydin introduced skew cyclic codes over $F_2 + vF_2$ (Abualrub and Aydin 2012). Gao defined skew cyclic codes over $F_p + vF_p$ and showed that obtained results are equivalent to either cyclic codes or quasi-cyclic codes (Gao 2013). As a generalization of skew cyclic codes, skew constacyclic codes over various rings have been widely studied (Jitman and Ling 2012, Li *et al*. 2020, Melakhesson *et al.* 2019). Al- Ashker et al. studied skew constacyclic codes over $F_p + vF_p$ (Al-Ashker *et al.* 2021). Melakhesson *et al*. defined linear skew constacyclic codes over $Z_q(Z_q + uZ_q)$ (Melakhessou *et al.* 2019).

The aim of this paper is to present and study skew constacyclic codes over the ring F_pR, where p is an odd prime and $R = F_p + vF_p$ with $v^2 = v$. The ring F_pR is a finite semi-local and not a chain ring. The paper is organized as follows: Section 2 starts with some basic properties of the ring R and give brief description of the linear codes over the ring R. Then, it continues by introducing the algebraic structure of skew polynomial rings as well as the basic results of skew constacyclic codes over the ring R. In section 3, linear codes are

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generalized to the skew constacyclic codes and examine the algebraic structures of these codes. In section 4, we describe the generator polynomials of the linear skew constacyclic codes over the ring F_pR. In section 5, we determine the Gray images of skew constacyclic codes over the ring F_pR.

2. Preliminaries

Consider the ring $R = F_p + vF_p$, where $v^2 = v$ and p is a number of odd prime. The ring R has two maximal ideals which are $< v >$ and $< 1 - v >$ such that both $R/< v >$ and $R/< 1 - v >$ are isomoprhic to F_p. These ideals are maximal ideals in the ring R. Thus, R is not a chain ring. By the Chinese Remainder Theorem, one gets $R =< v > \oplus < 1 - v >$ (Lac, 2008). So, each element of R can be state as

$$a + vb = vd + (1 - v)e, \quad (1)$$

where $a, b, d, e \in F_p$.

A code C of lenght n over R is a non-empty subset of R^n and also it is a linear code if it is a submodule of the R-module R^n. If $(c_0, c_1, \dots, c_{-1}) \in C$, then its polynomial representation is defined as $\varrho(C) = \sum_{i=0}^{n-1} c_i x^i$. In the rest of the paper, we assume that all the codes are linear codes.

The Euclidean inner product on R is defined as

$$< x, y >= \sum_{i=0}^{n-1} x_i y_i, \quad (2)$$

for $x = (x_0, x_1, \cdots, x_{n-1})$ and $y = (y_0, y_1, \cdots, y_{n-1})$ in R^n.

The dual code of C, denoted *by* $C^\perp$, is also R-linear code and defined as

$$C^\perp = \{y \in R^n \mid < x, y >= 0, \forall\, x \in C\}. \quad (3)$$

2.1 Skew polynomial rings over R

First, we recall the construction of the non-commutative ring $R[x, \theta]$ and some of its basic properties (Gao 2013). The skew polynomial set $R[x, \theta]$ is defined by

$$R[x, \theta] = \{f(x) = r_0 + r_1 x + r_2 x^2 + \dots + r_n x^n\}, \quad (4)$$

where $r_i \in F_p$ for all $i = 0, 1, \cdots, n$ and the automorphism θ of R is defined as

$$\theta(vd + (1 - v)e) = (1 - v)d + ve, \quad (5)$$

where $d, e \in F_p$. Note that $\theta^2(r) = r$ for all $r \in R$, and θ is a ring homomorphism with order 2.

The skew polynomial ring $R[x, \theta]$ is the set of polynomials over the ring R in which the additon is the usual adddition of polynomials and multiplicaton is defined as

$$(ax^i)(bx^i) = a\theta^i(b)x^{i+j}\ . \quad (6)$$

Multiplication can be extended to all elements in $R[x, \theta]$ by the laws of distribution and association. If θ is not an identity automorphism on R, then the ring $R[x, \theta]$ is not a commutative ring. Thus, we have the following results:

Theorem 2.1. (Gao 2013). The center $Z([R, \theta])$ of $R[x, \theta]$ *is* $F_p[x^2]$, where 2 is the order of θ.

Corollary 2.2. (Gao 2013). Let $f(x) = x^n - 1$. Then $f(x) \in Z([R, \theta])$ if and only if *n* is even.

Let $g(x), f(x) \in [R, \theta]$, then $g(x)$ is called a right (resp. left) divisor of $f(x)$ if there exists $q(x) \in [R, \theta]$ such that $f(x) = q(x)g(x)$ (resp. $f(x) = g(x)q(x)$).

Lemma 2.3. (Gao 2013). Let $f(x), g(x) \in [R, \theta]$ such that the leading coefficient of $g(x)$ is a unit. Then, there exists unique $q(x), r(x) \in [R, \theta]$ such that $f(x) = q(x)g(x) + r(x)$, where $r(x) = 0$ or $\deg(r(x)) < \deg(g(x))$.

Definition 2.4. Let θ be an automorphsm of R. A code C is an R –linear skew cyclic code (or θ-cyclic code) of length *n* if

(i) C is an R-submodule of R^n,
(ii) C is closed under the T_θ-cyclic shift, i.e.,

$$T_\theta(c) = (\theta(c_{n-1}), \theta(c_0), \dots, \theta(c_{n-2})), \quad (7)$$

where $c = (c_0,\ c_1, \dots, c_{n-1}) \in C$.

Lemma 2.5. (Al-Ashker and Abu-Jazar 2016) Let $\lambda = \alpha + v\beta$ be an element in the ring R, then λ is a unit of R if and only if $\alpha \neq 0$ and $\alpha + \beta \neq 0$, where $\alpha, \beta \in F_p$.

2.2 Skew constacyclic codes over R

In this part, we firstly introduce definition of skew constacyclic codes and then continue with its basic results over the ring R. Since we follow (Al-Ashker and Abu-Jazar 2016), the proofs of the theories will be omitted. Throughout the study, we denote $\alpha + v\beta$ as λ for simplicity, where $\alpha, \beta \in F_p$.

Definition 2.6. A subset C of R^n is called skew constacyclic code, or (θ, λ)-constacyclic code of length n over R if

(i) C is an R-submodule of R^n,
(ii) C is closed under the skew constacyclic shift, i.e.,

$$T_{\theta,\lambda}(c) = (\lambda\theta(c_{n-1}), \theta(c_0), \dots, \theta(c_{n-2})), \quad (8)$$

where λ is unit in R and $c = (c_0,\ c_1, \dots, c_{n-1}) \in C$.

In particular, if λ is 1 or -1, then C is a skew-cyclic code or skew negacyclic code over R, respectively.

In general, a code C of lenght n over R is skew constacyclic code if and only if $\varrho(C)$ is an ideal of $R[x]/< x^n - \lambda >$. Thus, skew constacyclic code can be define as left ideals in $[R,\theta]/< x^n - \lambda >$.

Theorem 2.7. (Al-Ashker and Abu-Jazar 2016). A code C of lenght n over R is skew constacyclic if and only if the skew polynomial representaton of C is a left ideal in $[R,\theta]/< x^n - \lambda >$.

3. F_pR-Linear Skew Constacyclic Codes

In this section, we generalize our studies to the skew constacyclic codes over the ring F_pR.

Definition 3.1. Define

$$F_pR = \{(d, a+vb) |\ d\ \in\ F_p, (a+vb) \in R\}. \tag{9}$$

Let C be a skew cyclic code over F_pR and m (resp. n) is the set of F_p (resp. R) coordnate positons.

Throughout the paper, we assume that m and n are odd positive integers.

Any codeword $c \in C$ has the form

$$c = (d_0, d_1, \dots, d_{m-1}, e_0, e_1, \dots, e_{n-1}\ ) \in {F_p}^m R^n. \tag{10}$$

For $e_i = (a_i + vb_i)$ for all $i = 0, 1, \dots, n-1$.

The ring homomorphism map is defned as

$$\delta: R \to\ F_p$$

$$a + vb\ \mapsto a \tag{11}$$

For any $r \in R$, a scalar multiplication $*$ defined by

$$r * (d, a+vb) = (\delta(r)a, r(a+vb)), \tag{12}$$

where $d \in F_p$ and $a + vb \in R$. It inherently extends to ${F_p}^m R^n$ as follows:

$$r * x = (\delta(r)d_0,\ \delta(r)d_1, \dots,\ \delta(r)d_{m-1},\ re_0, re_1, \dots, re_{n-1}), \tag{13}$$

where x=$(d_0, d_1, \dots, d_{m-1}, e_0, e_1, \dots, e_{n-1}\ )\ \in\ {F_p}^m R^n$ for $m, n\ \in\ \mathbb{N}$.

A non-empty subset C of ${F_p}^m R^n$ is called F_pR-linear code if C is an R-submodule of ${F_p}^m R^n$.

Lemma 3.2. The set ${F_p}^m R^n$ is an R-module with respect to the addition and scalar multiplcation.

Proof. Clear.

Definiton 3.3. The Euclidean inner product over the ring ${F_p}^m R^n$ is defined as

$$< x, y >= v \textstyle\sum_{i=0}^{m-1} x_i y_i + \sum_{j=0}^{n-1} x'_i y'_i, \tag{14}$$

where $x = (x_0, x_1, \dots, x_{m-1}, x'_0, x'_1, \dots, x'_{n-1})$ and $y = (y_0, y_1, \dots, y_{m-1}, y'_0, y'_1, \dots, y'_{n-1})$ in ${F_p}^m R^n$.

The dual code of C is also F_pR-linear and defined as

$$C^{\perp} = \{y \in\ {F_p}^m R^n \big|\ < x, y >= 0, \forall\ x \in C\}. \tag{15}$$

Definition 3.4. Let ψ be an automorphism of the finite field F_p. For any two elements $ax^i, bx^j \in F[x, \psi]$, multiplication is defined as

$$(ax^i)(bx^j) = a\psi^i(b)x^{i+j}. \tag{16}$$

In polynomial representation, a linear code of lenght m over F_p is a skew constacyclic code if and only if it is a left $F_p[x, \psi]$-submodule of F_p[x, ψ]/<$x^m - \delta(\lambda)$ >.

For shortly, $R_{m,n} \coloneqq F_p$[x, ψ]/<$x^m - \delta(\lambda)$ >× $R[x,\theta]/< x^n - \lambda >$.

We can define the polynomial representation of each codeword of skew constacylic codes as follows:

Definition.3.5. An element

$$(d_0, d_1, \dots, d_{m-1}, e_0, e_1, \dots, e_{n-1}\ )\ \in\ {F_p}^m R^n \tag{17}$$

can be defined with a module element consisting of two polynomials:

$$c(x) = \big(d(x), e(x)\big) \in R_{m,n}, \tag{18}$$

where $d(x) = d_0 + d_1x + \cdots + d_{m-1}x^{m-1}$ and $e(x) = e_0 + e_1x + \cdots + e_{n-1}x^{n-1}$.

Let $f(x) = f_0 + f_1x + \cdots + f_sx^s \in [R,\theta]$ and $\big(d(x), e(x)\big) \in R_{m,n}$, then multiplication operator on F_pR is defined as

$$f(x) * \big(\mathrm{d(x)}, \mathrm{e(x)}\big) = (\delta\big(f(x)\big)d(x), f(x) * e(x)), \tag{19}$$

where $\delta\big(f(x)\big) = \delta(f_0) + \delta(f_1)x + \cdots + \delta(f_s)x^s$. This multiplication is well-defined on $R_{m,n}$. Also, $\delta\big(f(x)\big)d(x)$ and f$(x) * e(x)$ are defined in F_p[x, ψ]/< $x^m - \delta(\lambda)$ > and $R[x,\theta]/< x^n - \lambda >$, respectively.

Lemma 3.6. $R_{m,n}$ is a left $R[x,\theta]$-module under the * multiplcation.

Proof. Clear.

Now, we are ready to introduce the skew constacyclic codes over the ring ${F_p}^m R^n$.

Definition 3.7. Let θ and ψ be automorphsms of R and F_p, respectively. A linear code C over ${F_p}^m R^n$ is called skew constacyclic code if

(i) C is an F_pR-submodule of ${F_p}^m R^n$,

(ii) C is closed under the skew constacyclic shift, i.e.,

$$S_{\theta,\lambda}(c) = (\delta(\lambda)\psi(d_{m-1}), \psi(d_0), \ldots, \psi(d_{m-2}), \lambda\theta(e_{n-1}), \theta(e_0), \ldots, \theta(e_{n-2})) \quad (20)$$

where λ is unit in R and $c = (d_0, d_1, \ldots, d_{m-1}, e_0, e_1, \ldots, e_{n-1}) \in C$.

Lemma 3.8. Let *C* be a code over F_pR with lenght *m+n*. *C* is skew constacyclic code if and only if *C* is a left $R[x,\theta]$ −submodule of $R_{m,n}$.

Proof. Let *C* be a skew constacyclic code over F_pR and $c = (d_0, d_1, \ldots, d_{m-1}, e_0, e_1, \ldots, e_{n-1}) \in C$, such that $c(x) = \big(d(x), e(x)\big)$ be a codeword of $R_{m,n}$. Then

$$x * c(x) = (\delta(\lambda)\psi(d_{m-1}) + \psi(d_0)x + \ldots + \psi(d_{m-2})x^{m-1},\ \lambda\theta(e_{n-1}) + \theta(e_0)x + \ldots + \theta(e_{n-2})x^{n-1}) \in C \quad (21)$$

Also, $x^2 * c(x) \in C$ and so on. Since *C* is a linear code, one gets f(x)$* c(x) \in C$, for any $f(x) \in R[x,\theta]$. Thus, *C* is a left $R[x,\theta]$-submodule of $R_{m,n}$.

On the other hand, assume that *C* is a left $R[x,\theta]$-submodule of $R_{m,n}$, then for any $c(x) \in C$, we have $x^i * c(x) \in C$, $i \in \mathbb{N}$. Hence, *C* is a skew constacyclic code.

Let *C* be a linear code over F_pR with length *m+n.* Then *C* is called seperable if $C = C_m \times C_n$, while considering C_m and C_n as punctures codes of *C* by erasing the coordinates outside the *m* and *n* components, respectively. If $C = C_m \times C_n$ is seperable, then $C^\perp = C_m^\perp \times C_n^\perp$.

Theorem 3.9. Let $C = C_m \times C_n$ be a linear code over F_pR, where C_m and C_n are linear codes over F_p (with length *m*) and R (with length *n*), respectively. Then, *C* is a skew constacyclic code if and only if C_m is a $\delta(\lambda)$-constacyclic code over F_p and C_n is a skew λ-constacyclic code over R.

Proof. Let $(d_0, d_1, \ldots, d_{m-1}) \in C_m$ and $(e_0, e_1, \ldots, e_{n-1}) \in C_n$. Asume that *C* is a skew constacyclic code, then we have

$$(\delta(\lambda)\psi(d_{m-1}), \psi(d_0), \ldots, \psi(d_{m-2}), \lambda\theta(e_{n-1}), \theta(e_0), \ldots, \theta(e_{n-2})) \in C. \quad (22)$$

Thus, $(\delta(\lambda)\psi(d_{m-1}), \psi(d_0), \ldots, \psi(d_{m-2})) \in C_m$ and $(\lambda\theta(e_{n-1}), \theta(e_0), \ldots, \theta(e_{n-2})) \in C_n$, as desired.

Conversely, by the hypothesis we have $(\delta(\lambda)\psi(d_{m-1}), \psi(d_0), \ldots, \psi(d_{m-2})) \in C_m$ and $(\lambda\theta(e_{n-1}), \theta(e_0), \ldots, \theta(e_{n-2})) \in C_n$. Since $C = C_m \times C_n$, the proof holds.

4. The Generator Set For F_pR Skew Constacyclic Codes

Let C_1 and C_2 be two linear codes over F_p and defined as

$$C_1 = \left\{ a \in F_p^m \middle| \begin{matrix} (1-v)b + va \in C, \\ \text{for some } b \in F_p^m \end{matrix} \right\}, \quad (23)$$

and

$$C_2 = \left\{ b \in F_p^m \middle| \begin{matrix} (1-v)b + va \in C, \\ \text{for some } a \in F_p^m \end{matrix} \right\}. \quad (24)$$

So, *C* can be uniquely state as $C = vC_1 \oplus (1-v)C_2$ (Al-Ashker and Abu-Jazar 2016). Also, F_pR is a non-zero left ideal in $R[x,\theta]/< x^n - \lambda >$. In this section, we consider $f_i(x) (i = 1, 2)$ as a set of all non-zero skew polynomials with minimal degree in F_p.

Theorem 4.1. (Al-Ashker and Abu-Jazar 2016) Let C be a $\lambda -$ constacyclic code over R of length n. If $C =< vf_1(x), (1-v)f_2(x) >$ with $f_1(x)|(x^n - (\alpha + \beta))$ and $f_2(x)|(x^n - \alpha)$, then $C_1 =< f_1(x) >$ and $C_2 =< f_2(x) >$.

Theorem 4.2. Let *C* be a linear skew constacyclic code of lenght *m+n* over F_pR. Then,

$$C =< (g(x), 0), (j(x), h(x)) >, \quad (25)$$

where $g(x)\big|(x^m - \delta(\lambda)\big)$, $h(x) = vf_1(x) + (1-v)f_2(x)$ with $f_1(x)|(x^n - (\alpha + \beta))$ and $f_2(x)|(x^n - \alpha)$.

Proof. Let *C* be an F_pR-linaer skew constacyclic code of lenght *m+n.* Define

$$\omega: C \to R[x,\theta]/< x^n - \lambda >$$

$$\big(f_1(x), f_2(x)\big) \mapsto f_2(x), \quad (26)$$

where $f_1(x) \in F_p[\text{x}, \psi]/< x^m - \delta(\lambda) >$ and $f_2(x) \in R[x,\theta]/< x^n - \lambda >$. For any $r(x) \in R[x,\theta]$, one has $\omega\Big(r(x) * \big(f_1(x), f_2(x)\big)\Big) = r(x) * \omega(f_1(x), f_2(x))$. So, ω is a left $R[x,\theta]$-module homomorphism whose image is a left $R[x,\theta]$-submodule of $R[x,\theta]/< x^n - \lambda >$. By Lemma 3.8 and Theorem 4.1, we have $\omega(C) =< h(x) > =< f_1(x) + (1-v)f_2(x) >$ with $f_1(x)|(x^n - (\alpha + \beta))$ and $f_2(x)|(x^n - \alpha)$. Define a set I as

$$I = \{g(x) \in F_p[\text{x}, \psi]/< x^m - \delta(\lambda) > |(g(x), 0) \in \ker(\omega)\}. \quad (27)$$

Obviously, I is an ideal of $F_p[\text{x}, \psi]/< x^m - \delta(\lambda) >$. Thus, I is a cyclic code in $F_p[\text{x}, \psi]/< x^m - \delta(\lambda) >$ which implies that $I =< g(x) >$, where $g(x)$ is a divisor of

$x^m - \delta(\lambda)$. For any element $(k(x), 0) \in \ker(\omega)$, one has $k(x) \in I =< g(x) >$. So, there exists a polynomial $l(x) \in R[x, \theta]$ such that $k(x) = \delta(l(x))g(x)$. Thus, $(k(x), 0) = l(x) * (g(x), 0)$ and this implies that $\ker(\omega)$ is a submodule of C generated by an element of the form $(g(x), 0)$, such that $\ker(\omega) =< (g(x), 0) >$, where $g(x) \in F_p[x, \psi]$ and $g(x)|(x^m - \delta(\lambda))$. By the first isomorphism theorem,

$$C/\ker(\omega) \cong < h(x) >. \quad (28)$$

Let $(j(x), h(x)) \in C$ such that $\omega(j(x), h(x)) = h(x)$. Then, C can be generated as a left $R[x, \theta]$-submodule of $R_{m,n}$ by two elements of the form

$((g(x), 0))$ and $(j(x), h(x))$. So, any element in C can be written as

$$r_1(x) * ((g(x), 0)) + r_1(x) * (j(x), h(x)), \quad (29)$$

where $r_1(x), r_2(x) \in R[x, \theta]$. Hence,

$$C =< (g(x), 0), (j(x), h(x)) >, \quad (30)$$

where $g(x)|(x^m - \delta(\lambda)), h(x) = vf_1(x) + (1 - v)f_2(x)$ with $f_1(x)|(x^n - (\alpha + \beta))$ and $f_2(x)|(x^n - \alpha)$.

Lemma.4.3. If $C =< (g(x), 0), (j(x), h(x)) >$ is a linear skew constacyclic code of lenght *m+n* over F_pR, then $deg(j(x)) < \deg(g(x))$.

Proof. Assume that $deg(j(x)) \geq \deg(g(x))$ and $deg(j(x) - g(x)) = s \in \mathbb{N}$. Let

$$D =< (g(x), 0), (j(x) - x^s g(x), h(x)) >. \quad (31)$$

Then, one gets $D \sqsubseteq C$. Also,

$$(j(x), h(x)) = (j(x) - x^s g(x), h(x)) + x^s * (g(x), 0)).$$

Thus, $C \sqsubseteq D$ and so $D = C$. This implies a contradiction. So, $deg(j(x)) < \deg(g(x))$.

5. Gray Images of Skew Constacyclic Codes Over F_pR

The Gray map between F_pR and ${F_p}^3$ is defined as

$\zeta\colon F_pR \to {F_p}^3$

$$\zeta((d, a + vb)) = (d, -b, 2a + b), \quad (32)$$

where $(d, a + vb) \in F_pR$. This map can be extended to the map $\zeta\colon {F_p}^m R^n \to {F_p}^{m+2n}$ such that

$$\zeta((d_0, d_1, \dots, d_{m-1}, a_0 + vb_0, a_1 + vb_1, \dots, a_{n-1} + vb_{n-1})) = (d_0, d_1, \dots, d_{m-1}, -b_0, -b_1, \dots, -b_{n-1}, 2a_0 + b_0, 2a_1 + b_1, \dots, 2a_{n-1} + b_{n-1}), \quad (33)$$

for all $(d_0, d_1, \dots, d_{m-1}) \in {F_p}^m$ and $(a_0 + vb_0, a_1 + vb_1, \dots, a_{n-1} + vb_{n-1}) \in R^n$.

If C is an F_pR-linear codes, then $\zeta(C)$ is also F_p-linear. The Hamming weight of a codeword c in ${F_p}^m R^n$ is the number of non-zero coordinates in c and denoted by $w_H(c)$. The Hamming distance between two codewords c_1 and c_2 in ${F_p}^m R^n$ is defined as $d_H(c_1, c_2) = w_H(c_1 - c_2)$ and Hammng distance for a code C is defined by $d_H(C) = \min\{d_H(c_1, c_2)|\ c_1 \neq c_2, \forall\, c_1, c_2 \in C\}$.

The Lee weight of $(d, a + vb) \in F_pR$ can be defined as

$$w_L((d, a + vb)) = w_H(d) + w_H(-b) + w_H(2a + b). \quad (34)$$

The Lee weigth of a codeword is the rational sum of Lee weigths of its components. The Lee distance between two codewords c_1 and c_2 in ${F_p}^m R^n$ is defined as

$$d_L(c_1, c_2) = w_L(c_1 - c_2). \quad (34)$$

Proposition 5.1. The Gray map ζ is an F_p-linear distance preserving map from ${F_p}^m R^n$ to ${F_p}^{m+2n}$.

Proof. Let $x = (d_0, d_1, \dots, d_{m-1}, e_0, e_1, \dots, e_{n-1})$ and $y = (d'_0, d'_1, \dots, d'_{m-1}, e'_0, e'_1, \dots, e'_{n-1})$ be two elements in ${F_p}^m R^n$, where $e_i = a_i + v\, b_i$ and $e'_i = a'_i + vb'_i$ for $i = 0, 1, \dots, n-1$. By the definiton,

$$\zeta(x + y) = \zeta((d_0, d_1, \dots, d_{m-1}, e_0, e_1, \dots, e_{n-1}) + (d'_0, d'_1, \dots, d'_{m-1}, e'_0, e'_1, \dots, e'_{n-1})) = (d_0 + d'_0, d_1 + d'_1, \dots, d_{m-1} + d'_{m-1}, -b_0 - b'_0, -b_1 - b'_1, \dots, -b_{n-1} - b'_{n-1}, 2(a_0 + a'_0) + b_0 + b'_0,\ 2(a_1 + a'_1) + b_1 + b'_1, \dots,\ 2(a_{n-1} + a'_{n-1}) + b_{n-1} + b'_{n-1},) = (d_0, d_1, \dots, d_{m-1}, -b_0, -b_1, \dots, -b_{n-1}, 2(a_0 + b_0), 2(a_1 + b_1), \dots, 2(a_{n-1} + b_{n-1})) + (d'_0, d'_1, \dots, d'_{m-1}, -b'_0, -b'_1, \dots, -b'_{n-1},\ 2(a'_0 + b'_0), 2(a'_1 + b'_1), \dots, 2(a'_{n-1} + b'_{n-1})) = \zeta(x) + \zeta(y). \quad (35)$$

Also, for any element $s \in F_p$,

$$\zeta(sx) = \zeta(sd_0, sd_1, \dots, sd_{m-1}, se_0, se_1, \dots, e_{n-1}) = (sd_0, sd_1, \dots, sd_{m-1}, -sb_0, -sb_1, \dots, -sb_{n-1}, s(2a_0 + b_0), s(2a_1 + b_1), \dots, s(2a_{n-1} + b_{n-1})) = s\zeta(x). \quad (36)$$

Hence, ζ is an F_p-linear map. Moreover, we can show that ζ is an F_p-linear distance preserving map as follows:

$$d_L(x, y) = w_L(x - y) = w_H(d - d') + w_H(\zeta(e - e')) = w_H(d - d') + w_H(\zeta(e) - \zeta(e')) = d_H(d, d') + d_H(\zeta(e), \zeta(e')) \quad (37)$$

Corollary 5.2. If *C* is an F_pR-linear code with parameters [*m+n*, *M*, d_L], then $\zeta(C)$ is a *q*-ary linear code with parameters [*m+2n*, $\log_q M$, d_L], where *M* denotes the number of codewords in *C*.

Proof. Clear.

Theorem 5.3. Let C be a linear code over F_pR. Then $\zeta(C^{\perp}) = \zeta(C)^{\perp}$.

Proof. Let $c_1 = (d_1, a_1 + vb_1) \in C$ and $c_2 = (d_2, a_2 + vb_2) \in C^{\perp}$. By the definiton, one gets $d_1d_2 = a_1a_2 = a_1b_2 + a_2b_1 + b_1b_2 = 0$. Thus,

$$< \zeta(c_1), \zeta(c_1) >=< (d_1, -b_1, 2a_1 + b_1), (d_2, -b_2, 2a_2 + b_2) >= d_1d_2 + 4a_1a_2 + 2(a_1b_2 + a_2b_1 + b_1b_2) = 0. \quad (38)$$

Hence, $\zeta(c_2) \in C^{\perp}$ and this gives that $\zeta(C^{\perp}) \subseteq \zeta(C)^{\perp}$. Since F_pR is a Frobeniues ring,

$$|\zeta(C^{\perp})| = \frac{p^{m+2n}}{|\zeta(C)|} = \frac{p^{m+2n}}{|C|} = |C^{\perp}|, \quad (39)$$

and the result follows.

If a code *C* is equal to its dual, then *C* is called self-dual code.

Corollary 5.4. Let C be a linear code over F_pR. If C is a self-dual code, so is $\zeta(C)$.

Proof. Clear.

Declaration of Ethical Standards

The author declares that they comply with all ethical standards.

Declaration of Competing Interest

The author have no conflicts of interest to declare regarding the content of this article.

Data Availability Statement

All data generated or analyzed during this study are included in this published article.

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6. Kaynaklar

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