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Some Applications Related to Differential Inclusions Based on the Use of a Weighted Space

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Abstract

In this paper, we present an existence theorem for the problem of discontinuous dynamical system related to ordinary differential inclusion, based on the use of the concepts related to weighted spaces introduced by Górka and Rybka, without using any fixed point theorem. The solution concept in this theorem is considered to belong to the weighted space. For comparison with the classical case and as an application of the theorem, we give an example problem that has such a solution but no continuously differentiable solution.

1. Introduction

In the mathematical modeling of systems with dynamic behavior in various fields of the real-world and in the qualitative and numerical analysis of these systems, differential equations with initial or boundary conditions and the existence and uniqueness of solutions and numerical approach techniques to solutions for these equations appear as important mathematical tools (see, e.g. [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]). (Ordinary) differential inclusions, which are generalized forms of ordinary differential equations and started to be studied after the advances in right-side discontinuous differential equations and solution methods for the problems related to these equations in the 1960s, have a similarly important role in applied mathematics since using directly in modeling and especially in the necessary and sufficient results of optimal control problems of discontinuous systems (see, e.g. [12], [13], [2], [14], [15], [16]). In the literature, differential inclusions specific to various fields such as engineering, biology, economics, and special types of differential inclusions that arise from the use of different notions are also encountered (see, e.g. [17], [7], [14], [16]). Fuzzy differential inclusions, measure differential inclusions, Volterra differential inclusions, and impulsive differential inclusions are a few of them.

Górka and Rybka in [18] obtained some results about the existence of a solution for ordinary differential equation with an initial condition based on the use of the weighted space equipped with the weighted norm. Here, they used Banach fixed point theorem under the boundedness assumption and the assumption of a special type of Lipschitz continuity (with $l(t)/t$ depending upon t).

In this paper, we present some results about the existence of a global solution for the discontinuous differential inclusion with an initial condition, based on the use of the weighted space, without boundedness assumption in nonconvex case. For this purpose, we construct a sequence, based on the uses of the weighted norm and approximations mentioned in [13], without using any fixed point theorem to derive the solution.

Since our results are true for the discontinuous ordinary differential equations as well, these results can be applied to the system described by the differential equation in [18] without boundedness assumption. In addition, an illustrative example satisfying the assumptions mentioned in the results is also given in this paper.

2. Preliminaries

For unexplained terminology and the basic results on the weighted spaces and differential inclusion theory we refer to [17], [19], [18], [15], [16]. For a fixed $b>0$, $C([0,b],\mathbb{R}^n)$ denotes Banach space of all continuous functions $g:[0,b]\to\mathbb{R}^n$ with the

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supremum norm $\|g\|_\infty = \sup_{t\in[0,b]} |g(t)|$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. Let us fixed $\alpha \in]0,1]$ and $r \in \mathbb{R}^n$. Now, let $g \in C([0,b],\mathbb{R}^n)$ and we put

$$|g|_{r,\alpha} := \sup_{t\in]0,b]} \frac{|g(t)-r|}{t^\alpha}.$$

The collection of all functions $g \in C([0,b],\mathbb{R}^n)$ satisfying $|g|_{r,\alpha} < \infty$ is denoted by $W_{r,\alpha} = W_{r,\alpha}([0,b],\mathbb{R}^n)$. It is clear that $g(0) = r$ whenever $g \in W_{r,\alpha}$, and that $|\cdot|_{0,\alpha}$ is a norm when $r = 0$. Note that the function ρ defined as

$$\rho(g,h) = |g-h|_{0,\alpha} \text{ for } g,h \in W_{r,\alpha}.$$

is a metric on $W_{r,\alpha}$. Moreover, $(W_{r,\alpha},\rho)$ is a complete metric space (for details, see [18]).
Let S be a subset of $\mathbb{R}^n$ and $\Phi : S \to \mathbb{R}^n$ a set-valued map. For $a \in \mathbb{R}^n$, we denote the projection of a onto S by $\pi(a,S)$, that is, $\pi(a,S) = \{s \in S : |a-s| = d(x,S)\}$ where $d(a,S) = \inf\{|a-s| : s \in S\}$. If $\pi(a,S)$ is nonempty, then each element of it is called the closest point in S to a. It is known that $\pi(a,S)$ is nonempty and compact if S is closed (for details, see [20], [15]). A single-valued function $\phi : S \to \mathbb{R}^n$ is said to be a measurable selection from Φ if ϕ is measurable in the usual sense and $\phi(s) \in \Phi(s)$ for all s in S. Let E and Z be nonempty bounded subsets of $\mathbb{R}^n$. The ball of radius δ around E is defined as

$$O_\delta(E) = \left\{ r \in \mathbb{R}^n : d(E,r) = \inf_{x\in E} |x-r| < \delta \right\}.$$

The Hausdorff distance between E and Z is defined as

$$d_H(E,Z) = \inf\{\delta > 0 : O_\delta(E) \supseteq Z, O_\delta(Z) \supseteq E\}.$$

Note that the existence of corresponding finite $\delta > 0$ follows from the boundedness of sets E,Z.
Let $m \geq 1$. let $\mathscr{L}$ and $\mathscr{B}^m$ be the collection of Lebesgue measurable subsets of $[0,b]$ and Borel subsets of $\mathbb{R}^m$, respectively. The smallest $\sigma-$algebra of subsets of $[0,b]\times\mathbb{R}^m$ generated by Cartesian products of sets in $\mathscr{L}$ and $\mathscr{B}^m$ is denoted by $\mathscr{L}\times\mathscr{B}^m$. By L^1_m and $\|\cdot\|_1$, we denote the space of all Lebesgue integrable functions from $[0,b]$ into $\mathbb{R}^m$ and the norm on L^1_m as usual, respectively. Let $\Psi : [0,b]\times\mathbb{R}^m \to \mathbb{R}^n$ a set-valued map. Ψ is said to be $\mathscr{L}\times\mathscr{B}^m$-measurable if the set $\Psi^{-1}(V)$ lies in $\mathscr{L}\times\mathscr{B}^m$ for all open subset V of $\mathbb{R}^n$. We say that Ψ is w-integrably bounded (with η) if there exists a non-negative function $\eta \in L^1_1$ with $av(\eta,b) < \infty$ satisfying $\Psi(s,y) \subseteq \eta(s)B$ for all $(s,y) \in [0,b]\times\mathbb{R}^m$, where $av(\eta,b) := \sup_{t\in]0,b]} \frac{1}{t}\int_0^t \eta(s)\,ds$ and B is the closed unit ball of $\mathbb{R}^n$. We say that Ψ satisfies the w-Kamke-type Lipschitz condition (with ℓ) if there exists a non-negative function $\ell \in L^1_1$ satisfying $av(\ell,b) < 1$ and

$$d_H(\Psi(t,y),\Psi(t,x)) \leq \frac{\ell(t)}{t}|y-x|$$

for any (t,x) and (t,y) in $]0,b]\times\mathbb{R}^n$.
We now consider the following Cauchy problem related to a discontinuous differential inclusion,

$$\dot{x}(t) \in F(t,x(t)),\ x(0) = r \tag{2.1}$$

where $F : [0,b]\times\mathbb{R}^n \to \mathbb{R}^n$ is a given set-valued map.
We say that the absolutely continuous function $x \in W_{r,\alpha}([0,b],\mathbb{R}^n)$ satisfying the initial condition r and the differential inclusion in (2.1) a.e. on $[0,b]$ is a (global) solution of the problem.
Throughout this paper, "a.e. on $[0,b]$" is denoted by "a.e" briefly. AC denotes the space of absolutely continuous functions from $[0,b]$ to $\mathbb{R}^n$. For any $g(\cdot) \in AC$, the function ϑ_g define by $\vartheta_g(t) = d(\dot{g}(t),F(t,g(t)))$ a.e.

Proposition 2.1. *(see, [15]) Suppose that a sequence $\{\phi_n\}$in $L^1([0,b],\mathbb{R}^n)$ converges to a function $\phi \in L^1([0,b],\mathbb{R}^n)$ in $\|\cdot\|_1$. Then there exists a subsequence of $\{\phi_n\}$ that converges pointwise to ϕ a.e.*

3. Main results

Theorem 3.1. *Let F be the $\mathscr{L}\times\mathscr{B}^m$-measurable set-valued map with nonempty closed values satisfying w-Kamke-type Lipschitz condition (with ℓ). Then for any $g \in W_{r,1}\cap AC$ satisfying $\vartheta_g \in L^1_1$ and $av(\vartheta_g,b) < \infty$, there exists a solution of the problem (2.1) in $B_\delta(g)$. Here, δ is a positive number satisfying $\delta < \frac{av(\vartheta_g,b)}{1-av(\ell,b)}$ and $B_\delta(g)$ is the open ball of $(W_{r,1},\rho)$ with radius δ.*

Proof. The main idea in the proof of this theorem would be to construct a Cauchy sequence $\{g_n\}$ (approximations) in the complete $(W_{r,1},\rho)$. Here, it will be determined on the basis of choosing $\dot{g}_n(t)$ as the closest point in $F(t,g_{n-1}(t))$ to $\dot{g}_{n-1}(t)$, and the desired solution will be obtained with the limit of the sequence. With this goal, let $g_0 = g \in W_{r,1}$. By using Proposition 2.3.2

in [16] and Corollary 8.2.13 in [17] together, it can be easily observed that there exists a measurable selection $\phi_0 = \phi_0(g_0(\cdot))$ from $\pi(\dot{g}_0(\cdot), F(\cdot, g_0(\cdot)))$. Since the inequality $|\phi_0(t)| \leq |\dot{g}_0(t)| + \vartheta_{g_0}(t)$ holds a.e. and $\vartheta_{g_0} \in L_1^1$, we get $\phi_0 \in L_n^1$. Thus we can define an operator I_0 for $t \in [0,b]$ as

$$I_0(t) = g_0(0) + \int_0^t \phi_0(s)ds.$$

Now put $g_1 = I_0$. It is clear that $g_1 \in AC$. Then $\dot{g}_1 = \phi_0$ and $|\dot{g}_1 - \dot{g}_0| = \vartheta_{g_0}$ a.e. It follows from the relation

$$|g_1(t) - g_0(t)| \leq \int_0^t |\dot{g}_1(s) - \dot{g}_0(s)|\, ds = \int_0^t \vartheta_{g_0}(s)ds \tag{3.1}$$

that $(|g_1(t) - g_0(0)|/t) \leq av(\vartheta_{g_0}, b) + |g_0|_{r,1}$ a.e. As $av(\vartheta_{g_0}, b) < \infty$ then $g_1 \in W_{r,1}$. Moreover, by using the above inequalities, the basic properties of the Hausdorff distance notion and the Lipschitz condition, we have

$$\begin{aligned}
\vartheta_{g_1}(t) &\leq |\dot{g}_1(t) - \dot{g}_0(t)| + \vartheta_{g_0}(t) + d_H(F(t, g_0(t)), F(t, g_0(t))) \\
&\leq 2\vartheta_{g_0}(t) + (\ell(t)/t)\,|g_1(t) - g_0(t)| \\
&\leq 2\vartheta_{g_0}(t) + (\ell(t)/t)\int_0^t \vartheta_{g_0}(s)ds \\
&\leq 2\vartheta_{g_0}(t) + \ell(t)\, av(\vartheta_{g_0}, b) \text{ a.e.}
\end{aligned}$$

So it can easily be concluded that $\vartheta_{g_1} \in L_1^1$ and $av(\vartheta_{g_1}, b) < \infty$.
In this way, by defining $g_n := I_{n-1}$ and ϕ_n and using induction on $n = 1, 2, ...$, we get a sequence $\{g_n\}$ in $W_{r,1}$. Let us prove that the sequence $\{g_n\}$ is Cauchy in $W_{r,1}$. By definition of $\{g_n\}$ and $\{\phi_n\}$, for $n = 0, 2, ...$ we get

$$\dot{g}_{n+1} = \phi_n,\ |\dot{g}_{n+1} - \dot{g}_n| = \vartheta_{g_n} \text{ a.e.}$$

From the equality $d(\dot{g}_n(t), F(t, g_{n-1}(t))) = 0$ a.e. for $n = 1, 2, ...,$

$$\begin{aligned}
\vartheta_{g_n} &\leq d(\dot{g}_n(t), F(t, g_{n-1}(t))) + d_H(F(t, g_{n-1}(t)), F(t, g_n(t))) \\
&\leq (\ell(t)/t)\,|g_n(t) - g_{n-1}(t)| \text{ a.e.}
\end{aligned} \tag{3.2}$$

Taking integral from both sides, we have

$$|g_{n+1}(t) - g_n(t)| \leq \left(\sup_{s \in]0,b]} \frac{|g_n(s) - g_{n-1}(s)|}{s}\right) \int_0^t \ell(s)\, ds. \tag{3.3}$$

Therefore, it can be easily verified that

$$\rho(g_{n+1}, g_n) \leq av(\ell, b)\, \rho(u_n, u_{n-1}). \tag{3.4}$$

Note that the last inequality implies

$$\rho(g_{n+1}, g_n) \leq (av(\ell, b))^n \rho(g_1, g_0). \tag{3.5}$$

From here, we derive that

$$\begin{aligned}
|g_n|_{x_0,1} &\leq \rho(g_n, g_0) + |g_0|_{x_0,1} \\
&\leq \rho(g_n, g_{n-1}) + ... + \rho(g_1, g_0) + |g_0|_{x_0,1} < \infty.
\end{aligned}$$

Thus $g_n \in W_{r,1}$. In addition, the relations $\rho(g_1, g_0) \leq \vartheta_{g_0}$(as a result of (3.1)) and (3.5) implies that,

$$\begin{aligned}
\rho(g_n, g_0) &\leq \rho(g_n, g_{n-1}) + ... + \rho(g_1, g_0) \\
&\leq \vartheta_{g_0} \sum_{i=0}^{n-1} (av(\ell, b))^i.
\end{aligned} \tag{3.6}$$

As $av(\ell, b) < 1$, the relation (3.5) implies that the sequence $\{g_n\}$ is Cauchy, $W_{r,1}$ being complete, it converges uniformly to some function $y \in W_{r,1}$. Taking into account (3.2) and (3.3), we get

$$\|\phi_n - \phi_{n-1}\|_1 \leq \rho(g_n, g_{n-1}) \int_0^b \ell(s)\, ds,$$

so that $\{\phi_n\}$ is a Cauchy sequence in L_n^1. Let ϕ be the limit of $\{\phi_n\}$. One can easily have $y(t) = r + \int_0^t \phi(s)\, ds$. Moreover,

$$\begin{aligned}
\vartheta_{g_{n+1}} &\leq |\dot{g}_{n+1}(t) - \dot{y}(t)| + d(\dot{y}(t), F(t, y(t))) + d_H(F(t, g_{n+1}(t)), F(t, y(t))) \\
&\leq |\phi_n(t) - \dot{y}(t)| + \vartheta_y(t) + (\ell(t)/t)\,|g_{n+1}(t) - y(t)| \text{ a.e.}
\end{aligned}$$

Thus,

$$\left|\vartheta_{g_{n+1}}(t)-\vartheta_{y}(t)\right|\leq\phi\left(t\right)\rho\left(g_{n+1},y\right)+\left|\phi_{n}(t)-\dot{y}(t)\right| \text{ a.e.} \tag{3.7}$$

From Proposition 2.1, there exists a subsequence $\left\{\phi_{n_k}(t)\right\}$ converging to $\phi(t)$ a.e. Replacing ϕ_n with the ϕ_{n_k} in (3.7), we derive that $\vartheta_y(t)=\lim_{k\to\infty}\vartheta_{g_{n_k+1}}(t)$ a.e. By the inequality (3.2) one can get,

$$\vartheta_y(t)\leq\ell(t)\lim_{k\to\infty}\rho\left(g_{n_k+1},g_{n_k}\right) \text{ a.e.}$$

which implies $\vartheta_y(t)=0$ a.e. From here, we conclude that $y\in W_{r,1}$ is a solution. Moreover, by using (3.6), one can easily have $\rho(y,g)<\delta$. □

Remark that the following Corollary is a consequence of Theorem 3.1 for $g\equiv r$.

Corollary 3.2. *Let F be the $\mathscr{L}\times\mathscr{B}^m$-measurable set-valued map with nonempty closed values satisfying w-Kamke-type Lipschitz condition (with ℓ). If $\vartheta_r\in L_1^1$ and $av(\vartheta_r,b)<\infty$, then the problem (2.1) has at least one solution (in $W_{r,1}$).*

Corollary 3.3. *Let F be the $\mathscr{L}\times\mathscr{B}^m$-measurable set-valued map with nonempty closed values satisfying w-Kamke-type Lipschitz condition (with ℓ). Suppose further that F is w-integrably bounded (with η). Then the problem (2.1) has at least one solution (in $W_{r,1}$).*

Proof. Let the function $h^*\equiv(h_1,h_2,...h_n):[0,T]\to\mathbb{R}^n$ defined by $h_i(t)=r_i+\int_0^t\eta(s)ds$. We choose $g\equiv h^*$. By hypotheses we get $\dot{g}=(\eta,\eta,...\eta)$, $\vartheta_g(t)\leq(1+\sqrt{n})\,\eta(t)$ a.e. and $g\in W_{r,1}\cap AC$. Thus, $\vartheta_g\in L^1$ and $av(\vartheta_g,b)<\infty$. By Theorem 3.1, we have desired conclusion. □

Remark 3.4. *Let $h:[0,b]\times\mathbb{R}^n\to\mathbb{R}^n$ be single-valued function. Consider F as set-valued map with value $F(s,z)=\{h(s,z)\}$. Then the problem (2.1) turns into the following Cauchy problem related to a discontinuous differential equation:*

$$\dot{z}(s)=h(s,z(s)),\ z(0)=r. \tag{3.8}$$

It is known that the uniqueness and existence results for the problem (2.1) can be obtained from hypotheses of Theorem 2.6 and Theorem 3.1 in [18]. Note that hypotheses of Corollary 3.2 are similar to these hypotheses except for the boundedness hypothesis (that is, for every $c>0$ there exists a non-negative function $m_c\in L_1^1$ such that $|z|<c$ implies $|h(s,z)|\leq m_c(s)$ for a.e.). It follows from Corollary 3.2 that the following existence result still holds without boundedness assumption. The uniqueness result can be obtained easily with the same proof in [18].

Corollary 3.5. *Let $h:[0,b]\times\mathbb{R}^n\to\mathbb{R}^n$ be the function satisfying the following:*

(a) h is $\mathscr{L}\times\mathscr{B}^m$-measurable,
(b) there exists a non-negative function $\ell\in L_1^1$ with $av(\ell,b)<1$ satisfying

$$|h(s,y)-h(t,z)|\leq(\ell(s)/s)\,|y-z|$$

for any (s,y) and (s,z) in $]0,b]\times\mathbb{R}^n$,
(c) $|h(\cdot,r)|\in L_1^1$ and $av(|h(\cdot,r)|,b)<\infty$.

Then the problem (3.8) has a unique solution (in $W_{r,1}$).

Example 3.6. *Let $r>0$, $b\in\left]\frac{r}{2},2r\right[$ and consider the following problem:*

$$\begin{aligned} h(s,z) &= \begin{cases} \frac{2}{2s+r}z & s>\frac{r}{2} \\ 0 & 0\leq s\leq\frac{r}{2} \end{cases},\ s\in[0,b] \\ \dot{z}(s) &= h(s,z),\ z(0)=r. \end{aligned}$$

The problem has no a continuously differentiable solution. It can be easily verified that h is $\mathscr{L}\times\mathscr{B}^m$-measurable, and that h satisfies the Lipschitz condition with l (defined by $l(s)=\frac{s}{r}$) given in Corollary 3.5. Moreover, $|h(\cdot,r)|$ is Riemann integrable on $[0,b]$ and $av(|h(\cdot,r)|,b)<\infty$. As the hypotheses of Corollary 3.2 are satisfied, the problem has a unique solution (in $W_{r,1}$). Note that the solution is the function $z:[0,b]\to\mathbb{R}$ defined by $z(s)=s+\frac{r}{2}$ if $\frac{r}{2}<s\leq b$ and $z(s)=r$ if $0\leq s\leq\frac{r}{2}$.

4. Conclusion

In this paper, an existence result for the discontinuous differential inclusion with an initial condition, where the solution lies in the weighted space, is given in Theorem 3.1. Here, unlike the classical existence results, the concepts related to the weighted space and the topology of this space are used in the nonconvex and unbounded case. As a consequence of the theorem, the existence result of the differential equations in [18] is generalized to differential inclusions without boundedness assumption. In addition, in the proof of the theorem, the approximations mentioned in [13] is used to be members of the weighted space. Considering recent studies using similar approximations in various fields related to differential inclusion theory (see, e.g. [21], [22], [23]), this paper will contribute to the theory by providing the generalized results based on the use of the concepts and the approximations related to the weighted space.

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