Istanbul Journal of Mathematics
ijmath 2023, 1 (1), 40–47
DOI: 10.26650/ijmath.2023.00001

RESEARCH ARTICLE

Complex extreme points and complex rotundity in Orlicz spaces equipped with the s-norm

E. Başar [1,4], B.H. Uysal [2,4*], Ş. Yaşar [3,4]

[1] Yeditepe University, Faculty of Arts and Sciences, Department of Mathematics, Ataşehir, 34755, İstanbul, Türkiye
[2] İstanbul University, Faculty of Science, Department of Mathematics, Vezneciler, 34134, İstanbul, Türkiye
[3] Gebze Technical University, Faculty of Basic Sciences, Department of Mathematics, Gebze, 41400, Kocaeli, Türkiye
[4] İstanbul University, Institue of Graduate Studies in Science, Süleymaniye, 34134, İstanbul, Türkiye

ABSTRACT
Let Φ be an Orlicz function and $L^{\Phi}(X,\Sigma,\mu)$ be the corresponding Orlicz space on a non-atomic, σ-finite, complete measure space (X,Σ,μ). It is known that extreme points which are connected with rotundity of the whole spaces are the most essential and important geometric notion in the geometric theory of Banach spaces. On the other hand, geometric theory of complex Banach spaces has significant applications that differ from the geometric theory of real Banach spaces. In this paper, we first describe the complex extreme points of unit ball of Orlicz spaces equipped with the s-norm where s is a strictly increasing outer function. We also give criteria for complex rotundity. Our study generalizes and unifies the results that have been obtained for the Orlicz norm and the p-Amemiya norm $(1 < p < \infty)$ separately.

Mathematics Subject Classification (2020): 46E30, 46B20

Keywords: Orlicz space, complex extreme points, complex strictly rotund, s-norm

1. INTRODUCTION

The notion of extreme points plays a crucial role for geometric theory of Banach spaces. Also, rotundity properties are very important in geometry of Banach spaces and its applications. Since the early 1980's, the investigations concerning the geometric theory of complex Banach spaces have been developed because it has significant applications that differ from the geometric theory of real Banach spaces. For instance, the notion of complex rotundity, which was introduced by Thorp, E., Whitley, R. (1967), has an important application in the theory of analytic functions. It is known that if f is a function from the unit disc of $\mathbb{C}$ into a complex Banach space X, f is analytic, i.e. $x^* \circ f$ is analytic in the classical sense for any $x^* \in X^*$ (the dual space of X) and the maximum of the function $F(z) = \|f(z)\|$ is attained in an interior point of unit disc, then F is a constant function. However, in the case when X is complex rotund, more can be deduced, namely that f is a constant function.

On the other hand, Orlicz spaces comprise an important class of Banach spaces that are a kind of generalization of Lebesgue spaces. The theory of Orlicz spaces has been greatly developed because of its important theoretical properties and value in applications. Some examples for applications of Orlicz spaces can be found in Arıs B., Öztop S., (2023) and Üster R. (2021). Structure of complex extreme points and complex rotundity in the class of Musielak–Orlicz spaces have been first studied by Wu, C.X., Sun, H. (1987) and Wu, C.X., Sun, H. (1987). Then Chen, L., Cui, Y. (2010) gave criteria for complex extreme points and complex rotundity in Orlicz function spaces equipped with the p-Amemiya norm.

Wisła, M. (2020), using the concept of an outer function, presented a general and universal method of introducing norms in Orlicz spaces that covered the classical Orlicz and Luxemburg norms, and p-Amemiya norms $(1 \leqslant p \leqslant \infty)$. After then, Başar E., Öztop, S., Uysal, B.H., Yaşar, Ş. (2023), classified s-norms with respect to the constant σ_s and described real extreme points as well.

Our first aim in this work is to describe the complex extreme points in Orlicz spaces equipped with s-norms where s is strictly increasing. Then we give criteria for complex rotundity by using description of extreme points.

The structure of this paper as follows. In Section 2, we provide necessary definitions. In Section 3, we recall some technical results for Orlicz spaces equipped with s-norms that will be used and we make some observations from these known results. In

Corresponding Author: B.H. Uysal **E-mail:** huseyinuysal@istanbul.edu.tr
Submitted: 17.04.2023 • **Last Revision Received:** 31.05.2023 • **Accepted:** 01.06.2023 • **Published Online:** 06.06.2023

Section 4, we first describe complex extreme points of unit ball in Orlicz spaces equipped with s-norms for a strictly increasing outer function s. Then we obtain a necessary and sufficient condition for complex rotundity.

2. PRELIMINARIES

A map $\Phi : \mathbb{R} \to [0, \infty]$ is said to be an Orlicz function if $\Phi(0) = 0$, Φ is not identically equal to zero, Φ is even and convex on the interval $(-b_\Phi, b_\Phi)$, and Φ is left continuous at b_Φ, where $b_\Phi = \sup\{u > 0 : \Phi(u) < \infty\}$. From these properties it follows that an Orlicz function Φ is continuous on $(-b_\Phi, b_\Phi)$, increasing on $[0, b_\Phi)$, and satisfies $\lim\limits_{u\to\infty} \Phi(u) = \infty$. If Φ is an Orlicz function, letting also $a_\Phi = \sup\{u \geqslant 0 : \Phi(u) = 0\}$, then $a_\Phi = 0$ means that Φ vanishes only at 0 while $b_\Phi = \infty$ means that Φ takes only finite values. In this work, we assume that Orlicz function satisfies $\lim\limits_{u\to\infty} \frac{\Phi(u)}{u} = \infty$.

For an Orlicz function Φ, we define its complementary function Ψ by the formula

$$\Psi(v) = \sup_{u\geq 0}\{u|v| - \Phi(u)\}.$$

It is well-known that the complementary function is an Orlicz function as well. Let p_+ denote the right derivative of an Orlicz function Φ and q_+ denote the right derivative of its complementary function Ψ with the conventions that $\lim\limits_{u\to\infty} p_+(u) = p_+(\infty)$ and $p_+(u) = \infty$ for all $u \geqslant b_\Phi$. If there exists a constant $K > 0$ such that $\Phi(2u) \leqslant K\Phi(u)$ for all $u \in \mathbb{R}$, we say that Orlicz function Φ satisfies the Δ_2 condition and we denote this by $\Phi \in \Delta_2$. We know that the pair (Φ, Ψ) satisfies Young's inequality, that is,

$$xy \leqslant \Phi(x) + \Psi(y) \qquad (x, y \in \mathbb{R}),$$

where equality holds when $y = p_+(x)$ or $x = q_+(y)$ for $x, y \in \mathbb{R}$ (Rao, M. M. and Ren, Z. D. (1991)).

Throughout the paper, we will assume that (X, Σ, μ) is a measure space with a σ-finite, non-atomic and complete measure μ and denote by $L^c(X, \Sigma, \mu)$ (for short, $L^c(X)$) the space of all μ-equivalence classes of complex-valued and Σ-measurable functions defined on X. In addition, we use the conventions $0 \cdot \infty = 0$, $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$.

For a given Orlicz function Φ we define on $L^c(X, \Sigma, \mu)$ a convex functional I_Φ by

$$I_\Phi(f) = \int\limits_X \Phi(|f(t)|)\, d\mu \;\text{ for any } f \in L^c(\mu).$$

The Orlicz space $L^\Phi(X, \Sigma, \mu)$ generated by an Orlicz function Φ is a linear space of measurable functions defined by Orlicz, W. (1932)

$$L^\Phi(X, \Sigma, \mu) = \{f \in L^c(X, \Sigma, \mu) : I_\Phi(\lambda f) < \infty \text{ for some } \lambda > 0\}.$$

We denote the Orlicz space $L^\Phi(X, \Sigma, \mu)$ shortly by L^Φ.

The Orlicz space L^Φ is usually equipped with the Orlicz norm (Orlicz, W. (1932))

$$\|f\|^o_\Phi = \sup\left\{\int_X |f(t)g(t)|\, d\mu : g \in L^\Psi,\ I_\Psi(g) \leq 1\right\},$$

where Ψ is the complementary function to Φ, or with the equivalent Luxemburg norm

$$\|f\|_\Phi = \inf\left\{\lambda > 0 : I_\Phi\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

Further, for all $1 \leq p \leq \infty$ the p-Amemiya norm is defined on L^Φ by

$$\|f\|_{\Phi,p} = \begin{cases} \inf\limits_{k>0} k^{-1}(1 + I_\Phi(kf)^p)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \inf\limits_{k>0} k^{-1} \max\{1, I_\Phi(kf)\}, & \text{if } p = \infty. \end{cases}$$

The family of p-Amemiya norms includes the Orlicz and Luxemburg norms (see Cui, Y., Duan, L., Hudzik, H. and Wisła, M. (2008)).

In 2020, the notion of the s-norm was introduced by M. Wisła and all of the following definitions can be found in Wisła, M. (2020).

Definition 2.1. A function $s : [0, \infty] \to [1, \infty]$ is called an outer function if it is convex and satisfies the inequality

$$\max\{u, 1\} \leqslant s(u) \leqslant u + 1$$

for all $u \geqslant 0$.

Let us note that an outer function s is continuous and increasing on $[0, \infty)$. Evidently $s(0) = 1$ and set $s(\infty) = \infty$.

Since it is convex, an outer function s has both right and left derivatives. Let s'_+ be the right derivative of s so that $s'_+ : [0, \infty) \to [0, 1]$ is an increasing function. Let ${s'_+}^{-1} : [0, 1] \to [0, \infty]$ be a general inverse of s'_+ as defined in (Wisła, M. 2020, p. 11). Then ${s'_+}^{-1}$ is an increasing function as well.

Let us give some examples of families of outer functions (see Wisła, M. (2020)).

Example 2.1. (i) For $1 \leqslant p \leqslant \infty$,

$$s_p(u) = \begin{cases} (1 + u^p)^{1/p}, & \text{if } 1 \leqslant p < \infty, \\ \max\{1, u\}, & \text{if } p = \infty. \end{cases} \tag{1}$$

(ii) For $0 \leqslant c \leqslant 1$,

$$s_c(u) = \max\{1, u + c\}. \tag{2}$$

(iii) For $1 \leqslant m \leqslant 2$,

$$s_m(u) = \begin{cases} (m - 1)u + 1, & \text{if } 0 \leqslant u \leqslant 1, \\ u + m - 1, & \text{if } u > 1. \end{cases} \tag{3}$$

Definition 2.2. Let s be an outer function and Φ be an Orlicz function. Then the s-norm of $f \in L^\Phi$ is defined by

$$\|f\|_{\Phi,s} = \inf_{k>0} \frac{1}{k} s(I_\Phi(kf)).$$

The Orlicz space equipped with the s-norm will be denoted by L^Φ_s.

Observe that each of the families given in Example 2.1 generates both the Orlicz norm and the Luxemburg norm. In (1), if we take $s = s_1$ then $\|f\|_{\Phi,s} = \|f\|^o_\Phi$; if $s = s_\infty$, then $\|f\|_{\Phi,s} = \|f\|_\Phi$; if $s = s_p$ for $1 < p < \infty$ then $\|f\|_{\Phi,s} = \|f\|_{\Phi,p}$ (see Cui, Y., Duan, L., Hudzik, H. and Wisła, M. (2008)). Similarly, in (2), $c = 0$ gives the Luxemburg norm and $c = 1$ the Orlicz norm. Further, in (3), $m = 1$ yields the Luxemburg norm and $m = 2$ the Orlicz norm.

It is known that the s-norm $\|\cdot\|_{\Phi,s}$ is equivalent to the Luxemburg norm $\|\cdot\|_\Phi$ with $\|f\|_\Phi \leqslant \|f\|_{\Phi,s} \leqslant 2\|f\|_\Phi$ for any $f \in L^\Phi_s$ (see Wisła, M. (2020)). Note that the Orlicz space L^Φ_s is a Banach space with the s-norm.

Definition 2.3. Let s be an outer function. For all $0 \leqslant v \leqslant 1$, define

$$w(v) = \int_0^v {s'_+}^{-1}(t)\, dt. \tag{4}$$

It is clear that w is a non-negative, increasing and continuous function on $[0, 1]$.

Definition 2.4. Let s be an outer function. For all $0 \leqslant u < \infty$ and $0 \leqslant v \leqslant \infty$,

$$\beta_s(u, v) = 1 - w\left(s'_+(u)\right) - v s'_+(u).$$

Denote also $\beta_s(kf) = \beta_s\left(I_\Phi(kf), I_\Psi(p_+(k|f|)\right)$ for all $f \in L^\Phi_s$.

Note that the function $k \mapsto \beta_s(kf)$ is decreasing on $[0, \infty)$.

Definition 2.5. Let s be an outer function and Φ be an Orlicz function. For $f \in L^\Phi \setminus \{0\}$ and $0 < k < \infty$, we define the following functions.

$$\begin{aligned} D &: L^\Phi_s \to \mathcal{P}([0, \infty)), & D(f) &= \{0 < k < \infty : I_\Phi(kf) < \infty\} \\ k^* &: L^\Phi_s \to (0, \infty], & k^*(f) &= \inf\{k \in D(f) : \beta_s(kf) \leqslant 0\} \\ k^{**} &: L^\Phi_s \to [0, \infty), & k^{**}(f) &= \sup\{k \in D(f) : \beta_s(kf) \geqslant 0\} \end{aligned}$$

It is easy to see that $0 < k^*(f) \leqslant k^{**}(f) \leqslant \infty$. Let us also define

$$K(f) := \{0 < k < \infty : k^*(f) \leqslant k \leqslant k^{**}(f)\}.$$

Obviously, $K(f) \neq \emptyset \Leftrightarrow k^*(f) < \infty$. If $k^*(f) < \infty$ for any $f \in L^\Phi_s \setminus \{0\}$, then the s-norm is called k^*-finite; if $k^{**}(f) < \infty$ for any $f \in L^\Phi_s \setminus \{0\}$, then the s-norm is called k^{**}-finite. Further, if $k^*(f) = k^{**}(f) < \infty$ for any $f \in L^\Phi_s \setminus \{0\}$, then the s-norm is called k-unique.

Definition 2.6. Let s be an outer function. Define the constant σ_s by

$$\sigma_s = \sup\{u \geqslant 0 : s(u) = 1\}.$$

Note that $0 \leqslant \sigma_s \leqslant 1$ and it is obvious that s is strictly increasing on $[\sigma_s, \infty)$. We focus on the cases of $\sigma_s > 0$ and $\sigma_s = 0$ in the rest of this paper. The key point in defining this constant is that the equality $\sigma_s = 0$ provides an inverse function for the outer function s since this function is strictly increasing on the entire interval $[0, \infty)$ whenever $\sigma_s = 0$.

Let $\mathcal{S}$ denote the set of outer functions and define the sets

$$\mathcal{S}_0 = \{s \in \mathcal{S} : \sigma_s = 0\} \qquad \text{and} \qquad \mathcal{S}_+ = \{s \in \mathcal{S} : \sigma_s > 0\}.$$

The constants σ_s of the outer functions in Example 2.1 are obtained as follows.
(i) For s_p of (1),

$$\sigma_{s_p} = \begin{cases} 0, & 1 \leqslant p < \infty, \\ 1, & p = \infty. \end{cases}$$

(ii) For s_c of (2),

$$\sigma_{s_c} = \sup\{u \geqslant 0 : u + c \leqslant 1\} = 1 - c.$$

Note that $0 \leqslant c \leqslant 1$.
(iii) For s_m of (3),

$$\sigma_{s_m} = \sup\{u \geqslant 0 : (m-1)u + 1 = 1\} = \begin{cases} 1, & m = 1, \\ 0, & 1 < m \leqslant 2. \end{cases}$$

As a consequence, we can classify the given outer functions as follows. The outer functions $s_p, s_c, s_m \in \mathcal{S}_0$ for $1 \leqslant p < \infty$, $c = 1$, $1 < m \leqslant 2$ and $s_p, s_c, s_m \in \mathcal{S}_+$ for $p = \infty$, $0 \leqslant c < 1$, $m = 1$.

3. AUXILIARY RESULTS

We recall some technical results that will be used in the rest of paper.

Lemma 3.1. (Chen, S. (1996), Proposition 5.17) *For any $\varepsilon > 0$, there exists $\delta \in (0, \frac{1}{2})$ such that if $u, v \in \mathbb{C}$ and*

$$|v| \geqslant \frac{\delta}{8} \max_j |u + jv|,$$

then

$$|u| \leqslant \frac{1 - 2\delta}{4} \sum_j |u + jv|,$$

where

$$\max_j |u + jv| = \max\{|u + v|, |u - v|, |u + iv|, |u - iv|\},$$

$$\sum_j |u + jv| = |u + v| + |u - v| + |u + iv| + |u - iv|.$$

Lemma 3.2. (Wisła, M. (2020), Lemma 3.2) *For every outer function s and Orlicz function Φ,*

$$\|f\|_{\Phi,\infty} \leq \|f\|_{\Phi,s} \leq \|f\|_{\Phi,1} \leq 2\|f\|_{\Phi,\infty}$$

for all $f \in L_s^{\Phi}$.

Lemma 3.3. (Cui, Y., Zhan, Y. (2019), Lemma 7) *If $\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty$ then $K(f) \neq \emptyset$ for any $f \in L_s^{\Phi} \setminus \{0\}$.*

Theorem 3.4. (Wisła, M. (2020), Theorem 7.3) *Let s be an outer function and Φ be an Orlicz function.*
(i) The s-norm is k^*-finite if and only if one of the following conditions is satisfied.
(a) Φ takes infinite values, i.e., $b_\Phi < \infty$,
(b) $w\left(s_+'(u)\right) = 1$ for some $0 < u < \infty$,
(c) $w(1) = 1$ and Φ is not linear on $[0, \infty)$,
(d) Φ does not admit an oblique asymptote.
(ii) The s-norm is k^{**}-finite if and only if one of the conditions (a), (c) or (d) is satisfied.
(iii) If Φ does not admit an oblique asymptote, then the s-norm is k^{**}-finite if and only if it is k^*-finite.

Theorem 3.5. (**Wisła, M.** (**2020**), **Theorem 6.1**) *Let s be an outer function and Φ be an Orlicz function. For all $f \in L_s^\Phi \setminus \{0\}$, we have*

$$k^*(f) = \inf\left\{k > 0 : \|f\|_{\Phi,s} = \frac{1}{k}s(I_\Phi(kf))\right\},$$
$$k^{**}(f) = \sup\left\{k > 0 : \|f\|_{\Phi,s} = \frac{1}{k}s(I_\Phi(kf))\right\}.$$

Corollary 3.1 (Wisła, M. (2020), **Corollary 6.2**). Let s and Φ be an outer and an Orlicz function, respectively. The followings hold for any $f \in L_s^\Phi \setminus \{0\}$.
(i) For every $k \in (0,\infty) \cap [k^*(f), k^{**}(f)]$, we have $\|f\|_{\Phi,s} = \frac{1}{k}s(I_\Phi(kf))$.
(ii) If $k^{**}(f) = \infty$, then $\|f\|_{\Phi,s} = \lim\limits_{k\to\infty} \frac{1}{k}s(I_\Phi(kf))$.

4. MAIN RESULTS

In this section, we will give some results for s-norms that generalize the results obtained for the Orlicz and the p-Amemiya norms ($1 < p < \infty$). Then, we will give our main results on complex extreme points of unit ball and complex rotundity of Orlicz space (Theorems 4.3 and Theorem 4.4).

Definition 4.1. (see Chen, S. (1996)) Let $B(L_s^\Phi)$ (resp. $S(L_s^\Phi)$) be the closed unit ball (resp. the unit sphere) of a Orlicz space L_s^Φ. A function $f \in S(L_s^\Phi)$ is called an complex extreme point of $B(L_s^\Phi)$ if for any non-zero $g \in L_s^\Phi$ implies $\max_{|\lambda|=1} \|f+\lambda g\|_{\Phi,s} > 1$. The set of all complex extreme points of $B(L_s^\Phi)$ is denoted by $\operatorname{Ext} B(L_s^\Phi)$. Orlicz space is called complex strictly rotund if every element of $S(L_s^\Phi)$ is a complex extreme point of $B(L_s^\Phi)$.

Lemma 4.2. *If $f \in B(L_s^\Phi)$, then $|f(t)| \leqslant b_\Phi$ μ-a.e. on X.*

Proof. Assume that $f \in B(L_s^\Phi)$. By Lemma 3.2, we have $\|f\|_{\Phi,\infty} \le 1$. Therefore, we obtain $I_\Phi(f) \leqslant 1$ (see Chen, S. (1996)). Hence, $\Phi(|f(t)|) < \infty$ for μ-a.e. $t \in X$. By definition of b_Φ, we have $|f(t)| \leqslant b_\Phi$ μ- a.e. on X.

Theorem 4.3. *Let $s \in \mathcal{S}_0$. Then $f \in S(L_s^\Phi)$ is a complex extreme point of the unit ball $B(L_s^\Phi)$ if and only if $\mu(\{t \in X : k|f(t)| < a_\Phi\}) = 0$ for any $k \in K(f)$.*

Proof. Necessity. Suppose that $f \in S(L_s^\Phi)$ with $\sigma_s = 0$ is a complex extreme point of the unit ball $B(L_s^\Phi)$. Let us prove for any $k \in K(f)$, $\mu(\{t \in X : k|f(t)| < a_\Phi\}) = 0$. Assume that there exists $k_0 \in K(f)$ such that $\mu(\{t \in X : k_0|f(t)| < a_\Phi\}) > 0$. Then we can find $d > 0$ and measurable subset A of X such that $\mu(A) > 0$ and

$$k_0|f(t)| + d \le a_\Phi$$

for any $t \in A$. Letting $g = \frac{d}{k_0}\chi_A$, we obtain $g \neq 0$ and for any $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$,

$$\begin{aligned}
\|f+\lambda g\|_{\Phi,s} \le \frac{1}{k_0}s(I_\Phi(k_0(f+\lambda g))) &= \frac{1}{k_0}s(I_\Phi(k_0 f\chi_{X\setminus A}) + I_\Phi(k_0 f\chi_A + \lambda d\chi_A)) \\
&\le \frac{1}{k_0}s(I_\Phi(k_0 f\chi_{X\setminus A}) + I_\Phi((k_0 f + d)\chi_A)) \\
&\le \frac{1}{k_0}s(I_\Phi(k_0 f\chi_{X\setminus A})) \\
&\le \frac{1}{k_0}s(I_\Phi(k_0 f)) = \|f\|_{\Phi,s} = 1.
\end{aligned}$$

This gives that $f \notin \operatorname{Ext} B(L_s^\Phi)$.

Sufficiency. Suppose that for any $k \in K(f)$, $\mu(\{t \in X : k|f(t)| < a_\Phi\}) = 0$. Let us prove $f \in S(L_s^\Phi)$ with $\sigma_s = 0$ is a complex extreme point of the unit ball $B(L_s^\Phi)$. Assume that $f \in S(L_s^\Phi)$ with $\sigma_s = 0$ is not a complex extreme point of the unit ball $B(L_s^\Phi)$. Therefore, there exist $\varepsilon_0 > 0$ and $g_0 \in L_s^\Phi$ with $\|g_0\|_{\Phi,s} > \varepsilon_0$ such that

$$\max_{|\lambda|\le 1} \|f+\lambda g_0\|_{\Phi,s} \le 1. \tag{5}$$

By Lemma 3.1, there exists $\delta_0 \in (0, \frac{1}{2})$ such that if $u, v \in \mathbb{C}$ and

$$|v| \ge \frac{\varepsilon_0}{8}\max_j |u + jv|,$$

then we have

$$|u| \leq \frac{1-2\delta_0}{4} \sum_j |u + jv|.$$

Define $A = \{t \in X \ : \ |g_0(t)| \geq \frac{\varepsilon_0}{8} \max_j |f(t) + jg_0(t)|\}$. We obtain by using (5)

$$\|g_0\chi_{X\setminus A}\|_{\Phi,s} < \frac{\varepsilon_0}{8} \| \max_j |f + jg_0| \|_{\Phi,s} \leq \frac{\varepsilon_0}{8} \sum_j \|f + jg_0\|_{\Phi,s} \leqslant \frac{\varepsilon_0}{2}.$$

Consequently, we have $\|g_0\chi_A\|_{\Phi,s} > \frac{\varepsilon_0}{2}$ which shows that $\mu(A) > 0$. For any $t \in A$, we obtain

$$|f(t)| \leq \frac{1-2\delta_0}{4} \sum_j |f(t) + jg_0(t)|.$$

By Lemma 3.3, we can take any $k \in K(\frac{1}{4}\sum_j |f + jg_0|)$ and we have by (5)

$$\begin{aligned} 1 \geq \left\|\tfrac{1}{4}\textstyle\sum_j |f + jg_0|\right\|_{\Phi,s} &= \tfrac{1}{k}s\left(I_\Phi\left(\tfrac{k}{4}\textstyle\sum_j |f + jg_0|\right)\right) \geq \tfrac{1}{k}s\left(I_\Phi\left(k\tfrac{1-2\delta_0}{4}\textstyle\sum_j |f + jg_0|\right)\right) \\ &\geq \tfrac{1}{k}s(I_\Phi(kf)) \geq \|f\|_{\Phi,s} = 1 \end{aligned}$$

which implies that $\|f\|_{\Phi,s} = \frac{1}{k}s(I_\Phi(kf)) = \|\frac{1}{4}\sum_j |f + jg_0|\|_{\Phi,s} = 1$ and $k \geq 1$. Since $k|f(t)| \geq a_\Phi$ for μ-a.e. $t \in X$, we obtain that

$$\frac{1}{1-2\delta_0}k|f(t)| \geq \frac{a_\Phi}{1-2\delta_0}, \quad \mu - \text{a.e.} t \in A,$$

we conclude that $I_\Phi(k\frac{|f|}{1-2\delta_0}\chi_A) \geq \Phi(\frac{a_\Phi}{1-2\delta_0})\mu(A) > 0$. Let us define $b = \Phi(\frac{a_\Phi}{1-2\delta_0})\mu(A)$. To complete the proof, we consider the following two cases.

Case 1. Let assume that $I_\Phi(k(\frac{1}{4}\sum_j |f + jg_0|)) \geq 2\delta_0 b$. In this case, we obtain the following contradiction

$$\begin{aligned} 1 = \|f\|_{\Phi,s} &= \frac{1}{k}s(I_\Phi(kf\chi_A) + I_\Phi(kf\chi_{X\setminus A})) \\ &\leq \frac{1}{k}s\left(I_\Phi\left(k\left(\frac{1-2\delta_0}{4}\sum_j |f + jg_0|\right)\chi_A\right) + I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\chi_{X\setminus A}\right)\right) \\ &\leq \frac{1}{k}s\left((1-2\delta_0)I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\chi_A\right) + I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\chi_{X\setminus A}\right)\right) \\ &\leq \frac{1}{k}s\left(I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\right) - 2\delta_0 I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\chi_A\right)\right) \\ &\leq \frac{1}{k}s\left(I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\right) - 2\delta_0 I_\Phi\left(k\frac{|f|}{1-2\delta_0}\sum_j |f + jg_0|\chi_A\right)\right) \\ &\leq \frac{1}{k}s\left(I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\right) - 2\delta_0 b\right) \\ &< \frac{1}{k}s(I_\Phi(k(\frac{1}{4}\sum_j |f + jg_0|))) = \|\frac{1}{4}\sum_j |f + jg_0|\|_{\Phi,s} = 1. \end{aligned}$$

Therefore, we obtain a contradiction.

Case 2. Let assume that $I_\Phi(k(\frac{1}{4}\sum_j |f + jg_0|)) < 2\delta_0 b$. By using the fact that for all outer functions $s(u) \leq 1 + u$ for any $u \in \mathbb{R}$.

$$
\begin{aligned}
1 = \|f\|_{\Phi,s} &= \frac{1}{k}s(I_\Phi(kf\chi_A) + I_\Phi(kf\chi_{X\setminus A})) \le \frac{1}{k}(1 + I_\Phi(kf\chi_A) + I_\Phi(kf\chi_{X\setminus A})) \\
&\le \frac{1}{k}\left(1 + I_\Phi\left(k\left(\frac{1-2\delta_0}{4}\sum_j |f + jg_0|\right)\chi_A\right) + I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\chi_{X\setminus A}\right)\right) \\
&\le \frac{1}{k}\left(1 + (1-2\delta_0)I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\chi_A\right) + I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\chi_{X\setminus A}\right)\right) \\
&\le \frac{1}{k}\left(1 + I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\right) - 2\delta_0 I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\chi_A\right)\right) \\
&\le \frac{1}{k}\left(1 + I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\right) - 2\delta_0 I_\Phi\left(k\frac{|f|}{1-2\delta_0}\sum_j |f + jg_0|\chi_A\right)\right) \\
&\le \frac{1}{k}\left(1 + I_\Phi\left(k\left(\frac{1}{4}\sum_j |f + jg_0|\right)\right) - 2\delta_0 b\right) < \frac{1}{k} \le 1
\end{aligned}
$$

Therefore, we obtain a contradiction.

The following theorem gives us necessary and sufficient condition for being complex rotundity of Orlicz spaces when $s \in \mathcal{S}_0$.

Theorem 4.4. *Let $s \in \mathcal{S}_0$. Then L_s^Φ is complex rotund if and only if $a_\Phi = 0$.*

Proof. Necessity. Suppose that L_s^Φ with $\sigma_s = 0$ is complex strictly rotund. Let us prove $a_\Phi = 0$. Assume that $a_\Phi > 0$. Then take $c \in (0, a_\Phi)$. Choose measurable subset A of X and $f \in S(L_s^\Phi)$ such that $\mu(A) > 0$ and $\operatorname{supp} f = X \setminus A$. Take $k \in K(f)$, and define

$$
g(t) = \begin{cases} \frac{c}{k}, & t \in A, \\ f(t), & t \in X \setminus A. \end{cases}
$$

Since $\operatorname{supp} f = X \setminus A$, we obtain $\|g\|_{\Phi,s} \ge \|f\|_{\Phi,s} = 1$. On the other hand,

$$
\begin{aligned}
\|g\|_{\Phi,s} \le \frac{1}{k}s(I_\Phi(kg)) &= \frac{1}{k}s(I_\Phi(c\chi_A) + I_\Phi(kf\chi_{X\setminus A})) \\
&= \frac{1}{k}s(I_\Phi(kf\chi_{X\setminus A})) = \|f\|_{\Phi,s} = 1.
\end{aligned}
$$

Thus, $\|g\|_{\Phi,s} = 1$. However, for $t \in A$, we have $k|g(t)| = c < a_\Phi$, which implies that $g \notin \operatorname{Ext} B(L_s^\Phi)$ by Theorem 4.3.

Sufficiency. Suppose that $a_\Phi = 0$. Let us prove L_s^Φ with $\sigma_s = 0$ is complex strictly rotund. Assume that $f \in S(L_s^\Phi)$ is not a complex extreme point of the unit ball $B(L_s^\Phi)$. It follows from Theorem 4.3 that $\mu(\{t \in X : k|f(t)| < a_\Phi\}) > 0$ for some $k \in K(f)$. Then there exists $t_0 \in X$ such that $a_\Phi > k|f(t_0)| \ge 0$, which contradicts with $a_\Phi = 0$.

5. CONCLUSION

In this work, we characterize complex extreme points and complex rotundity of Orlicz Spaces equipped with the s-norms for $\sigma_s = 0$.

Peer Review: Externally peer-reviewed.
Author Contribution: All authors have contributed equally.
Conflict of Interest: Authors declared no conflict of interest.
Financial Disclosure: Authors declared no financial support.

ACKNOWLEDGEMENTS

We are grateful to anonymous referees for careful reading of the manuscript and for helpful comments.

LIST OF AUTHOR ORCIDS

E. Başar	https://orcid.org/0000-0002-7300-6177
B.H. Uysal	https://orcid.org/0000-0002-2522-3417
Ş. Yaşar	https://orcid.org/0000-0001-9708-1107

REFERENCES

Arıs, B., Öztop, S., 2023, Wiener amalgam spaces with respect to Orlicz spaces on the affine group. J. Pseudo-Differ. Oper. Appl., 14(23).

Başar E., Öztop, S., Uysal, B.H., Yaşar, Ş. 2023, Extreme points in Orlicz spaces equipped with s-norms and its closedness, Math. Nachr. 00, 1–21.

Chen, L., Cui, Y., 2010, Complex extreme points and complex rotundity in Orlicz function spaces equipped with the p-Amemiya norm, Nonlinear Anal., 73(5), 1389-1393.

Chen, S., 1996, Geometry of Orlicz space. Dissertationes Math., Harbin University, China.

Cui, Y., Duan, L., Hudzik, H. and Wisła, M., 2008, Basic theory of p-Amemiya norm in Orlicz spaces ($1 \leq p \leq \infty$): Extreme points and rotundity in Orlicz spaces endowed with these norms, Nonlinear Anal., 69(5), 1796-1816.

Cui, Y., Zhan, Y., 2019, Strongly extreme points and middle point locally uniformly convex in Orlicz spaces equipped with s-norm, Journal of Funct. Spac., 2019, 1-7.

Orlicz, W., 1932, Über eine gewisse klasse von Räumen vom typus B, Bull. Int. Acad. Polon. Sér. A, 207-220.

Rao, M. M. and Ren, Z. D., 1991, Theory of Orlicz Spaces, Marcel Dekker, New York.

Üster, R., 2021, Multipliers for the weighted Orlicz spaces of a locally compact abelian group, Results in Mathematics, 76 (4), Paper No. 183.

Thorp, E., Whitley, R., 1967, The strong maximum modulus theorem for analytic functions into a Banach space, Proc. Amer. Math. Soc., 18, 640-646.

Wisła, M., 2020, Orlicz spaces equipped with s-norms, J. Math. Anal. Appl., 483(2), 123659-123689.

Wu, C.X., Sun, H., 1987, On complex extreme points and complex strict convexities of Musielak–Orlicz spaces, J. Sys. Sci. Math. Scis., 7, 7-13.

Wu, C.X., Sun, H., 1988 On complex uniform convexity of Musielak–Orlicz spaces, J. Northeast. Math., 4, 389-396.