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Bases of fixed point subalgebras on nilpotent Leibniz algebras

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Abstract

Let K be a field of characteristic zero, $X = \{x_{1,} x_2, \dots, x_n\}$ and $R_m = \{r_{1,}, \dots, r_m\}$ be two sets of variables, F be the free left nitpotent Leibniz algebra generated by X, and $K[R_m]$ be the commutative polynomial algebra generated by R_m over the base field K. The fixed point subalgebra of an automorphism φ is the subalgebra of F consisting of elements that are invariant under the automorphism. In this work, we consider specific automorphisms of F and determine the fixed point subalgebras of these automorphisms. Then, we find bases of these fixed point subalgebras. In addition, we get generators of these subalgebras as a free $K[R_m]$ -module.

Keywords: *Nilpotent Leibniz algebras, fixed point, automorphism.*

Nilpotent Leibniz cebirlerinde sabit nokta altcebirlerinin bazları

Öz

K karakteristiği 0 olan bir cisim, $X = \{x_{1,} x_2, \dots, x_n\}$ ve $R_m = \{r_{1,}, \dots, r_m\}$ iki değişkenler kümesi, F, K cismi üzerinde X tarafından üretilen bir serbest sol nilpotent Leibniz cebiri ve $K[R_m]$, K cismi üzerinde R_m tarafından üretilen komutatif polinomlar cebiri olsunlar. F nin bir φ otomorfizminin sabit nokta altcebiri, F nin bu otomorfizm altında invaryant kalan elemanlarını içeren altcebiridir. Bu çalışmada F nin bazı özel otomorfizmleri ele alınarak bu otomorfizmlerin sabit nokta altcebirleri belirlenmiştir. Sonra, bu sabit nokta altcebirlerinin baz kümeleri elde edilmiştir. Daha sonra bu altcebirlerin serbest $K[R_m]$-modülü olarak üreteçleri verilmiştir.

Anahtar kelimeler: *Nilpotent Leibniz cebirleri, sabit nokta, otomorfizm*

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1. Introduction

The problem of determining the fixed points of endomorphisms on free Leibniz algebras is a significant topic in the theory of Leibniz algebras. Leibniz algebras were first introduced by Bloh [1], in 1965, and later rediscovered by Loday and Pirashvili in 1993 [2, 3]. These algebras provide a non-antisymmetric generalization of Lie algebras, and their applications are given in various papers. Mikhalev and Umirbaev worked on subalgebras of free Leibniz algebras [4]. Drensky and Cattaneo in their work from 1993, described the free nilpotent Leibniz algebras of class 2 [5]. Additionally, Abanina and Mishchenko investigated the variety of left nilpotent Leibniz algebras of class 3 defined by the polynomial identity $[x_1,[x_2,[x_3,x_4]]]=0$ [6]. On the relatively free Leibniz algebras, for more details see the works [7-11]. In [12], Drensky and Papistas obtained a generating set of the automorphism group of free nilpotent Leibniz algebras and they show that the fixed points subalgebra is not finitely generated. The earlier work on fixed points in free algebras has been obtained by Formanek [13]. Bryant and Drensky have made notable contributions to understanding the fixed point subalgebras of finite groups acting on free Lie algebras in [14, 15], respectively. They established that under certain assumptions, the fixed point subalgebra of a free Lie algebra of finite rank n(with $n \geq 2$) is not finitely generated. In [16], Bryant and Papistas extended these results, expanding our understanding of fixed points in free Lie algebras. In [17], Ekici and Parlak Sönmez applied the problem to fixed points subalgebras for a single endomorphism of free metabelian Lie algebras. The fixed point subalgebra of a single endomorphism φ consists of elements that remain unchanged under φ, i.e., $\varphi(x) = x$ for all x in the subalgebra.

Let F be a free left nilpotent Leibniz algebra generated by a finite set $\{x_{1,} x_2, \dots, x_n\}$ and $K[R_m]$, be the polynomial algebra generated by a set $R_m = \{r_{1,}, \dots, r_m\}$ over the field K of characteristic zero. In this work, we focus on fixed point subalgebras of a single automorphism of the free left nilpotent Leibniz algebra F. First, we determine the fixed point subalgebras under certain automorphisms of F. Then, we find the bases of these fixed point subalgebras. At the end, we give the free generating sets of these subalgebras as a $K[R_m]$-module.

2. Preliminaries

A Leibniz algebra Lover a field K is a non-associative algebra equipped with a bracket operation $[,]: L \times L \to L$ that satisfies the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

for all elements x, y, z in L. If we further impose the condition $[x,x] = 0$ for all $x \in L$ the Leibniz identity becomes equivalent to the Jacobi identity, which is a fundamental property of Lie algebras $\gamma_1(L) = [L,L]$ is the derived subalgebra of L. Denote by $Ann(L)$ the ideal of L generated by elements $\{[a,a]: a \in L\}$. The factor algebra $L_{Lie} = L/Ann(L)$ then becomes a Lie algebra. In particular, it is known that $r_a = 0$ if and only if a belongs to $Ann(L)$ (see [3]). The Leibniz identity enables us to express any commutator as a linear combination of left-normed commutators. We introduce the notation

$$[x_1, x_2] = x_1 r_2$$

$$[[x_1,\dots,x_{n-1}],x_n]=[x_1,\dots,x_{n-1},x_n]=[x_1,x_2]r_3r_4\dots r_n$$

where r_i represents the adjoint operator adx_i acting from the right by commutator multiplication. By utilizing these notations, we can simplify the representation of commutators and express them in terms of left-normed commutators. This reduction allows us to study the properties and relationships within the Leibniz algebra more effectively.

Let F_m be the free Leibniz algebra of rank m over a field K of characteristic 0 freely generated by the set $\{x_{1,} x_2,\dots,x_m\}$, where $m\geq 2$. Loday and Pirashvili described the structure and properties of free Leibniz algebras in 1993 [3]. Let N be a variety of Leibniz algebras defined by the identity of left nilpotency $[x,[y,z]]=0$. The left nilpotency polynomial identity is equivalent to $[x,y,z]=[x,z,y]$. We consider the relatively free Leibniz algebra in the variety N, denoting this algebra as $F_m(N)$. $F_m(N)$ is a free left nilpotent Leibniz algebra of class two with finite rank m. Clearly $F_m(N)=\frac{F_m}{\gamma_2(F_m)}$, where $\gamma_2(F_m)=[F_m,[F_m,F_m]]$. The elements of the free Leibniz algebra F_m and their corresponding images in $F_m(N)$ are represented using the same letters. The Leibniz identity together with the left nilpotent identity implies that

$$[x_{i_1},x_{i_2}\dots,x_{i_k}]=[x_{i_1},x_{i_{\sigma(2)}}\dots,x_{i_{\sigma(k)}}]$$

where σ is a permutation of $2,\dots,k$. Hence the commutative polynomial algebra $K[R_m]=K[r_{1,},\dots,r_m]$ acts on $F_m(N)$ as a right module by the rule

$ax_i=[a,x_i]$,
where $a\in F_m(N)$. Denote by Ω_m, the augmentation ideal of $K[R_m]$ that consists of all polynomials without constant term. In [5], Drensky and Piacentini Cattaneo described the structure of $F_m(N)$ and they give a basis of $F_m(N)$,

$$\{x_{i_{1,}}[x_{i_1},\dots,x_{i_k}]:1\leq i_1\leq m,1\leq i_2\leq\dots\leq i_k\leq m,k=2,3,\dots\}.$$

In [12], Drensky and Papistas obtained a generating set of the automorphism group of $F_m(N)$. Then, they showed that the fixed points subalgebra

$$F_m(N)^S=\{v\in F_m(N): g(v)=v\ for\ all\ g\in S\}$$

is not finitely generated, where S is an arbitrary nontrivial finite subgroup of the automorphism group of $F_m(N)$. Certain findings concerning fixed points of a finite group of automorphisms can be applied to the context of fixed points for individual endomorphisms. The fixed point subalgebra of an endomorphism φ as the set of elements in $F_m(N)$ that remain unchanged under the action of φ which is defined by

$$Fix\varphi=\{v\in F_m(N): \varphi(v)=v\}$$

An element v of $F_m(N)$ is called a fixed point of φ if $\varphi(v)=v$. The trivial fixed point is always present, represented by the element 0 in $F_m(N)$. This is because for any endomorphism φ, we have $\varphi(0)=0$. In the present article, we obtain the basis of non-trivial fixed point subalgebras of some automorphisms of $F_m(N)$ for finite rank.

3. Results and discussion

In this section, we determine the basis of the fixed point subalgebras for specific endomorphisms of $F_m(N)$. We apply the condition $\varphi(v) = v$ to each element v of $F_m(N)$ and get some equations. By solving these equations, we determine the coefficients or representations of the basis elements that remain fixed under the endomorphisms.

Theorem 3.1. Let φ be an endomorphism of $F_m(N)$ defined by

$$\varphi: \begin{array}{l} x_1 \to x_1 + u \\ x_i \to x_i, i \neq 1 \end{array}$$

where $0 \neq u \in \gamma_1(F_m(N))$. Then, the subalgebra $Fix\varphi$ has a basis

$$\{x_{i_1}, [x_{i_1}, \dots, x_{i_k}] : 2 \leq i_1 \leq m, 1 \leq i_2 \leq \dots \leq i_k \leq m, k = 2,3, \dots\}$$

as a K-space.

Proof. Let $v \in Fix\varphi$. Then

$$v = \sum_{i=1}^{m} \alpha_i x_i + \sum_{i=1}^{m} x_i w_i(R_m),$$

where $\alpha_i \in K$ and $w_i(R_m)$ belongs to the augmentation ideal Ω_m of $K[R_m]$, $i = 1, \dots, m$. Then

$$\begin{aligned} \varphi(v) &= \sum_{i=1}^{m} \alpha_i x_i + \alpha_1 u + \sum_{i=1}^{m} x_i w_i(R_m) + u w_1(R_m) \\ &= v + u(\alpha_1 + w_1(R_m)) \end{aligned}$$

Hence $\alpha_1 = w_1(R_m) = 0$, which completes the proof.

Corollary 3.2. The subalgebra $Fix\varphi$ is the free right $K[R_m]$-module of rank $m-1$ with the generators $x_2, \dots, x_m$.

Proof. By Theorem 3.1, the subalgebra $Fix\varphi$ has a basis

$$\{x_{i_1}, x_{i_1} r_{i_2} \dots r_{i_k} : 2 \leq i_1 \leq m, 1 \leq i_2 \leq \dots \leq i_k \leq m, k = 2,3, \dots\}$$

Hence, $Fix\varphi$ is generated by $\{x_{i_1} : 2 \leq i_1 \leq m\}$ as a free right $K[R_m]$-module.

In the following corollary, we give fixed point subalgebra of a non-tame automorphism as an application of Theorem 3.1 for rank two. This automorphism is an element of free generating set of automorphism group of $F_2(N)$ that was defined by Drensky and Papistas [12].

Corollary 3.3. Let φ be an automorphism of $F_2(N)$ defined by

$$\varphi : \begin{matrix} x_1 \to x_1 + [x_1, x_2] \\ x_2 \to x_2 \end{matrix},$$

Then, the subalgebra $Fix\varphi$ has a basis

$$\{x_{2\,,}\, [x_2, x_{i_1} \dots, x_{i_k}] : 1 \leq i_1 \leq \dots \leq i_k \leq 2\}$$

as a K-space.

Theorem 3.4. Let τ be an inner automorphism of $F_m(N)$ defined by

$$\tau\colon\ x_i \to x_i + [u, x_i], i = 1, \dots, m$$

where $0 \neq u \in \gamma_1(F_m(N))$. Then the subalgebra $Fix\tau$ has a basis

$$\{[x_i, x_j] - [x_j, x_i]: 1 \leq i < j \leq m, 1 \leq i \leq i_1 \leq \dots \leq i_k \leq m\}$$

Proof. Let $v \in Fix\tau$. Then

$$v = \sum_{i=1}^{m} \alpha_i\, x_i + \sum_{i=1}^{m} x_i\, w_i(R_m)$$

where $\alpha_i \in K$ and $w_i(R_m)$ belongs to the augmentation ideal Ω_m of $K[R_m]$, $i = 1,2, \dots, m$. Then

$$\tau(v) = v + u \sum_{i=1}^{m} r_i(\alpha_i + w_i(R_m)).$$

Hence $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$ and $\sum_{i=1}^{m} r_i w_i(R_m) = 0$. The polynomial $w_i(R_m)$ cannot depend on r_i only. Hence every monomial in $\sum_{i=1}^{m} r_i w_i(R_m)$ depends on at least two variables. The vector space of m-tuples $(r_1 w_1(R_m), \dots, r_m w_m(R_m))$ such that $\sum_{i=1}^{m} r_i w_i(R_m) = 0$ is spanned by m-tuples of monomials $(\beta_1 v(R_m), \dots, \beta_m v(R_m))$ where $\sum_{i=1}^{m} \beta_i = 0$ and $\beta_i = 0$ if $\boldsymbol{r}_i$ does not participate in $v(R_m)$. Such m-tuples are linear combinations of

$$(0, \dots, 0, r_i r_j w(R_m), 0, \dots, 0, -r_j r_i w(R_m), 0, \dots, 0)$$

when $v(R_m) = r_i r_j w(R_m)$ for some monomial $w(R_m)$. Every such m-tuple corresponds to the element of the derived subalgebra $\gamma_1(F_m(N))$ is

$$([x_i, x_j] - [x_j, x_i])w(R_m).$$

Then, these elements span $Fix\tau$. For $i \leq j \leq k$, we obtain

$$([x_j, x_k] - [x_k, x_j])r_i = ([x_i, x_k] - [x_k, x_i])r_j - ([x_i, x_j] - [x_j, x_i])r_k.$$

Therefore, we omit the generators

$([x_j, x_k] - [x_k, x_j])w(R_m)$,

where $w(R_m)$ consists of the elements r_i for $i < j < k$. Hence the basis of $Fix\tau$ is

$\{[x_i, x_j] - [x_j, x_i]: 1 \leq i < j \leq m, 1 \leq i \leq i_1 \leq \cdots \leq i_k \leq m\}$.

Corollary 3.5. The subalgebra $Fix\tau$ is the free right $K[R_m]$-module with the generators

$\{[x_i, x_j] - [x_j, x_i]: 1 \leq i < j \leq m\}$,

Proof. By Theorem 3.4, the subalgebra $Fix\tau$ has a basis

$\{([x_i, x_j] - [x_j, x_i])r_{i_1} \dots r_{i_k}: 1 \leq i < j \leq m, 1 \leq i \leq i_1 \leq \cdots \leq i_k \leq m\}$.

Hence, $Fix\tau$ is generated by $\{[x_i, x_j] - [x_j, x_i]: 1 \leq i < j \leq m\}$ as a free right $K[R_m]$-module.

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