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ON LEAP ZAGREB INDICES OF A SPECIAL GRAPH OBTAINED BY SEMIGROUPS

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Abstract. In 2013, Das et al. defined the monogenic semi-group graphs [10]. And, various topological indices of the monogenic semi-group graphs have been calculated so far [3, 21]. The aim of this study is to continue to create formulas for the topological indices of these special graphs. In this study, we give exact formulae for various the leap Zagreb indices of this special algebraic graph obtained from monogenic semigroups.

1. Introduction and Preliminaries

Let $\mathscr{F}$ be a simple connected graph with vertex set $V(\mathscr{F})$ and edge set $E(\mathscr{F})$, where $V(\mathscr{F}) = \{a^i : 1 < i < n\}$. In graph $\mathscr{F}$, the degree of a vertex a^i is defined as the number of vertices that incident to the vertex a^i and is denoted by $d_{\mathscr{F}}(a^i)$. The distance between any two vertices a^i and a^j in a graph $\mathscr{F}$, denoted as $d(a^i, a^j)$, is the length of the shortest path between these vertices. The eccentricity of the vertex a^i is the maximum distance from a^i to any vertex. That is, $ecc(a^i) = max\{d(a^i, a^j) : a^j \in V(\mathscr{F})\}$.

In a graph $\mathscr{F}$, the k-distance degree of a vertex a^i, denoted as $d_k(a^i/\mathscr{F})$, is defined as the number of vertices at a distance of k from a^i [25]. Clearly, $d_1(a^i/\mathscr{F}) = d_{\mathscr{F}}(a^i)$.

Topological indices are important tools used in the study of chemical and physical properties of molecules, especially in QSAR and QSPR researchs [12]. Many publications have been made about the Zagreb indices (especially the first Zagreb index $(M_1(\mathscr{F}))$[14] and the second Zagreb index $(M_2(\mathscr{F}))$[15]), which is one of the oldest topological indices. These indices are defined as follows :

$$M_1(\mathscr{F}) = \sum_{a^i \in V(\mathscr{F})} d^2_{\mathscr{F}}(a^i) \quad and \quad M_2(\mathscr{F}) = \sum_{a^i a^j \in E(\mathscr{F})} d_{\mathscr{F}}(a^i) d_{\mathscr{F}}(a^j).$$

Inspired by Zagreb indices, the leap Zagreb indices were defined by Naji et al. in 2017 [22]. For more information on the leap Zagreb indices, we prefer references

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[7, 9, 24, 28]. The first leap Zagreb index, the second leap Zagreb index and the third leap Zagreb index are defined as follows:

$$LM_1(\mathscr{F}) = \sum_{a^i \in V(\mathscr{F})} d_2^2(a^i)$$
$$LM_2(\mathscr{F}) = \sum_{a^i a^j \in E(\mathscr{F})} d_2(a^i) d_2(a^j)$$
$$LM_3(\mathscr{F}) = \sum_{a^i a^j \in E(\mathscr{F})} (d_2(a^i) + d_2(a^j)).$$

The F-leap index of a graph $\mathscr{F}$ is defined by Kulli [17] as follows:

$$LF(\mathscr{F}) = \sum_{a^i \in V(\mathscr{F})} d_2^3(a^i).$$

The leap eccentric connectivity index (LEC) is defined as follows in an unpublished work by Pawar et al. [23, 16, 18, 26, 27][1]

$$LEC(\mathscr{F}) = \sum_{a^i \in V(\mathscr{F})} d_2(a^i) ecc(a^i).$$

For more information on graph theory, we prefer references [13].

The concept of a zero-divisor graph defined on a commutative ring R was first introduced by Beck [8] in 1988. In his study, he presented results on the coloring of this graph. Anderson and Livingston [5] continued this study, and their definition of zero-divisor graphs has been widely accepted. Anderson and Livingston defined the zero-divisor graph as follows:

Let $\mathscr{R}$ be a commutative ring with identity. Let $\mathscr{Z}(\mathscr{R})$ denote its set of zero-divisors. The vertex set of the zero-divisor graph $\Gamma(\mathscr{R})$ consists of the elements of $\mathscr{Z}(\mathscr{R})$. Two distinct vertices x and y are adjacent if and only if their product is zero.

The concept of zero-divisor graphs defined on a commutative ring has been generalized by Demeyer [11] et al. to define zero-divisor graphs of a commutative semigroups. Many studies have been conducted and are still ongoing regarding the concept of zero-divisor graphs constructed on various algebraic structures, such as Cayley graphs, total graphs, annihilator graphs, power graphs, etc. [4, 6]

In this direction, Das et al. [10] introduced a new graph obtained from multiplicative semigroups in 2013. They defined this algebraic graph as follows:

Definition 1.1. [10] Let $\mathscr{S}_{\mathscr{A}} = \{a, a^2, a^3, \ldots, a^n\}$ be a monogenic semigroup (with zero). The vertex set of this graph consists of the elements of $\mathscr{S}_{\mathscr{A}}$, except for zero. Any two vertices a^i and a^j are adjacent if and only if $i + j \geq n$.

They investigated some graph parameters of this graph in [3]. For more properties of the monogenic semigroup graphs, we can refer to [1, 2, 19, 20]. In this paper, we present exact formulas for the leap Zagreb indices, F-leap Zagreb index, and leap eccentric connectivity index of monogenic semigroup graphs with given order.

2. Leap Zagreb Indices of Monogenic Semigroup Graphs

In this section, we will give our basic results. First of all, let's give some lemmas that we will use in the proofs of theorems. Here and later, we will prefer the notations $d_1(a^i)$ and $d_2(a^i)$ over $d_1(a^i/\Gamma(\mathscr{S}_{\mathscr{A}}))$ and $d_2(a^i/\Gamma(\mathscr{S}_{\mathscr{A}}))$ for $1 \leq i \leq n$, respectively.

The degree sequence of the monogenic semigroup graph is given in the following lemma.

Lemma 2.1. [10] *Let $\mathscr{S}_{\mathscr{A}} = \{a, a^2, a^3, \ldots, a^n\}$ be a monogenic semigroup (with zero). Then, the degree sequence of the monogenic semigroup graph is given as*

$$\left.\begin{array}{l} d_1(a) = 1, d_1(a^2) = 2, \cdots, d_1(a^{\lfloor \frac{n}{2} \rfloor}) = \left\lfloor \dfrac{n}{2} \right\rfloor, d_1(a^{\lfloor \frac{n}{2} \rfloor + 1}) = \left\lfloor \dfrac{n}{2} \right\rfloor, \\ d_1(a^{\lfloor \frac{n}{2} \rfloor + 2}) = \left\lfloor \dfrac{n}{2} \right\rfloor + 1, \cdots, d_1(a^{n-1}) = n-2, d_1(a^n) = n-1. \end{array}\right\}$$

Lemma 2.2. [10] *Let $\mathscr{S}_{\mathscr{A}} = \{a, a^2, a^3, \ldots, a^n\}$ be a monogenic semigroup (with zero). Then*

$$diam(\Gamma(\mathscr{S}_{\mathscr{A}})) = 2.$$

Lemma 2.3. [22] *Let $\mathscr{G}$ be a connected graph with n vertices. Then for any vertex $v \in V(\mathscr{G})$*

$$d_2(v) \leq n - 1 - d_1(v).$$

Equality holds if and only if G has diameter at most two.

With the help of Lemma 2.1, Lemma 2.2 and Lemma 2.3 we give the sequence of 2-distance degrees of vertices in $\Gamma(\mathscr{S}_{\mathscr{A}})$ in the following lemma.

Lemma 2.4. *Let $\mathscr{S}_{\mathscr{A}} = \{a, a^2, a^3, \ldots, a^n\}$ be a monogenic semigroup (with zero). Then, the 2-distance degree sequence of the monogenic semigroup graphs is given as*

$$\left.\begin{array}{l} d_2(a) = n-2, d_2(a^2) = n-3, \cdots, d_2(a^{\lfloor \frac{n}{2} \rfloor}) = n - 1 - \left\lfloor \dfrac{n}{2} \right\rfloor, d_2(a^{\lfloor \frac{n}{2} \rfloor + 1}) = n - 1 - \left\lfloor \dfrac{n}{2} \right\rfloor, \\ d_2(a^{\lfloor \frac{n}{2} \rfloor + 2}) = n - 2 - \left\lfloor \dfrac{n}{2} \right\rfloor, \cdots, d_2(a^{n-1}) = 1, d_2(a^n) = 0. \end{array}\right\}$$

Let us give the well-known equation that will appear here in the proofs of our theorems with the following lemma.

Lemma 2.5. *For the natural number n, we have*

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2}, & n \text{ even} \\ \frac{n-1}{2}, & n \text{ odd}. \end{cases} \tag{2.1}$$

We are now ready to calculate the first leap zagreb index of monogenic semigroup graphs.

Theorem 2.6. *Let $\Gamma(\mathscr{S}_{\mathscr{A}})$ be monogenic semigroup graphs. Then, we have*

$$LM_1(\Gamma(\mathscr{S}_{\mathscr{A}})) = \begin{cases} \frac{4n^3 - 15n^2 + 14n}{12}, & n \text{ even} \\ \frac{4n^3 - 15n^2 + 20n - 9}{12}, & n \text{ odd}. \end{cases}$$

Proof. From definition of the first leap Zagreb index and by Lemma 2.4, we have

$$\begin{aligned} LM_1(\Gamma(\mathscr{S}_{\mathscr{A}})) = & \sum_{a^i \in V(\Gamma(\mathscr{S}_{\mathscr{M}}))} d_2^2(a^i) \\ =& d_2^2(a) + d_2^2(a^2) + \cdots + d_2^2(a^{\lfloor \frac{n}{2} \rfloor}) + d_2^2(a^{\lfloor \frac{n}{2} \rfloor} + 1) + d_2^2(a^{\lfloor \frac{n}{2} \rfloor + 2}) + \\ &+ \cdots + d_2^2(a^{n-1}) + d_2^2(a^n) \\ =& (n-2)^2 + (n-3)^2 + ... + \Big(n - 1 - \Big\lfloor \frac{n}{2} \Big\rfloor\Big)^2 + \Big(n - 1 - \Big\lfloor \frac{n}{2} \Big\rfloor\Big)^2 \\ &+ \Big(n - 2 - \Big\lfloor \frac{n}{2} \Big\rfloor\Big)^2 + \cdots + 2^2 + 1^2 + 0^2 \\ =& \frac{(n-2)(n-1)(2n-3)}{6} + (n - 1 - \Big\lfloor \frac{n}{2} \Big\rfloor)^2 \end{aligned} \tag{2.2}$$

There are two possible situations from here. With (2.1) we get the desired result. Thus, the proof is completed. □

With the following theorem, we give the exact formula for the second leap Zagreb index of monogenic semigroup graphs.

Theorem 2.7. *Let $\Gamma(\mathscr{S}_{\mathscr{A}})$ be monogenic semigroup graphs. Then, we have*

$$LM_2(\Gamma(\mathscr{S}_{\mathscr{A}})) = \begin{cases} \frac{n^4 - 4n^3 + 2n^2 + 4n - 3}{48}, & n \text{ odd} \\ \frac{n^4 - 4n^3 + 2n^2 + 4n}{48}, & n \text{ even.} \end{cases}$$

Proof. There are two possible cases for the values n is odd or even.

Let n be odd. In this case, from definition of the second leap Zagreb index and by Lemma 2.4, we have

$$\begin{aligned} LM_2(\Gamma(\mathscr{S}_{\mathscr{A}})) = & \sum_{a^i a^j \in E(\Gamma(\mathscr{S}_{\mathscr{M}}))} d_2(a^i) d_2(a^j) \\ =& d_2(a^n) d_2(a) + d_2(a^n) d_2(a^2) + \cdots + d_2(a^n) d_2(a^{n-2}) + d_2(a^n) d_2(a^{n-1}) + \\ &+ d_2(a^{n-1}) d_2(a^2) + \cdots + d_2(a^{n-1}) d_2(a^{n-3}) + d_2(a^{n-1}) d_2(a^{n-2}) + \\ &+ \cdots + \\ &+ d_2(a^{\frac{n+1}{2}+2}) d_2(a^{\frac{n+1}{2}-2}) + d_2(a^{\frac{n+1}{2}+2}) d_2(a^{\frac{n+1}{2}-1}) + d_2(a^{\frac{n+1}{2}+2}) d_2(a^{\frac{n+1}{2}}) + \\ &+ d_2(a^{\frac{n+1}{2}+2}) d_2(a^{\frac{n+1}{2}+1}) + \\ &+ d_2(a^{\frac{n+1}{2}+1}) d_2(a^{\frac{n+1}{2}-1}) + d_2(a^{\frac{n+1}{2}+1}) d_2(a^{\frac{n+1}{2}}) \end{aligned}$$

Consequently, the second leap Zagreb index of $\Gamma(\mathscr{S}_{\mathscr{A}})$ is written as given in the following

$$LM_2(\Gamma(\mathscr{S}_{\mathscr{A}})) = LM_{2,n} + LM_{2,n-1} + \cdots + LM_{2,\frac{n+1}{2}+2} + LM_{2,\frac{n+1}{2}+1}$$

When calculating these operations, we use $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ from (2.2) for n odd. Then, we have

$$\begin{aligned} LM_{2,n} =& d_2(a^n) d_2(a) + d_2(a^n) d_2(a^2) + \cdots + d_2(a^n) d_2(a^{n-2}) + d_2(a^n) d_2(a^{n-1}) \\ =& 0.(n-2) + 0(n-3) + ... + 0.2 + 0.1 + 0(n - 1 - \frac{n-1}{2}) \\ =& 0. \end{aligned}$$

In the case of applying operations similar to $LM_{2,n-1}$, $LM_{2,n-2}$, $\cdots$, $LM_{2,\frac{n+1}{2}+2}$ and $LM_{2,\frac{n+1}{2}+1}$; we get

$$\begin{aligned}
LM_{2,n-1} =& d_2(a^{n-1})d_2(a^2) + \cdots + d_2(a^{n-1})d_2(a^{n-3}) + d_2(a^{n-1})d_2(a^{n-2}) \\
=& 1.(n-3) + 1.(n-4) + \cdots + 1(n-1-\frac{n-1}{2}) + \cdots + 1.3 + 1.2 + \\
& + 1(n-1-\frac{n-1}{2}) \\
=& \sum_{q=2}^{n-3} 1.q + 1.\frac{n-1}{2},
\end{aligned}$$

$$\begin{aligned}
LM_{2,n-2} =& d_2(a^{n-2})d_2(a^3) + \cdots + d_2(a^{n-2})d_2(a^{n-2}) + d_2(a^{n-2})d_2(a^{n-3}) \\
=& 2.(n-4) + 2.(n-5) + \cdots + 2.(n-1-\frac{n-1}{2}) + \cdots + 2.4 + 2.3 + \\
& + 2.(n-1-\frac{n-1}{2}) \\
=& \sum_{q=3}^{n-4} 2.q + 2.\frac{n-1}{2},
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
LM_{2,\frac{n+1}{2}+2} =& d_2(a^{\frac{n+1}{2}+2})d_2(a^{\frac{n+1}{2}-2}) + d_2(a^{\frac{n+1}{2}+2})d_2(a^{\frac{n+1}{2}-1}) + d_2(a^{\frac{n+1}{2}+2})d_2(a^{\frac{n+1}{2}}) + \\
& + d_2(a^{\frac{n+1}{2}+2})d_2(a^{\frac{n+1}{2}+1}) \\
=& \frac{n-5}{2}.\frac{n+1}{2} + \frac{n-5}{2}.\frac{n-1}{2} + \frac{n-5}{2}.\frac{n-3}{2} + \frac{n-5}{2}.(n-1-\frac{n-1}{2}) \\
=& \sum_{q=\frac{n-5}{2}}^{\frac{n+1}{2}} \frac{n-5}{2}.q + \frac{n-5}{2}.\frac{n-1}{2},
\end{aligned}$$

and finally

$$\begin{aligned}
LM_{2,\frac{n+1}{2}+1} =& d_2(a^{\frac{n+1}{2}+1})d_2(a^{\frac{n+1}{2}-1}) + d_2(a^{\frac{n+1}{2}+1})d_2(a^{\frac{n+1}{2}}) \\
=& \frac{n-3}{2}.\frac{n-1}{2} + \frac{n-3}{2}.\frac{n-1}{2} \\
=& \sum_{q=\frac{n-1}{2}}^{\frac{n-1}{2}} \frac{n-3}{2}.q + \frac{n-3}{2}.\frac{n-1}{2},
\end{aligned}$$

Hence

$$LM_{2,n} + LM_{2,n-1} + \cdots + LM_{2,\frac{n+1}{2}+2} + LM_{2,\frac{n+1}{2}+1} = \sum_{i=1}^{\frac{n-1}{2}-1} \sum_{q=r+1}^{n-2-r} r.q + \sum_{s=1}^{\frac{n-1}{2}-1} s.(\frac{n-1}{2}).$$

If similar operations are performed in case n is even, the following sum is obtained

$$LM_{2,n} + LM_{2,n-1} + \cdots + LM_{2,\frac{n}{2}+2} + LM_{2,\frac{n}{2}+1} = \sum_{r=1}^{\frac{n}{2}-1} \sum_{q=r+1}^{n-2-r} r.q + \sum_{s=1}^{\frac{n}{2}-1} s.(\frac{n}{2}-1)$$

So as desired. □

Theorem 2.8. *Let $\Gamma(\mathscr{S}_{\mathscr{A}})$ be monogenic semigroup graphs. Then, we have*

$$LM_3(\Gamma(\mathscr{S}_{\mathscr{A}})) = \begin{cases} \frac{2n^3-3n^2-2n+3}{12}, & n \text{ odd} \\ \frac{2n^3-3n^2-2n}{12}, & n \text{ even.} \end{cases}$$

Proof. There are two possible cases for the values n is odd or even.

Let n be odd. In this case, from definition of the third leap Zagreb index and by Lemma 2.4, we have

$$\begin{aligned} LM_3(\Gamma(\mathscr{S}_{\mathscr{A}})) = & \sum_{a^i a^j \in E(\Gamma(\mathscr{S}_{\mathscr{A}}))} [d_2(a^i) + d_2(a^j)] \\ =& (d_2(a^n) + d_2(a)) + (d_2(a^n) + d_2(a^2)) + \cdots + (d_2(a^n) + d_2(a^{n-2})) \\ &+ (d_2(a^n) + d_2(a^{n-1})) + (d_2(a^{n-1}) + d_2(a^2)) + \cdots + \\ &+ (d_2(a^{n-1}) + d_2(a^{n-3})) + (d_2(a^{n-1}) + d_2(a^{n-2})) + \\ &+ \cdots + \\ &+ (d_2(a^{\frac{n+1}{2}+2}) + d_2(a^{\frac{n+1}{2}-2})) + (d_2(a^{\frac{n+1}{2}+2}) + d_2(a^{\frac{n+1}{2}-1})) + \\ &+ (d_2(a^{\frac{n+1}{2}+2}) + d_2(a^{\frac{n+1}{2}})) + (d_2(a^{\frac{n+1}{2}+2}) + d_2(a^{\frac{n+1}{2}+1})) + \\ &+ (d_2(a^{\frac{n+1}{2}+1}) + d_2(a^{\frac{n+1}{2}-1})) + (d_2(a^{\frac{n+1}{2}+1}) + d_2(a^{\frac{n+1}{2}})). \end{aligned}$$

Consequently, the third leap Zagreb index of $\Gamma(\mathscr{S}_{\mathscr{A}})$ is written as given in the following

$$LM_3(\Gamma(\mathscr{S}_{\mathscr{A}})) = LM_{3,n} + LM_{3,n-1} + LM_{3,n-2} + \cdots + LM_{3,\frac{n+1}{2}+2} + LM_{3,\frac{n+1}{2}+1}$$

When calculating these operations, we use $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ from (2.2) for n odd. Then, we have

$$\begin{aligned} LM_{3,n} =& (d_2(a^n) + d_2(a)) + (d_2(a^n) + d_2(a^2)) + \cdots + (d_2(a^n) + d_2(a^{n-2})) \\ &+ (d_2(a^n) + d_2(a^{n-1})) \\ =& (0 + n - 2) + (0 + n - 3) + \cdots + (0 + 2) + (0 + 1) + (0 + (n - 1 - \frac{n-1}{2})) \\ =& \sum_{q=1}^{n-2} (0 + q) + (0 + \frac{n-1}{2}). \end{aligned}$$

In the case of applying operations similar to $LM_{3,n-1}$,$LM_{3,n-2}$, $\cdots$, $LM_{3,\frac{n+1}{2}+2}$ and $LM_{3,\frac{n+1}{2}+1}$; we get

$$\begin{aligned} LM_{3,n-1} =& (d_2(a^{n-1}) + d_2(a^2)) + \cdots + (d_2(a^{n-1}) + d_2(a^{n-3})) + (d_2(a^{n-1}) + d_2(a^{n-2})) \\ =& (1 + n - 3) + (1 + n - 4) + \cdots + (1 + (n - 1 - \frac{n-1}{2})) + \cdots + (1 + 3) + \\ &+ (1 + 2) + (1 + (n - 1 - \frac{n-1}{2})) \\ =& \sum_{q=2}^{n-3} (1 + q) + (1 + \frac{n-1}{2}), \end{aligned}$$

$$
\begin{aligned}
LM_{3,n-2} =&(d_2(a^{n-2})+d_2(a^3))+\cdots+(d_2(a^{n-2})+d_2(a^{n-2}))+(d_2(a^{n-2})+d_2(a^{n-3}))\\
=&(2+(n-4))+(2+(n-5))+\cdots+(2+(n-1-\frac{n-1}{2}))+\cdots+(2+4)+\\
&+(2+3)+(2+(n-1-\frac{n-1}{2}))\\
=&\sum_{q=3}^{n-4}(2+q)+(2+\frac{n-1}{2}),
\end{aligned}
$$

$$\vdots$$

$$
\begin{aligned}
LM_{3,\frac{n+1}{2}+2} =&(d_2(a^{\frac{n+1}{2}+2})+d_2(a^{\frac{n+1}{2}-2}))+(d_2(a^{\frac{n+1}{2}+2})+d_2(a^{\frac{n+1}{2}-1}))+(d_2(a^{\frac{n+1}{2}+2})+d_2(a^{\frac{n+1}{2}}))+\\
&+(d_2(a^{\frac{n+1}{2}+2})+d_2(a^{\frac{n+1}{2}+1}))\\
=&(\frac{n-5}{2}+\frac{n+1}{2})+(\frac{n-5}{2}+\frac{n-1}{2})+(\frac{n-5}{2}+\frac{n-1}{2})+(\frac{n-5}{2}+\frac{n-3}{2})\\
=&\sum_{q=\frac{n-5}{2}}^{\frac{n+1}{2}}(\frac{n-5}{2}+q)+(\frac{n-5}{2}+\frac{n-1}{2}),
\end{aligned}
$$

and finally

$$
\begin{aligned}
LM_{3,\frac{n+1}{2}+1} =&(d_2(a^{\frac{n+1}{2}+1})+d_2(a^{\frac{n+1}{2}-1}))+(d_2(a^{\frac{n+1}{2}+1})d_2(a^{\frac{n+1}{2}}))\\
=&(\frac{n-3}{2}+\frac{n-1}{2})+(\frac{n-3}{2}+\frac{n-1}{2})\\
=&\sum_{q=\frac{n-1}{2}}^{\frac{n-1}{2}}(\frac{n-3}{2}+q)+(\frac{n-3}{2}+\frac{n-1}{2}),
\end{aligned}
$$

Hence

$$LM_{3,n}+LM_{3,n-1}+\cdots+LM_{3,\frac{n+1}{2}+2}+LM_{3,\frac{n+1}{2}+1}=\sum_{r=0}^{\frac{n-1}{2}-1}\sum_{q=r+1}^{n-2-r}(r+q)+\sum_{s=0}^{\frac{n-1}{2}-1}(s+\frac{n-1}{2})$$

If similar operations are performed in case n is even, the following sum is obtained

$$LM_{3,n}+LM_{3,n-1}+\cdots+LM_{3,\frac{n}{2}+2}+LM_{3,\frac{n}{2}+1}=\sum_{r=0}^{\frac{n}{2}-2}\sum_{q=r+1}^{n-2-r}(r+q)+\sum_{s=0}^{\frac{n}{2}-1}(s+\frac{n}{2}-1)$$

So as desired. □

Theorem 2.9. *Let $\Gamma(\mathscr{S}_{\mathscr{A}})$ be monogenic semigroup graphs. Then, we have*

$$LF(\Gamma(\mathscr{S}_{\mathscr{A}}))=\begin{cases}\frac{2n^4-11n^3+20n^2-12n}{8}, & n \text{ even}\\ \frac{2n^4-11n^3+23n^2-21n+7}{8}, & n \text{ odd.}\end{cases}$$

Proof. From definition of the first leap Zagreb index and by Lemma 2.4, we have

$$
\begin{aligned}
LF(\Gamma(\mathscr{S}_{\mathscr{A}})) = & \sum_{a^i \in V(\Gamma(\mathscr{S}_{\mathscr{A}}))} d_2^3(a^i) \\
=& d_2^3(a) + d_2^3(a^2) + \cdots + d_2^3(a^{\lfloor \frac{n}{2} \rfloor}) + d_2^3(a^{\lfloor \frac{n}{2} \rfloor} + 1) + d_2^3(a^{\lfloor \frac{n}{2} \rfloor + 2}) + \\
& + \cdots + d_2^3(a^{n-1}) + d_2^3(a^n) \\
=& (n-2)^3 + (n-3)^3 + \ldots + \left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right)^3 + \left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right)^3 \\
& + \left(n - 2 - \left\lfloor \frac{n}{2} \right\rfloor\right)^3 + \cdots + 2^3 + 1^3 + 0^3 \\
=& \left(\frac{(n-2)(n-1)}{2}\right)^2 + \left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right)^3 \qquad (2.3)
\end{aligned}
$$

There are two possible situations from here. With (2.1) we get the desired result. Thus, the proof is completed. □

Theorem 2.10. *Let $\Gamma(\mathscr{S}_{\mathscr{A}})$ be monogenic semigroup graphs. Then, we have*

$$
LEC(\Gamma(\mathscr{S}_{\mathscr{A}})) = \begin{cases} n^2 - 2n, & n \text{ odd} \\ n^2 - 2n + 1, & n \text{ even.} \end{cases}
$$

Proof. By definition of the monogenic semigroup graph, we see that $ecc(a^n) = 1$ and the eccentries of the other vertices are two the except of the vertex a^n. Thus, from definition of the leap eccentric connectivity index and Lemma 2.4, we have

$$
\begin{aligned}
LEC(\Gamma(\mathscr{S}_{\mathscr{A}})) = & \sum_{a^i \in V(\Gamma(\mathscr{S}_{\mathscr{A}}))} d_2(a^i).ecc(a^i) \\
=& d_2(a)ecc(a) + d_2(a^2)ecc(a^2) + \cdots + d_2(a^{\lfloor \frac{n}{2} \rfloor})ecc((a^{\lfloor \frac{n}{2} \rfloor})) \\
& + d_2(a^{\lfloor \frac{n}{2} \rfloor + 1})ecc(a^{\lfloor \frac{n}{2} \rfloor + 1}) + d_2(a^{\lfloor \frac{n}{2} \rfloor + 2})ecc(a^{\lfloor \frac{n}{2} \rfloor + 2}) + \\
& + \cdots + d_2(a^{n-1})ecc(a^{n-1}) + d_2(a^n)ecc(a^n) \\
=& (n-2).2 + (n-3).2 + \ldots + \left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right).2 + \left(n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right).2 \\
& + \left(n - 2 - \left\lfloor \frac{n}{2} \right\rfloor\right).2 + \cdots + 2.2 + 1.2 + 0.1 \\
=& (n-2)(n-1) + (n - 1 - \left\lfloor \frac{n}{2} \right\rfloor).2 \qquad (2.4)
\end{aligned}
$$

There are two possible situations from here. With (2.1) we get the desired result. Thus, the proof is completed. □

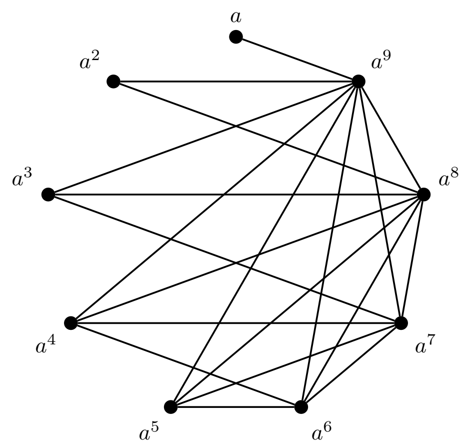

FIGURE 1. The graph of $\Gamma(\mathscr{S}_{\mathscr{A}})$

Example 2.11. Let us consider the monogenic semigroup graphs $\Gamma(\mathscr{S}_{\mathscr{A}})$ with nine vertices as Figure 1. Then we get

- $LM_1(\Gamma(\mathscr{S}_{\mathscr{A}})) = \frac{4.9^3-15.9^2+20.9-9}{12} = 156$ (by Theorem 2.5)
- $LM_2(\Gamma(\mathscr{S}_{\mathscr{A}})) = \frac{9^4-4.9^3+2.9^2+4.9-3}{48} = 80$ (by Theorem 2.6)
- $LM_3(\Gamma(\mathscr{S}_{\mathscr{A}})) = \frac{2.9^3-3.9^2-2.9+3}{12} = 25$ (by Theorem 2.7)
- $LF(\Gamma(\mathscr{S}_{\mathscr{A}})) = \frac{2.9^4-11.9^3+23.9^2-21.9+7}{8} = 848$ (by Theorem 2.8)
- $LEC(\Gamma(\mathscr{S}_{\mathscr{A}})) = 9^2 - 2.9 = 63$ (by Theorem 2.9)

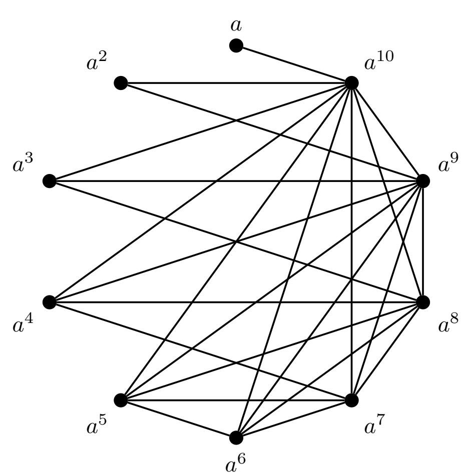

FIGURE 2. The graph of $\Gamma(\mathscr{S}_{\mathscr{A}})$

Example 2.12. Let us consider the monogenic semigroup graphs $\Gamma(\mathscr{S}_{\mathscr{A}})$ with ten vertices as Figure 2. Then we get

- $LM_1(\Gamma(\mathscr{S}_{\mathscr{A}})) = \frac{4.10^3-15.10^2+14.10}{12} = 220$ (by Theorem 2.5)

- $LM_2(\Gamma(\mathscr{S}_{\mathscr{A}})) = \frac{10^4-4.10^3+2.10^2+4.10}{48} = 130$ (by Theorem 2.6)
- $LM_3(\Gamma(\mathscr{S}_{\mathscr{A}})) = \frac{2.10^3-3.10^2-2.10}{12} = 140$ (by Theorem 2.7)
- $LF(\Gamma(\mathscr{S}_{\mathscr{A}})) = \frac{2.10^4-11.10^3+20.10^2-12.10}{8} = 1360$ (by Theorem 2.8)
- $LEC(\Gamma(\mathscr{S}_{\mathscr{A}})) = 10^2 - 2.10 + 1 = 81$ (by Theorem 2.9)

3. Conclusion

Topological indices are important tools that are widely used in revealing the chemical and physical properties of molecules, especially in QSAR and QSPR research. Leap indices have an important place among topological indices. In this study, we calculated the leap indices of monogenic semigroup graphs in terms of number of the vertices.

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