Fractalization of Fractional Integral and Composition of Fractal Splines

A. Gowrisankar a,1

a Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India.

ABSTRACT The present study perturbs the fractional integral of a continuous function \( f \) defined on a real compact interval, say \( (I^\alpha f) \) using a family of fractal functions \( (I^\alpha f)^\alpha \) based on the scaling parameter \( \alpha \). To elicit this phenomenon, a fractal operator is proposed in the space of continuous functions, an analogue to the existing fractal interpolation operator which perturbs \( f \) giving rise to \( \alpha \)-fractal function \( f^\alpha \). In addition, the composition of \( \alpha \)-fractal function with the linear fractal function is discussed and the composition operation on the fractal interpolation functions is extended to the case of differentiable fractal functions.

INTRODUCTION

The launch of fractal interpolation function has initiated a new theory of approximation concerning the naturally existing functions with non-differentiable nature. Rooted from the remark of Barnsley in (Barnsley 1986), Navascués has explored the approximation of continuous functions defined on a real closed interval by a class of \( \alpha \)-fractal functions, where \( \alpha \) is the appropriately chosen scaling parameter, in (Navascués 2005). Non-smooth analogue of prescribed continuous function can be achieved with the choice of non-differentiable base function. Further, Navascués has pioneered the fractal operator to associate each prescribed function to its class of \( \alpha \)-fractal functions. The theme of proposing a fractal operator has fruitfully enabled the fractal theory to connect with various mathematical fields not limited to operator theory. While constructing \( \alpha \)-fractal function, the base function choice is significant since the fractal operator is dependent on the boundedness of the base function. Literature survey acknowledges various interesting discussions on \( \alpha \)-fractal functions, for instance, the derivative of \( \alpha \)-fractal function is explored and its respective fractal operator is studied in (Navascués and Sebastián 2006).

The Riemann-Liouville (RL) fractional integral of affine fractal functions has been investigated in (Pan 2014). The quadratic fractal function’s fractional integral with constant and function scalings has been discussed in (Gowrisankar and Prasad 2019). The fractional integral as well as the fractional derivative of different kinds of fractal interpolation functions have been discussed by several authors (for additional information refer, (Pan 2014; Gowrisankar and Prasad 2019; Ruan et al. 2009; Priyanka and Gowrisankar 2021a)). The aforementioned results on \( \alpha \)-fractal function and its fractional order integral, naturally arises a question: Is it possible to generate a class of fractal functions such that the fractional integral of a continuous function is interpolated? To answer this question, the present paper initiates the construction of self-referential functions for the fractional integral of continuous functions.

In recent times, fractional calculus has been receiving remarkable attention among the fractal community. The Riemann-Liouville (RL) fractional integral of affine fractal functions has been investigated in (Pan 2014). The quadratic fractal function’s fractional integral with constant and function scalings has been discussed in (Gowrisankar and Prasad 2019). The fractional integral as well as the fractional derivative of different kinds of fractal interpolation functions have been discussed by several authors (for additional information refer, (Pan 2014; Gowrisankar and Prasad 2019; Ruan et al. 2009; Priyanka and Gowrisankar 2021a)).

The construction procedure follows Navascués’s \( \alpha \)-fractal function in (Navascués and Sebastián 2006) and such a construction is guaranteed with the continuity of fractional integral. In addition, a fractal operator is defined to assign the fractional integral of a continuous function to its fractal version. The boundedness of

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Corresponding author

gowrisankarga@gmail.com
the fractional integral discussed in (Samko et al. 1993) instigates to discuss the boundedness of the proposed operator. The base function of the newly constructed fractal function, (i.e) the fractional integral of base function of the alpha-fractal function, is chosen appropriately to explicitly estimate the bound of the operator.

The recent works on fractal functions reported in (Navacués et al. 2022; Massopust 2022b; a; Dai and Liu 2023) show the curiosity of young researchers to develop more generalized and flexible fractal interpolation functions. In (Priyanka and Gowrisankar 2021b), authors have demonstrated that the resultant functions on the evaluation of the fractional integral of alpha-fractals are again alpha-fractals functions obeying the end point conditions. The work by Dai and Liu (Dai and Liu 2023) is also noticeable, in which the composite fractal function is introduced along with the discussion of its fractal dimension. In this direction, the present paper investigates the composition of alpha-fractal function as well as the composition of fractal spline. Further, it is observed that the composition operator also renders new fractal functions like the case of fractional integral operator, which is discussed in (Priyanka and Gowrisankar 2021b). With this end, the paper directly enters the discussion on the fractal perturbation of continuous functions in the following section.

FRACTALIZATION OF CONTINUOUS FUNCTIONS

Let \( N \geq 2 \) and \( \mathbb{N}_N \) denote the initial set of natural numbers of length \( N \). Consider the interpolation data set,

\[
\{(x_j, y_j) \in [x_1, x_{N+1}] \times \mathbb{R} : j \in \mathbb{N}_{N+1}\}.
\]

Let \( l_j \) be the set of \( N \) homeomorphisms from \( I = [x_1, x_{N+1}] \) to \( l_j = [x_j, x_{j+1}] \), \( j \in \mathbb{N}_N \) satisfying

\[
|f_j(s) - f_j(t)| \leq \lambda_j |s - t|, \quad \lambda_j \in [0, 1),
\]

\[
f_j(x_1) = x_j, \quad f_j(x_{N+1}) = x_{j+1}, \quad j \in \mathbb{N}_N.
\]

Define the maps \( f_j : \mathcal{X} := I \times \mathbb{R} \rightarrow \mathbb{R} \) to be continuous in the first argument and Lipschitz continuous in the second argument with Lipschitz constant \( \alpha_j < 1 \) such that

\[
f_j(x_1, y_1) = y_j, \quad f_j(x_{N+1}, y_{N+1}) = y_{j+1}, \quad j \in \mathbb{N}_N.
\]

The space of continuous functions defined on the interval \( I \) reserves the notation \( C(I) \). Let \( G = \{h \in C(I) : h(x_1) = y_1, h(x_{N+1}) = y_{N+1}\} \). For \( h_1, h_2 \in C(I) \), the metric \( d \) defined by \( d(h_1, h_2) = \sup_{x \in I} |h_1(x) - h_2(x)| \) completes \( (G, \delta) \). Further, in (Barnsley 1986), the Read-Bajrakaravicius operator (RB), \( T \), is defined on \( (G, \delta) \) by

\[
T_h(x) = f_j(l_j^{-1}(x), h(l_j^{-1}(x))), \quad j \in \mathbb{N}_N.
\]

The continuity properties of \( l_j \) and \( f_j \) make easier to verify the continuity of \( T \) as follows

\[
d(T_{g_1}, T_{g_2}) \leq |\alpha|_{\text{inf}} \delta_{1, g_2}, \quad g_1, g_2 \in C(I)
\]

where \(|\alpha|_{\text{inf}} = \max\{|\alpha_j| : j \in \mathbb{N}_N| < 1 \) and \( \alpha = \{a_1, a_2, \ldots, a_N\}\). The choice of \( a_j \) makes the operator \( T \) contractive on the space \( (G, \delta) \). Hence, with the aid of Banach contraction principle, it is concluded that \( T \) has a unique fixed point, say \( g \), satisfying \( g(x_j) = y_j \), for all \( j \in \mathbb{N}_{N+1} \) and from Eqn.(1), it follows that

\[
g(x) = f_j(l_j^{-1}(x), g(l_j^{-1}(x))), \quad j \in \mathbb{N}_N.
\]

Thus, the finite collection of contractive maps \( w_j \) together with the complete metric space \((\mathcal{X}, d)\) forms a hyperbolic Iterated Function System (IFS) and it is denoted by

\[
\{\mathcal{X}; w_j(x, y) = (l_j(x), f_j(x, y)) : j \in \mathbb{N}_N\}.
\]

Let \( H(\mathcal{X}) := \{A \subset \mathcal{X} : A \neq \emptyset \) and \( \text{compact}\). The Hausdorff metric \( h_d \) is defined on \( H(\mathcal{X}) \) by

\[
h_d(A, B) = \max\{d(A, B), d(B, A)\},
\]

where \( d(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y) \), then the pair \((H(\mathcal{X}), h_d)\) is a complete metric space whenever the metric space \((\mathcal{X}, d)\) is complete. A Hutchinson-Barnsley operator \( W \) is defined as a self-map on \( H(\mathcal{X}) \) by

\[
W(C) = \bigcup_{j=1}^{N} w_j(C),
\]

where \( C \in H(\mathcal{X}) \). By the Banach principle of fixed point, there exists a unique \( G_\mathcal{X} \) in \( H(\mathcal{X}) \) such that

\[
G_\mathcal{X} = \lim_{n \to \infty} W^n(C),
\]

where \( W^n \) is the \( n \)-fold self-composition of \( W \). Moreover, this set \( G_\mathcal{X} \) is the graph of the function \( g \) obeying the self-referential equation (2). In this construction, the function \( g \) is called the Fractal Interpolation Function (FIF) associated with the IFS (3). The interested readers may consult (Barnsley 1986; Agathiyan et al. 2022; Gowrisankar and Uthayakumar 2016) for more details on FIFS.

The following is the review of construction of alpha-fractal function explored by Navacués in (Navacués 2005). Slightly deviating from the theme of fractal interpolation function approximating the given interpolation data sharing complex behaviour, Navacués has generated a class of continuous functions with fractal properties to approximate \( f \in C(I) \). For \( f \in C(I) \), let \( \{(x_j, f(x_j)) : j \in \mathbb{N}_{N+1}\} \) be the interpolation points. A partition \( \Delta := \{x_1, x_2, \ldots, x_{N+1}\} \) is considered such that \( x_1 < x_2 < \cdots < x_{N+1} \) and the continuous function \( b : I \to \mathbb{R} \) is taken as the base function equal to \( f \) only at the endpoints \( x_1 \) and \( x_{N+1} \), i.e.,

\[
b(x_1) = f(x_1), \quad b(x_{N+1}) = f(x_{N+1}), \quad b \neq f.
\]

Let \( a_j \in (-1, 1), \quad j \in \mathbb{N}_N \). Consider the maps

\[
l_j(x) = a_j x + b_j, \quad f_j(x, f(x)) = a_j f(x) + q_j(x), \quad j \in \mathbb{N}_N.
\]

Then, the attractor of the IFS (3) involving the maps in (5) and (6) is the graph of the fractal interpolation function say, \( f_{a,b}^\Delta \), corresponding to \( f \) with respect to scale vector \( a \), partition \( \Delta \) and base function \( b \). In addition, the function \( f_{a,b}^\Delta \) is the fixed point of the RB operator \( F_\Delta \) defined on \( C(I) \), where \( C(I) \) is the space of continuous functions \( h \) obeying \( h(x_1) = f(x_1), \quad h(x_{N+1}) = f(x_{N+1}) \). The operator \( F_\Delta \) is described as

\[
F_\Delta h(x) = f(x) + a_i(h - b) \circ l_i^{-1}(x), \quad x \in I, \quad j \in \mathbb{N}_N.
\]

Then, \( f_{a,b}^\Delta \) obeys

\[
f_{a,b}^\Delta(x) = f(x) + a_j(f^a - b) \circ l_j^{-1}(x), \quad x \in I, \quad j \in \mathbb{N}_N.
\]

**Definition 1.** The function \( f_{a,b}^\Delta := f^a \) satisfying the self-referential equation (7) is the fractal perturbation of \( f \) and it is known as the alpha-fractal function corresponding to \( a, \Delta \) and \( b \).
According to Eqn.\((7)\), \(f^\alpha\) interpolates \(f\) at each \(x_i\) (i.e.) \(f^\alpha(x_i) = f(x_i)\), for all \(i \in \mathbb{N}_{N+1}\). Also, \(f^\alpha\) equals the prescribed function \(f\) when all the scaling factors are taken to be zero. In addition, from Eqn.\((7)\), the uniform distance between \(f\) and \(f^\alpha\) can be deduced as follows.

\[
\|f^\alpha - f\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\|f - b\|_\infty.
\]

Let \(C[a,b]\) be equipped with sup norm

\[
\|f\|_\infty = \max\{|f(x)| : x \in [a,b]\}.
\]

Consider the linearly dependent base function \(b\) on \(f\), \(b = Lf\), where \(L : C[a,b] \rightarrow C[a,b]\) is a linear operator and bounded, its operator norm is given by

\[
\|L\| := \sup\{\|Lf\|_\infty : \|f\|_\infty \leq 1\}
\]

and \(Lf(x_1) = x_1\), \(Lf(x_{N+1}) = x_{N+1}\) with \(L \neq \text{Identity.}\)

**Remark 1.** The present study proceeds with \(Lf = f \circ c\), where \(c\) is an increasing as well as continuous function such that \(c(x_1) = x_1\), \(c(x_{N+1}) = x_{N+1}\) and \(c \neq \text{Identity.}\) For this particular choice of \(b = f \circ c\), \(\|b\|_\infty = \|Lf\|_\infty = \|f\|_\infty\) with operator norm \(\|L\| = 1\).

**Lemma 1.** (Navascués 2010) For any \(f \in C(1)\) and \(b = Lf\), the following inequality holds

\[
\|f^\alpha - f\|_\infty \leq \frac{|\alpha|_\infty |Id - L|_\infty}{1 - |\alpha|_\infty}\|f\|_\infty,
\]

where \(Id\) is the identity operator.

**Note 1.** If \(Lf = f \circ c\), the inequality \((\cdot)\) becomes

\[
\|f^\alpha - f\|_\infty \leq \frac{2|\alpha|_\infty}{1 - |\alpha|_\infty}\|f\|_\infty.
\]

In (Navascués 2005), a fractal interpolation operator \(F^\alpha : C(1) \rightarrow C(1)\) is introduced to fractalize each continuous function as

\[F^\alpha(f) = f^\alpha, f \in C(1)\]

**Theorem 1.** (Navascués 2010) For any bounded and linear operator \(L\) with sup norm, the following holds

\[
\|F^\alpha(f)\|_\infty \leq \left(1 + \frac{|\alpha|_\infty |Id - L|_\infty}{1 - |\alpha|_\infty}\right)\|f\|_\infty.
\]

In analogue to the above discussed operator, various fractal operators have been proposed to the fractalize the given continuous functions, see for instance (Navascués and Sebastián 2006; Priyanaka and Gowrisankar 2012b).

**Fractal Perturbation of Fractional Integral of a Continuous Function**

In order to define a new class of \(a\)-fractal functions to approximate the fractional integral of \(f \in C(1)\), this section commences with the definition of RL fractional integral of a continuous function.

**Definition 2.** (Samko et al. 1993) Let \(f\) be the integrable function on \([a, b] \subset \mathbb{R}\) and \(\nu > 0\) be a real number. Then, the Riemann-Liouville (RL) fractional integral of \(f\) is defined by

\[
(I^\nu(f))(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t-s)^{\nu-1}f(s)ds, \ (t > a),
\]

here the notation \(\Gamma(\cdot)\) denotes the Gamma function.

In (Samko et al. 1993), it is proved that the fractional integral operator \((I^\nu)^f\) is bounded in \(L_p\) space with \(1 \leq p \leq \infty\) and it is precisely provided in the following lemma.

**Lemma 2.** For \(\nu > 0\), the RL fractional integral operator is bounded such that

\[
\|I^\nu f\| \leq K\|f\|_\infty, \text{ where } K = \frac{x_{N+1} - x_1}{\nu \Gamma(\nu)}.
\]

Using the above lemma, the uniform distance between the germ function \(f\) and its fractional integral \((I^\nu)^f\) can be estimated as follows.

**Lemma 3.** The distance between \(f\) and \((I^\nu)^f\) with respect to the uniform norm is given by

\[
\|f - (I^\nu)^f\|_\infty \leq (1 + K)\|f\|_\infty,
\]

where \(K = \frac{x_{N+1} - x_1}{\nu \Gamma(\nu)}\). Then,

\[
\|f - (I^\nu)^f\|_\infty \leq (1 + K)\|f\|_\infty.
\]

The following lemma ensures the continuity of the fractal order integral \((I^\nu)^f\) which is proved by Pan in reference (Pan 2014).

**Lemma 4.** Let \(\nu > 0\) and \(f \in C[a,b]\). Then \((I^\nu)^f \in C[a,b]\).

From Lemma 4, it is straight forward to define a family of fractal functions to approximate \((I^\nu)^f\).

Let \(\{x_j, (I^\nu)^f(x_j)\}\) be the interpolation data with partition \(\Delta\) and scale vector \(a\). To define a new family of self-referential functions, consider the base function as the fractional integral of \(b\), expressed by

\[
(I^\nu b)(t) = \frac{1}{\Gamma(\nu)} \int_a^x (t-s)^{\nu-1}b(s)ds,
\]

such that

\[
(I^\nu b)(x_1) = (I^\nu f)(x_1),
(I^\nu b)(x_{N+1}) = (I^\nu f)(x_{N+1})
\]

and \((I^\nu)^f \neq (I^\nu)^f\). In correspondence with the new continuous functions \((I^\nu)^f\) and \((I^\nu)^b\), the maps defined in \((5)\) becomes,

\[
l_j(x) = a_jx + b_j, f_j(x, y) = a_j y + (I^\nu)^f l_j(x) - a_j(I^\nu f)(b)(x), j \in \mathbb{N}_N.\]

The attractor of the IFS with the maps in \((8)\) is the graph of the new kind of \(a\)-fractal function say, \((I^\nu)^\alpha\) associated with \((I^\nu)^f\). It can be verified that \((I^\nu)^\alpha(x_j) = (I^\nu f)(x_j)\) for all \(j \in \mathbb{N}_{N+1}\). Besides, \((I^\nu)^\alpha\) is a unique fixed point of the RB operator \(\mathcal{S}_a\) with the change of arguments such that

\[
(I^\nu)^\alpha(x) = (I^\nu f)(x) + a_j ((I^\nu)^f - (I^\nu)^\alpha) \circ l_j^{-1}(x), x \in I, j \in \mathbb{N}_N.
\]

The function \((I^\nu)^\alpha\) is the \(a\)-fractal function of the RL fractional integral of \(f \in C(1)\) approximating \((I^\nu)^f\) with respect to base.
function \((I^\alpha b)\), partition \(\Delta\) and scaling parameter \(a\). With an aim to estimate the error, now consider the mapping

\[
T : \mathbb{R} \times C(I) \to C(I)
\]

\[
(a, I^\alpha f) \to \mathcal{T}_a(I^\alpha f)
\]

where \(R = [0, t] \times [0, t] \times [0, t] \times \cdots \times [0, t] \subseteq \mathbb{R}^N, 0 \leq t < 1, t\) is fixed. For \(x \in I_f\), define

\[
\mathcal{T}_a(I^\alpha f)(x) = I^\alpha_j (l^{-1}_j(x), (I^\alpha f) \circ l^{-1}_j(x))
\]

\[
= a_j(I^\alpha f) \circ l^{-1}_j(x) + q_j^\alpha \circ l^{-1}_j(x)
\]

with

\[
q_j^\alpha (x) = (I^\alpha f) \circ l^{-1}_j(x) - a_j(I^\alpha b)(x).
\]

The uniform distance between the functions \((I^\alpha f)\) and \((I^\alpha f)^a\) is estimated in the following theorem.

**Theorem 2.** If \(b\) is a bounded linear operator, then the below inequality holds

\[
\| (I^\alpha f)^a - (I^\alpha f) \|_\infty \leq \frac{2\mathcal{K}|a|_\infty}{1 - |a|_\infty} \| f \|_\infty,
\]

where \(\mathcal{K} = \frac{x(M - 1)}{\delta(t)}\).

**Proof.** Let \((I^\alpha f) \in C_f(I)\). Then for each \(x \in I_f\),

\[
\| \mathcal{T}_a(I^\alpha f)(x) - \mathcal{T}_b(I^\alpha f)(x) \| = |a_j(I^\alpha f) \circ l^{-1}_j(x) + q_j^\alpha \circ l^{-1}_j(x) - (I^\alpha f) \circ l^{-1}_j(x) - q_j \circ l^{-1}_j(x)|
\]

\[
\leq |a_j(I^\alpha f) \circ l^{-1}_j(x) - (I^\alpha f) \circ l^{-1}_j(x)| + |q_j^\alpha \circ l^{-1}_j(x) - q_j \circ l^{-1}_j(x)|
\]

From Eqn.(.), the second term is rewritten as

\[
\|\mathcal{T}_a(I^\alpha f) - \mathcal{T}_b(I^\alpha f)\|_\infty \leq |a_j - q_j| |\| I^\alpha f \|_\infty + |I^\alpha f| \circ l^{-1}_j(x) - a_j(I^\alpha b)(x)\|
\]

\[
\leq |a_j - q_j| |\| I^\alpha f \|_\infty + |(I^\alpha f) \circ l^{-1}_j(x) - a_j(I^\alpha b)(x)\|.
\]

Theorem 3. \(F^{\alpha,p}\) is bounded on \(C(I)\). Moreover,

\[
\| F^{\alpha,p}(I^\alpha f) \|_\infty \leq \left( 1 + \frac{2K|a|_\infty}{1 - |a|_\infty} \right) K \| f \|_\infty,
\]

where \(K = \frac{x(M - 1)}{\delta(t)}\).

**Proof.** From Theorem 2, one has

\[
\| (I^\alpha f)^a - (I^\alpha f) \|_\infty \leq \frac{2K|a|_\infty}{1 - |a|_\infty} \| f \|_\infty.
\]

The above theorem is a prelude to discuss the boundedness of the fractal operator \(F^{\alpha,p}\) which is explored in the following section.

**FRACTAL OPERATOR ASSOCIATED WITH THE FRACTIONAL INTEGRAL**

This section proposes a fractal operator to send each continuous function \(I^\alpha f\) to its fractal version \((I^\alpha f)^a\) where the function \((I^\alpha f)^a\) is the \(a\)-fractional function of the RL fractional integral of a prescribed continuous function \(f\) as discussed in the previous section. To be concise, for a fixed scale vector \(v\) and a fixed fractional order \(v > 0\), there exists an operator

\[
F^{\alpha,p} : C(I) \to C(I)
\]

\[
I^\alpha f \mapsto (I^\alpha f)^a.
\]

The linearity of \(b\) assures the linearity of \(F^{\alpha,p}\). For fixed scalars \(\lambda\) and \(\mu\), it can be verified that

\[
F^{\alpha,p}(\lambda I^\alpha f + \mu I^\alpha g) = \lambda F^{\alpha,p}(I^\alpha f) + \mu F^{\alpha,p}(I^\alpha g).
\]

**Theorem 3.** \(F^{\alpha,p}\) is bounded on \(C(I)\). Moreover,

\[
\| F^{\alpha,p}(I^\alpha f) \|_\infty \leq \left( 1 + \frac{2K|a|_\infty}{1 - |a|_\infty} \right) K \| f \|_\infty,
\]

with \(K = \frac{x(M - 1)}{\delta(t)}\). Then,

\[
\| (I^\alpha f)^a - (I^\alpha f) \|_\infty \leq \frac{2K|a|_\infty}{1 - |a|_\infty} \| f \|_\infty,
\]

which provides the required bound of the operator \(F^{\alpha,p}\),

\[
\| F^{\alpha,p}(I^\alpha f) \|_\infty \leq \left( 1 + \frac{2K|a|_\infty}{1 - |a|_\infty} \right) K \| f \|_\infty.
\]

Hence, the required inequality.

Next, the bound for the perturbation error between \(f\) and \((I^\alpha f)^a\) is explored in the following theorem.

**Theorem 4.** For any \(f \in C(I)\), the following inequality

\[
\| f - (I^\alpha f)^a \|_\infty \leq \left( 1 + \frac{2K|a|_\infty}{1 - |a|_\infty} \right) \| f \|_\infty,
\]

holds with \(K = \frac{x(M - 1)}{\delta(t)}\).
Proof. One can have
\[
\|f - (I^\alpha f)^a\|_\infty \leq \|f - I^\alpha f + I^\alpha f - (I^\alpha f)^a\|_\infty
\]
\[
\leq \|f - I^\alpha f\|_\infty + \|I^\alpha f - (I^\alpha f)^a\|_\infty.
\]
Using Lemma 4 and Theorem 2, the above inequality is reduced to
\[
\|f - (I^\alpha f)^a\|_\infty \leq (1 + K_\alpha)\|f\|_\infty + 2K\|\alpha\|_\infty 1 - \|\alpha\|_\infty \|f\|_\infty.
\]
Thus, the required result follows immediately. \(\square\)

**Remark 2.** In [Priyanka and Gowrisankar 2021b], a fractal operator \(F^\alpha\) has been proposed to associate the given function \(f \in C(1)\) to the Riemann-Liouville fractional integral of its fractal version, namely \(I^\alpha (F^\alpha f)\), and discussed some of its elementary properties. Whereas, here the fractal operator \(F^\alpha\) is defined on \(C(1)\) to associate the fractional integral of \(f \in C(1)\) to its fractal version, namely \((I^\alpha f)^a\).

**COMPOSITE FRACTAL FUNCTIONS**

This section discusses the composition of fractal functions and demonstrates that the compositions are again fractal functions.

**Composition of \(\alpha\)-fractal Function**

Let \(f = [y_1, y_{N+1}] \subset \mathbb{R}\) and \(l_{1,j} : I \to \mathbb{R}\) be the homeomorphic maps defined by \(l_{1,j}(x) = a_{1,j}x + b_{1,j}\) satisfying
\[
d(l_{1,j}(a), l_{1,j}(b)) \leq r_d(a, b), 0 \leq r_1 < 1, a, b \in I,
\]
where \(d\) is a Euclidean metric or its equivalent metric and
\[
l_{1,j}(x_1) = x_1, l_{1,j}(x_{N+1}) = x_{j+1}, j \in \mathbb{N}_N.
\]
(11)

Let \(K_1 = l \times J\). Define the continuous functions \(F_{1,j} : K_1 \to \mathbb{R}\) to be contraction with respect to second variable satisfying
\[
F_{1,j}(x_1, y_1) = y_{j,} F_{1,j}(x_{N+1}, y_{N+1}) = y_{j+1}, j \in \mathbb{N}_N.
\]
(12)

The general form of the maps \(F_{1,j}\) is given by
\[
F_{1,j}(x, y) = a_{1,j}y + q_j(x),
\]
where \(a_j = (a_1, a_2, \ldots, a_{N+1})\) is the free parameter chosen in the interval \([0, 1]\), which scales the graph vertically and referred as vertical scaling factor, \(q_j\) is a suitable continuous function satisfying
\[
q_j(x_1) = y - a_1y, q_j(x_1) = y_{j+1} - a_1y_{N+1}.
\]
The system \(K_1; (l_{1,j}, F_{1,j}) : j \in \mathbb{N}_N\) is a IFS and its attractor \(G_I\) is the graph of fractal interpolation function \(h : I \to \mathbb{R}\) interpolating the data set \(\{(x_j, y_j) : j \in \mathbb{N}_{N+1}\}\) such that \(h(x_j) = y_j, j \in \mathbb{N}_{N+1}\) In [Dai and Liu 2023], the functional equation of \(h\) is provided by
\[
h(x) = F_{1,j}(l_{1,j}^{-1}(x), F(l_{1,j}^{-1}(x))),
\]
or
\[
h(l_{1,j}(x)) = a_1h(x) + q_j(x), x \in I, j \in \mathbb{N}_N.
\]
On the other hand, if the data set \(\{(x_j, f(x_j)) : j \in \mathbb{N}_{N+1}\}\) is given to approximate, where \(f\) is a continuous function, the following choice of \(q_j(x) = f \circ l_{1,j}(x) - a_1b(x)\) generates an \(\alpha\)-fractal function satisfying
\[
f^a(l_{1,j}(x)) = a_1f^a(x) + f \circ l_{1,j}(x) - a_1b(x)
\]
and \(f^a(x_j) = f(x_j), \forall j \in \mathbb{N}_{N+1}\), here \(b\) is the base function obeying the conditions provided in (4). Let \(N^f = \{f^a(x_1), f^a(x_{N+1})\}\) and \(N_j^f = \{f^a(x_j), f^a(x_{j+1})\}, j \in \mathbb{N}_N\). Now, to interpolate the data set \(\{(f^a(x_j), z_j) : j \in \mathbb{N}_{N+1}\}, z_j \in \mathbb{R}\) for all \(j \in \mathbb{N}_{N+1}\), a new fractal interpolation function \(h : \mathbb{N} \to \mathbb{R}\) is constructed with the maps \(m_{1,j}\) and \(G_{1,j}\) defined below which respectively obey the conditions of \(l_{1,j}\) and \(F_{1,j}\)
\[
m_{1,j}(x) = c_{1,j}(x) + d_{1,j},
\]
\[
G_{1,j}(f^a(x), z) = a_jz + p_j(f(x)), j \in \mathbb{N}_N,
\]
where \(p_j\) is a linear polynomial of \(x\) satisfying \(p_j(f^a(x_0)) = z_j, p_j(f^a(x_{N+1})) = z_{j+1}\). Note that the domain of \(h\) agrees with \(f^a(I)\), thus it is possible to composite \(g\) with \(f^a\). Similar to the composite fractal interpolation function discussed in [Dai and Liu 2023], the composite \(\alpha\)-fractal function \(h^a\) can be defined such that \(h^a(x_j) = z_j\) and its associated functional equation is expressed by
\[
h^a(x) = G_{1,j}(m_{1,j}^{-1}(f^a(x)), h(m_{1,j}^{-1}(f^a(x))))), f^a(x) \in \mathbb{N}_I, j \in \mathbb{N}_N.
\]
From the above equation, it is seen that the composite function \(h^a\) interpolates \(\{x_j, z_j\} : j \in \mathbb{N}_{N+1}\}. For instance, consider the \(\alpha\)-fractal function \(f^a\) corresponding to the germ function \(f_1(x) = x^2 + 2x\) and base function \(b_1(x) = 3x\) with \(a = (0.5, -0.5, 0.5)\). Its graphical illustration is provided in Fig. 1(a). The linear fractal interpolation function \(h_1\) corresponding to the data set \(\{(f_1^a(x_j), z_j) = \{(0, 0), (0.25, 0.2), (0.5, 0.5), (1.25, 0.5)\}\} is represented in Fig. 1(b). The composite \(\alpha\)-fractal function \(h_1^a(\frac{f_2}{f_2})\) is provided in Fig. 1(c). Considering the height function \(f_2(x) = x^2 + x\) and base function \(b_2(x) = x\) with the scalings \(\alpha = (-0.7, -0.7, 0.7)\). The graph of another \(\alpha\)-fractal function \(f_2^a\) approximating \(f_2\) is provided in Fig. 2(a). The data set \(\{(f_2^a(x_j), z_j) = \{(0, 0), (0.25, 0.2), (0.84, 0.5), (2, 0.25)\}\} is approximated using the linear \(FIF h_2\) and it is graphically illustrated in Fig. 2(b). Fig. 2(c) represents the graph of the composite \(\alpha\)-fractal function \(h_2^a(f_2^a)\).
Figure 1 Graphical illustration of (a) $\alpha$-fractal function $f_1^\alpha$, (b) linear FIF $h_1$ and (c) its composition $h_1(f_1^\alpha)$

Figure 2 Graphical illustration of (a) $\alpha$-fractal function $f_2^\alpha$, (b) linear FIF $h_2$ and (c) its composition $h_2(f_2^\alpha)$
Remark 3. In addition to the differentiability of \( q_j \), for the existence of a differentiable fractal interpolation function, it is important to make sure the scaling parameter \( a_j \) obeys Eqn. (14). Then, for each \( k = 1, 2, \ldots, n \), the fractal spline \( h^k \) : \( I \to \mathbb{R} \) interpolates a new data set \( \{(x_j, y_{jk}) \in I \times \mathbb{R} : j \in \mathbb{N}_{N+1} \} \) and its functional equation is given by

\[
h^k(x) = F_{1,jk}(x, h^k(I_{1,jk}(x))),
\]

(14)

For each \( k = 1, 2, \ldots, n \), let \( \{(y_{jk}, z_{jk}) : j \in \mathbb{N}_{N+1} \} \) be the new set of interpolation data, where \( y_{jk} < y_{jk-1} < \ldots < y_{j}, \) \( j \) is a partition of \( I = [y_{jk}, y_{N+1,k}] \) and \( z_{jk} \in \mathbb{R} = [z_{jk}, z_{N+1,k}] \subset \mathbb{R} \). Let \( I_j = [y_{jk}, y_{j+1,k}] \), \( R_j = [z_{jk}, z_{j+1,k}] \) for \( j \in \mathbb{N} \). To interpolate the data set \( \{(y_{jk}, z_{jk}) : j \in I \times R \} \), for each \( k = 1, 2, \ldots, n \), an another fractal interpolation function \( g \) is constructed similar to the FIF \( h \). Set \( K_2 = I_j \times R_j \). Let \( F_{2,jk} : I_j \to I_{jk} \times R_{jk} \) for each \( k = 1, 2, \ldots, n \), obeying

\[
I_{1,jk} = a_{1,jk} y + b_{1,jk},
\]

\[
I_{2,jk}(y_{jk}) = y_{jk}, \quad I_{2,jk}(y_{N+1,k}) = y_{j+1,k},
\]

\[
d(F_{2,jk}(s,t_j), F_{2,jk}(s,t_{j-1})) \leq r_{2,j} d(t_j, t_{j-1}),
\]

(15)

\[
d(F_{2,jk}(y_{jk}, z_{jk}), F_{2,jk}(y_{jk+1}, z_{jk+1})) = z_{j+1,k}, \quad j \in \mathbb{N}.
\]

The attractor \( G_{\mathbb{N}} \) of the hyperbolic IFS

\[
\{K_2; (I_{2,jk}, F_{2,jk}) : j \in \mathbb{N}_{N+1} \}
\]

is the graph of \( \mathbb{S} : I_j \to \mathbb{R} \) such that \( g(y_{jk}) = z_{jk} \), for \( j \in \mathbb{N}_{N+1} \)

(16)

and for each \( k = 1, 2, \ldots, n \), obeying

\[
g(y) = F_{2,jk}(I_{2,jk}(y), g(I_{2,jk}(y))), \quad y \in I_{jk}, \quad j \in \mathbb{N}, \quad k = 1, 2, \ldots, n
\]

is the functional equation of \( F \) and \( g \).

\[
\mathbb{F}_{\mathbb{N}} h^k(I) = I_j.
\]

(17)

Therefore, the map \( F_{2,jk} \) satisfies the contraction ratio \( r_{2,j} \). Thus, \( F_{2,jk} \) is the composite fractal interpolation function \( g^k \).

\[
\mathbb{F}_{\mathbb{N}} h^k(I) = I_j.
\]

(18)

which corresponds to the composite fractal interpolation function \( g(h^k) \).

Theorem 5. Let \( h^k \) be the differentiable fractal function generated by the IFS (13). Then the IFS defined in (18) is a self-similar fractal interpolation function \( g(h^k) \) satisfying

\[
g(h^k(x_j)) = z_{jk}, \quad j \in \mathbb{N}, \quad k = 1, 2, \ldots, n.
\]

Proof. Let \( h^* \) be the FIF generated by the IFS (18) such that

\[
h^*(x) = F_{1,jk}(I_{1,jk}(x), h^*(1_{j}(x))), \quad x \in I_j.
\]

(19)

(20)

Therefore, \( g(h^k(x_j)) \) satisfies the contraction ratio \( r_{2,j} \). Thus, \( g(h^k) \) is a self-similar fractal interpolation function.
Remark 4. Theorem 5 has illustrated that composite of fractal function with a non-differentiable fractal function provided a fractal function of non-differentiable nature. Similar to this construction, one can generate the composite a-fractal spline and explore its corresponding fractal operator.

Remark 5. Encompassing the recent trend of fractional calculus, one can investigate the fractional integral and fractional derivative of composite fractal functions as well as verify for the resultant functions to be again attractors of new IFS.

CONCLUSION

As the fractional integral of a continuous function \((I^\alpha f)\) enjoys the continuity, a new family of fractal functions \((I^\alpha f)^\alpha\) is generated in the present paper. In this regard, a fractal operator is also proposed and its bound is estimated as \(\left(1 + \frac{2|a|}{T - |a|}\right) K\|f\|_\infty\), where \(K = \frac{2|a|}{(T - |a|)}\), with the proper choice of bounded linear base function. In addition, the composition of \(a\)-fractal function is discussed. The concept of composition operation is studied to the case of differentiable fractal function \(h^{(k)}\). The composition of differentiable fractal function \(h^{(k)}\) with a non-differentiable fractal function \(g\) yielded a non-differentiable fractal function \(g(h^{(k)})\) satisfying the necessary end point conditions. The composite fractal functions can be employed for approximating complex real data generated from multiple functions. For instance, in engineering the composite functions can establish a concrete relationship between different physical quantities, especially in unit conversions.

Availability of data and material

Not applicable.

Conflicts of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

Ethical standard

The author has no relevant financial or non-financial interests to disclose.

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