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Fractalization of Fractional Integral and Composition of Fractal Splines

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ABSTRACT The present study perturbs the fractional integral of a continuous function f defined on a real compact interval, say $(\mathcal{I}^v f)$ using a family of fractal functions $(\mathcal{I}^v f)^\alpha$ based on the scaling parameter α. To elicit this phenomenon, a fractal operator is proposed in the space of continuous functions, an analogue to the existing fractal interpolation operator which perturbs f giving rise to α-fractal function f^α. In addition, the composition of α-fractal function with the linear fractal function is discussed and the composition operation on the fractal interpolation functions is extended to the case of differentiable fractal functions.

KEYWORDS
Fractional integral
α-fractal function
Error estimation
Composite fractal functions

INTRODUCTION

The launch of fractal interpolation function has initiated a new theory of approximation concerning the naturally existing functions with non-differentiable nature. Rooted from the remark of Barnsley in (Barnsley 1986), Navascués has explored the approximation of continuous functions defined on a real closed interval by a class of α-fractal functions, where α is the appropriately chosen scaling parameter, in (Navascués 2005). Non-smooth analogue of prescribed continuous function can be achieved with the choice of non-differentiable base function. Further, Navascués has pioneered the fractal operator to associate each prescribed function to its class of α-fractal functions. The theme of proposing a fractal operator has fruitfully enabled the fractal theory to connect with various mathematical fields not limited to operator theory. While constructing α-fractal function, the base function choice is significant since the fractal operator is dependent on the boundedness of the base function. Literature survey acknowledges various interesting discussions on α-fractal functions, for instance, the derivative of α-fractal function is explored and its respective fractal operator is studied in (Navascués and Sebastián 2006).

The Riemann-Liouville fractional integral of α-fractal function has been discussed for the α-fractal functions with both constant and variable scalings in (Priyanka and Gowrisankar 2021b). Further, a fractional operator is defined to assign the continuous function to the fractional integral of its fractal version. For more works on α-fractal functions, the readers are recommended to consult (Balasubramani *et al.* 2020; Akhtar *et al.* 2017; Banerjee *et al.* 2023). While analysing fractals and fractal functions, the study of their fractal dimension is an ever interesting topic. Falconer has discussed the dimension theory for the fractal interpolation functions in (Falconer 2004). The dimensional analysis for the graphs of α-fractal functions is investigated in (Akhtar *et al.* 2016). Beyond the theoretical framework, fractal dimension has been estimated for various physical phenomena. For more fascinating work on the fractal dimension, the readers may visit (Banerjee *et al.* 2021; Fortin *et al.* 1992; Sanjuán 2021; Çimen *et al.* 2020).

In recent times, fractional calculus has been receiving remarkable attention among the fractal community. The Riemann-Liouville (RL) fractional integral of affine fractal functions has been investigated in (Pan 2014). The quadratic fractal function's fractional integral with constant and function scalings has been discussed in (Gowrisankar and Prasad 2019). The fractional integral as well as the fractional derivative of different kinds of fractal interpolation functions have been discussed by several authors (for additional information refer, (Pan 2014; Gowrisankar and Prasad 2019; Ruan *et al.* 2009; Priyanka and Gowrisankar 2021a)). The aforementioned results on α-fractal function and its fractional order integral, naturally arises a question: Is it possible to generate a class of fractal functions such that the fractional integral of a continuous function is interpolated? To answer this question, the present paper initiates the construction of self-referential functions for the fractional integral of continuous functions.

The construction procedure follows Navascués's α-fractal function in (Navascués and Sebastián 2006) and such a construction is guaranteed with the continuity of fractional integral. In addition, a fractal operator is defined to assign the fractional integral of a continuous function to its fractal version. The boundedness of

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the fractional integral discussed in (Samko *et al.* 1993) instigates to discuss the boundedness of the proposed operator. The base function of the newly constructed fractal function, (i.e) the fractional integral of base function of the α-fractal function, is chosen appropriately to explicitly estimate the bound of the operator.

The recent works on fractal functions reported in (Navascués *et al.* 2022; Massopust 2022b,a; Dai and Liu 2023) show the curiosity of young researchers to develop more generalized and flexible fractal interpolation functions. In (Priyanka and Gowrisankar 2021b), authors have demonstrated that the resultant functions on the evaluation of the fractional integral of α-fractal functions are again α-fractal functions obeying the end point conditions. The work by Dai and Liu(Dai and Liu 2023) is also noticeable, in which the composite fractal function is introduced along with the discussion of its fractal dimension. In this direction, the present paper investigates the composition of α-fractal function as well as the composition of fractal spline. Further, it is observed that the composition operator also renders new fractal functions like the case of fractional integral operator, which is discussed in (Priyanka and Gowrisankar 2021b). With this end, the paper directly enters the discussion on the fractal perturbation of continuous functions in the following section.

FRACTALIZATION OF CONTINUOUS FUNCTIONS

Let $N \geq 2$ and $\mathbb{N}_N$ denote the initial set of natural numbers of length N. Consider the interpolation data set,

$$\{(x_j, y_j) \in [x_1, x_{N+1}] \times \mathbb{R} : j \in \mathbb{N}_{N+1}\}.$$

Let l_j be the set of N homeomorphisms from $I = [x_1, x_{N+1}]$ to $I_j = [x_j, x_{j+1}]$, $j \in \mathbb{N}_N$ satisfying

$$|l_j(s) - l_j(t)| \leq \lambda_j |s - t|,\ \lambda_j \in [0, 1),$$
$$l_j(x_1) = x_j,\ l_j(x_{N+1}) = x_{j+1},\ j \in \mathbb{N}_N.$$

Define the maps $F_j : \mathcal{X} := I \times \mathbb{R} \to \mathbb{R}$ to be continuous in the first argument and Lipschitz continuous in the second argument with Lipschitz constant $\alpha_j < 1$ such that

$$F_j(x_1, y_1) = y_j,\ F_j(x_{N+1}, y_{N+1}) = y_{j+1},\ j \in \mathbb{N}_N.$$

The space of continuous functions defined on the interval I reserves the notation $C(I)$. Let $G = \{h \in C(I) : h(x_1) = y_1, h(x_{N+1}) = y_{N+1}\}$. For $h_1, h_2 \in C(I)$, the metric δ, defined by $\delta(h_1, h_2) = \max\{|h_1(x) - h_2(x)| : x \in I\}$, completes (G, δ). Further, in (Barnsley 1986), the Read-Bajrakteravic operator (RB), $\mathbb{T}$ is defined on (G, δ) by

$$\mathbb{T}h(x) = F_j(l_j^{-1}(x), h(l_j^{-1}(x)),\ j \in \mathbb{N}_N. \quad (1)$$

The continuity properties of l_j and F_j make easier to verify the continuity of $\mathbb{T}$ as follows

$$\delta(\mathbb{T}g_1, \mathbb{T}g_2) \leq |\alpha|_\infty \delta(g_1, g_2),\ g_1, g_2 \in C(I)$$

where $|\alpha|_\infty = \max\{|\alpha_j| : j \in \mathbb{N}_N\} < 1$ and $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$. The choice of α_k makes the operator $\mathbb{T}$ contractive on the space (G, δ). Hence, with the aid of Banach contraction principle, it is concluded that $\mathbb{T}$ has a unique fixed point, say g, satisfying $g(x_j) = y_j$, for all $j \in \mathbb{N}_{N+1}$ and from Eqn.(1), it follows that

$$g(x) = F_j(l_j^{-1}(x), g(l_j^{-1}(x)),\ j \in \mathbb{N}_N. \quad (2)$$

Using the maps l_j and F_j, define contractive transformations w_j from $\mathcal{X}$ to $I_j \times \mathbb{R}$ as

$$w_j(x, y) = (l_j(x), F_j(x, y)),\ (x, y) \in \mathcal{X},\ j \in \mathbb{N}_N.$$

Thus, the finite collection of contractive maps w_j together with the complete metric space $(\mathcal{X}, d)$ forms a hyperbolic Iterated Function System (IFS) and it is denoted by

$$\{\mathcal{X}; w_j(x, y) = (l_j(x), f_j(x, y)) : j \in \mathbb{N}_N\}. \quad (3)$$

Let $H(\mathcal{X}) := \{\mathbb{A} \subset \mathcal{X} : \mathbb{A} \neq \emptyset \text{ and compact}\}$. The Hausdorff metric h_d is defined on $H(\mathcal{X})$ by

$$h_d(\mathbb{A}, \mathbb{B}) = \max\{d(\mathbb{A}, \mathbb{B}), d(\mathbb{B}, \mathbb{A})\},$$

where $d(\mathbb{A}, \mathbb{B}) = \sup_{a \in \mathbb{A}} \inf_{b \in \mathbb{B}}\{d(a, b)\}$, then the pair $(H(\mathcal{X}), h_d)$ is a complete metric space whenever the metric space $(\mathcal{X}, d)$ is complete. A Hutchinson-Barnsley operator W is defined as a self-map on $H(\mathcal{X})$ by

$$W(\mathbb{C}) = \bigcup_{j=1}^{N} w_j(\mathbb{C}),$$

where $\mathbb{C} \in H(\mathcal{X})$. By the Banach principle of fixed point, there exists a unique $\mathbb{G}_g$ in $H(\mathcal{X})$ such that

$$\mathbb{G}_g = \lim_{n \to \infty} W^{\circ n}(\mathbb{C}),$$

where $W^{\circ n}$ is the n-fold self-composition of W. Moreover, this set $\mathbb{G}_g$ is the graph of the function g obeying the self-referential equation (2). In this construction, the function g is called the *Fractal Interpolation Function* (FIF) associated with the IFS (3). The interested readers may consult (Barnsley 1986; Agathiyan *et al.* 2022; Gowrisankar and Uthayakumar 2016) for more details on FIFs.

The following is the review of construction of α-fractal function explored by Navacués in (Navascués 2005). Slightly deviating from the theme of fractal interpolation function approximating the given interpolation data sharing complex behaviour, Navacués has generated a class of continuous functions with fractal properties to approximate $f \in C(I)$. For $f \in C(I)$, let $\{(x_j, f(x_j)) : j \in \mathbb{N}_{N+1}\}$ be the interpolation points. A partition $\Delta := \{x_1, x_2, \ldots, x_{N+1}\}$ is considered such that $x_1 < x_2 < \cdots < x_{N+1}$ and the continuous function $b : I \to \mathbb{R}$ is taken as the base function equal to f only at the endpoints x_1 and x_{N+1}. i.e.,

$$b(x_1) = f(x_1),\ b(x_{N+1}) = f(x_{N+1}), \text{ and } b \neq f. \quad (4)$$

Let $\alpha_j \in (-1, 1)$, $j \in \mathbb{N}_N$. Consider the maps

$$l_j(x) = a_j x + b_j,\ F_j(x, f(x)) = \alpha_j f(x) + q_j(x),\ j \in \mathbb{N}_N, \quad (5)$$

where

$$q_j(x) = f(l_j(x)) - \alpha_j b(x). \quad (6)$$

Then, the attractor of the IFS (3) involving the maps in (5) and (6) is the graph of the fractal interpolation function say, $f^\alpha_{\Delta,b}$ corresponding to f with respect to scale vector α, partition Δ and base function b. In addition, the function $f^\alpha_{\Delta,b}$ is the fixed point of the RB operator $\mathcal{T}_\alpha$ defined on $C_f(I)$, where $C_f(I)$ is the space of continuous functions h obeying $h(x_1) = f(x_1),\ h(x_{N+1}) = f(x_{N+1})$. The operator $\mathcal{T}_\alpha$ is described as

$$\mathcal{T}_\alpha h(x) = f(x) + \alpha_j (h - b) \circ l_j^{-1}(x),\ x \in I,\ j \in \mathbb{N}_N.$$

Then, $f^\alpha_{\Delta,b}$ obeys

$$f^\alpha_{\Delta,b}(x) = f(x) + \alpha_j (f^\alpha - b) \circ l_j^{-1}(x),\ x \in I,\ j \in \mathbb{N}_N. \quad (7)$$

Definition 1. *The function $f^\alpha_{\Delta,b} := f^\alpha$ satisfying the self-referential equation (7) is the fractal perturbation of f and it is known as the α-fractal function corresponding to α, Δ and b.*

According to Eqn.(7), f^α interpolates f at each x_j (i.e.) $f^\alpha(x_j) = f(x_j)$, for all $j \in \mathbb{N}_{N+1}$. Also, f^α equals the prescribed function f when all the scaling factors are taken to be zero. In addition, from Eqn.(7), the uniform distance between f and f^α can be deduced as follows.

$$\|f^\alpha - f\|_\infty \leq \frac{|\alpha|_\infty}{1-|\alpha|_\infty}\|f-b\|_\infty.$$

Let $C[a,b]$ be equipped with sup norm

$$\|f\|_\infty = \max\{|f(x)| : x \in [a,b]\}.$$

Consider the linearly dependent base function b on f, $b = Lf$, where $L : C[a,b] \to C[a,b]$ is a linear operator and bounded, its operator norm is given by

$$\|L\| := sup\{\|Lf\|_\infty : \|f\|_\infty \leq 1\}$$

and $Lf(x_1) = x_1,\ Lf(x_{N+1}) = x_{N+1}$ with $L \neq Identity$.

Remark 1. *The present study proceeds with $Lf = f \circ c$, where c is an increasing as well as continuous function such that $c(x_1) = x_1$, $c(x_{N+1}) = x_{N+1}$ and $c \neq$ Identity. For this particular choice of $b = f \circ c$, $\|b\|_\infty = \|Lf\|_\infty = \|f\|_\infty$ with operator norm $\|L\| = 1$.*

Lemma 1. *(Navascués 2010) For any $f \in C(I)$ and $b = Lf$, the following inequality holds*

$$\|f^\alpha - f\|_\infty \leq \frac{|\alpha|_\infty\|Id - L\|_\infty}{1-|\alpha|_\infty}\|f\|_\infty,$$

where Id is the identity operator.

Note 1. *If $Lf = f \circ c$, the inequality () becomes*

$$\|f^\alpha - f\|_\infty \leq \frac{2|\alpha|_\infty}{1-|\alpha|_\infty}\|f\|_\infty.$$

In (Navascués 2005), a fractal interpolation operator $\mathcal{F}^\alpha : C(I) \to C(I)$ is introduced to fractalize each continuous function as

$$\mathcal{F}^\alpha(f) = f^\alpha,\ f \in C(I).$$

Theorem 1. *(Navascués 2010) For any bounded and linear operator L with sup norm, the following holds*

$$\|\mathcal{F}^\alpha(f)\|_\infty \leq \left(1 + \frac{|\alpha|_\infty\|Id - L\|_\infty}{1-|\alpha|_\infty}\right)\|f\|_\infty.$$

In analogue to the above discussed operator, various fractal operators have been proposed to the fractalize the given continuous functions, see for instance (Navascués and Sebastián 2006; Priyanka and Gowrisankar 2021b).

FRACTAL PERTURBATION OF FRACTIONAL INTEGRAL OF A CONTINUOUS FUNCTION

In order to define a new class of α-fractal functions to approximate the fractional integral of $f \in C(I)$, this section commences with the definition of RL fractional integral of a continuous function.

Definition 2. *(Samko et al. 1993) Let f be the integrable function on $[a,b] \subset \mathbb{R}$ and $v > 0$ be a real number. Then, the Riemann-Liouville (RL) fractional integral of f is defined by*

$$(\mathcal{I}^v f)(t) = \frac{1}{\Gamma(v)}\int_a^t (t-s)^{v-1} f(s)ds,\ (t > a),$$

here the notation $\Gamma(\cdot)$ denotes the Gamma function.

In (Samko *et al.* 1993), it is proved that the fractional integral operator $(\mathcal{I}^v f)$ is bounded in L_p space with $1 \leq p \leq \infty$ and it is precisely provided in the following lemma.

Lemma 2. *For $v > 0$, the RL fractional integral operator is bounded such that*

$$\|\mathcal{I}^v f\| \leq \mathcal{K}\|f\|_\infty, \text{ where } \mathcal{K} = \frac{x_{N+1} - x_1}{v\Gamma(v)}.$$

Using the above lemma, the uniform distance between the germ function f and its fractional integral $\mathcal{I}^v f$ can be estimated as follows.

Lemma 3. *The distance between f and $\mathcal{I}^v f$ with respect to the uniform norm is given by*

$$\|f - \mathcal{I}^v f\|_\infty \leq (1 + \mathcal{K})\|f\|_\infty,$$

where $\mathcal{K} = \frac{x_{N+1}-x_1}{v\Gamma(v)}$.

Proof. By the definition of uniform norm,

$$\|f - \mathcal{I}^v f\|_\infty \leq \|f\|_\infty + \|\mathcal{I}^v f\|_\infty.$$

From Lemma 2, it follows that $\|\mathcal{I}^v f\|_\infty \leq \mathcal{K}\|f\|_\infty$, where $\mathcal{K} = \frac{x_{N+1}-x_1}{v\Gamma(v)}$. Then,

$$\begin{aligned}\|f - \mathcal{I}^v f\|_\infty &\leq \|f\|_\infty + \mathcal{K}\|f\|_\infty \\ &\leq (1 + \mathcal{K})\|f\|_\infty.\end{aligned}$$

□

The following lemma ensures the continuity of the fractional order integral $\mathcal{I}^v f$ which is proved by Pan in reference (Pan 2014).

Lemma 4. *Let $v > 0$ and $f \in C[a,b]$. Then $\mathcal{I}^v f \in C[a,b]$.*

From Lemma 4, it is straight forward to define a family of fractal functions to approximate $\mathcal{I}^v f$.

Let $\{x_j, \mathcal{I}^v f(x_j)\}$ be the the interpolation data with partition Δ and scale vector α. To define a new family of self-referential functions, consider the base function as the fractional integral of b, expressed by

$$(\mathcal{I}^v b)(t) = \frac{1}{\Gamma(v)}\int_{x_1}^x (t-s)^{v-1} b(s)ds,$$

such that

$$(\mathcal{I}^v b)(x_1) = (\mathcal{I}^v f)(x_1),$$
$$(\mathcal{I}^v b)(x_{N+1}) = (\mathcal{I}^v f)(x_{N+1})$$

and $\mathcal{I}^v b \neq \mathcal{I}^v f$. In correspondence with the new continuous functions $(\mathcal{I}^v f)$ and $(\mathcal{I}^v b)$, the maps defined in (5) becomes,

$$l_j(x) = a_j x + b_j,\ F_j(x,y) = \alpha_j y + (\mathcal{I}^v f) l_j(x) - \alpha_j(\mathcal{I}^v b)(x),\ j \in \mathbb{N}_N. \tag{8}$$

The attractor of the IFS with the maps in (8) is the graph of the new kind of α-fractal function say, $(\mathcal{I}^v f)^\alpha$ associated with $(\mathcal{I}^v f)$. It can be verified that $(\mathcal{I}^v f)^\alpha(x_j) = (\mathcal{I}^v f)(x_j)$ for all $j \in \mathbb{N}_{N+1}$. Besides, $(\mathcal{I}^v f)^\alpha$ is a unique fixed point of the RB operator $\mathscr{T}_\alpha$ with the change of arguments such that

$$(\mathcal{I}^v f)^\alpha(x) = \mathcal{I}^v f(x) + \alpha_j((\mathcal{I}^v f)^\alpha - \mathcal{I}^v b) \circ l_j^{-1}(x)),\ x \in I,\ j \in \mathbb{N}_N.$$

The function $(\mathcal{I}^v f)^\alpha$ is the α-fractal function of the RL fractional integral of $f \in C(I)$ approximating $(\mathcal{I}^v f)$ with respect to base function $(\mathcal{I}^v b)$, partition Δ and scaling parameter α. With an aim to estimate the error, now consider the mapping

$$\begin{aligned} T :& R \times C(I) \to C(I) \\ &(\alpha, \mathcal{I}^v f) \to \mathscr{T}_\alpha(\mathcal{I}^v f) \end{aligned}$$

where $R = [0,t] \times [0,t] \times [0,t] \times \cdots \times [0,t] \subset \mathbb{R}^N, 0 \le t < 1$, t is fixed. For $x \in I_j$, define

$$\begin{aligned} \mathscr{T}_\alpha(\mathcal{I}^v f)(x) &= F_j^{\alpha_j}(l_j^{-1}(x), (\mathcal{I}^v f) \circ l_j^{-1}(x)) \\ &= \alpha_j (\mathcal{I}^v f) \circ l_j^{-1}(x) + q_j^{\alpha_j} \circ l_j^{-1}(x) \end{aligned}$$

with

$$q_j^{\alpha_j}(x) = (\mathcal{I}^v f) \circ l_j(x) - \alpha_j (\mathcal{I}^v b)(x).$$

The uniform distance between the functions $(\mathcal{I}^v f)$ and $(\mathcal{I}^v f)^\alpha$ is estimated in the following theorem.

Theorem 2. *If b is a bounded linear operator, then the below inequality holds*

$$\|(\mathcal{I}^v f)^\alpha - (\mathcal{I}^v f)\|_\infty \le \frac{2\mathcal{K}|\alpha|_\infty}{1 - |\alpha|_\infty}\|f\|_\infty,$$

where $\mathcal{K} = \frac{x_{N+1} - x_1}{v\Gamma(v)}$.

Proof. Let $(\mathcal{I}^v f) \in \mathcal{C}_f(I)$. Then for each $x \in I_j$,

$$\begin{aligned} |\mathscr{T}_\alpha(\mathcal{I}^v f)(x) - &\mathscr{T}_\beta(\mathcal{I}^v f)(x)| \\ &= |\alpha_j (\mathcal{I}^v f) \circ l_j^{-1}(x) + q_j^{\alpha_j} \circ l_j^{-1}(x) - \beta_j (\mathcal{I}^v f) \circ l_j^{-1}(x) \\ &\quad - q_j^{\beta_j} \circ l_j^{-1}(x)| \\ &\le |\alpha_j (\mathcal{I}^v f) \circ l_j^{-1}(x) - \beta_j (\mathcal{I}^v f) \circ l_j^{-1}(x)| \\ &\quad + |q_j^{\alpha_j} \circ l_j^{-1}(x) - q_j^{\beta_j} \circ l_j^{-1}(x)| \end{aligned}$$

From Eqn.(), the second term is rewritten as

$$\begin{aligned} \|\mathscr{T}_\alpha&(\mathcal{I}^v f) - \mathscr{T}_\beta(\mathcal{I}^v f)\|_\infty \\ &\le |\alpha_j - \beta_j| \|\mathcal{I}^v f\|_\infty + |(\mathcal{I}^v f) \circ l_j(x) - \alpha_j (\mathcal{I}^v b)(x) \qquad (9) \\ &\quad - (\mathcal{I}^v f) \circ l_j(x) + \beta_j (\mathcal{I}^v b)(x)| \\ &\le |\alpha - \beta|_\infty \|\mathcal{I}^v f\|_\infty + |\alpha_j - \beta_j| \|\mathcal{I}^v b\|_\infty \\ &\le 2|\alpha - \beta|_\infty \|\mathcal{I}^v f\|_\infty. \qquad (10) \end{aligned}$$

Meanwhile, $(\mathcal{I}^v f)$ is the fixed point of $\mathscr{T}_\alpha$ corresponding to $q_j^{\alpha_j}(x) = (\mathcal{I}^v f) \circ l_j(x) - \alpha_j (\mathcal{I}^v b)(x)$. Then,

$$\begin{aligned} \|(\mathcal{I}^v f)^\alpha - (\mathcal{I}^v f)^\beta\|_\infty = \|&\mathscr{T}_\alpha(\mathcal{I}^v f)^\alpha - \mathscr{T}_\alpha(\mathcal{I}^v f)^\beta + \mathscr{T}_\alpha(\mathcal{I}^v f)^\beta \\ &- \mathscr{T}_\beta(\mathcal{I}^v f)^\beta\|_\infty \end{aligned}$$

Since $\mathscr{T}_\alpha$ is contractive with contractivity factor α and applying the inequality (9),

$$\begin{aligned} \|(\mathcal{I}^v f)^\alpha - (\mathcal{I}^v f)^\beta\|_\infty &\le |\alpha|_\infty \|(\mathcal{I}^v f)^\alpha - (\mathcal{I}^v f)^\beta\|_\infty + 2|\alpha \\ &\quad - \beta|_\infty \|(\mathcal{I}^v f)^\beta\|_\infty \\ &= \frac{2|\alpha - \beta|_\infty \|(\mathcal{I}^v f)^\beta\|_\infty}{1 - |\alpha|_\infty}. \end{aligned}$$

Setting $\beta = 0 \in \mathbb{R}^N$ and using the property $(\mathcal{I}^v f)^0 = (\mathcal{I}^v f)$, observe that

$$\begin{aligned} \|(\mathcal{I}^v f)^\alpha - (\mathcal{I}^v f)\|_\infty &= \frac{2|\alpha|_\infty \|(\mathcal{I}^v f)\|_\infty}{1 - |\alpha|_\infty} \\ &= \frac{2\mathcal{K}|\alpha|_\infty}{1 - |\alpha|_\infty}\|f\|_\infty. \end{aligned}$$

□

The above theorem is a prelude to discuss the boundedness of the fractal operator $\mathcal{F}^{\alpha,v}$ which is explored in the following section.

FRACTAL OPERATOR ASSOCIATED WITH THE FRACTIONAL INTEGRAL

This section proposes a fractal operator to send each continuous function $\mathcal{I}^v f$ to its fractal version $(\mathcal{I}^v f)^\alpha$ where the function $(\mathcal{I}^v f)^\alpha$ is the α-fractal function of the RL fractional integral of a prescribed continuous function f discussed in the previous section. To be concise, for a fixed scale vector α and a fixed fractional order $v > 0$, there exists an operator

$$\begin{aligned} \mathcal{F}^{\alpha,v} &: C(I) \to C(I) \\ &\mathcal{I}^v f \longmapsto (\mathcal{I}^v f)^\alpha. \end{aligned}$$

The linearity of b assures the linearity of $\mathcal{F}^{\alpha,v}$. For fixed scalars λ and μ, it can be verified that

$$\mathcal{F}^{\alpha,v}(\lambda \mathcal{I}^v f + \mu \mathcal{I}^v g) = \lambda \mathcal{F}^{\alpha,v}(\mathcal{I}^v f) + \mu \mathcal{F}^{\alpha,v}(\mathcal{I}^v g).$$

Theorem 3. *$\mathcal{F}^{\alpha,v}$ is bounded on $C(I)$. Moreover,*

$$\|\mathcal{F}^{\alpha,v}(\mathcal{I}^v f)\|_\infty \le \left(\frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty}\right)\mathcal{K}\|f\|_\infty,$$

where $\mathcal{K} = \frac{x_{N+1} - x_1}{v\Gamma(v)}$.

Proof. From Theorem 2, one has

$$\|(\mathcal{I}^v f)^\alpha - (\mathcal{I}^v f)\|_\infty \le \frac{2\mathcal{K}|\alpha|_\infty}{1 - |\alpha|_\infty}\|f\|_\infty,$$

with $\mathcal{K} = \frac{x_{N+1} - x_1}{v\Gamma(v)}$. Then,

$$\begin{aligned} \|(\mathcal{I}^v f)^\alpha\|_\infty - \|(\mathcal{I}^v f)\|_\infty &\le \frac{2\mathcal{K}|\alpha|_\infty}{1 - |\alpha|_\infty}\|f\|_\infty \\ \|(\mathcal{I}^v f)^\alpha\|_\infty &\le \|(\mathcal{I}^v f)\|_\infty + \frac{2\mathcal{K}|\alpha|_\infty}{1 - |\alpha|_\infty}\|f\|_\infty \\ &\le \mathcal{K}\|f\|_\infty + \frac{2\mathcal{K}|\alpha|_\infty}{1 - |\alpha|_\infty}\|f\|_\infty, \end{aligned}$$

which provides the required bound of the operator $\mathcal{F}^{\alpha,v}$,

$$\begin{aligned} \|\mathcal{F}^{\alpha,v}(\mathcal{I}^v f)\|_\infty &\le \left(1 + \frac{2|\alpha|_\infty}{1 - |\alpha|_\infty}\right)\mathcal{K}\|f\|_\infty \\ &= \left(\frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty}\right)\mathcal{K}\|f\|_\infty. \end{aligned}$$

Hence, the required inequality. □

Next, the bound for the perturbation error between f and $(\mathcal{I}^v f)^\alpha$ is explored in the following theorem.

Theorem 4. *For any $f \in C(I)$, the following inequality*

$$\|f - (\mathcal{I}^v f)^\alpha\|_\infty \le \left(1 + \mathcal{K} + \frac{2\mathcal{K}|\alpha|_\infty}{1 - |\alpha|_\infty}\right)\|f\|_\infty,$$

holds with $\mathcal{K} = \frac{x_{N+1} - x_1}{v\Gamma(v)}$.

Proof. One can have

$$\begin{aligned}\|f-(\mathcal{I}^{v}f)^{\alpha}\|_{\infty}&=\|f-\mathcal{I}^{v}f+\mathcal{I}^{v}f-(\mathcal{I}^{v}f)^{\alpha}\|_{\infty}\\&\leq\|f-\mathcal{I}^{v}f\|_{\infty}+\|\mathcal{I}^{v}f-(\mathcal{I}^{v}f)^{\alpha}\|_{\infty}.\end{aligned}$$

Using Lemma 4 and Theorem 2, the above inequality is reduced to

$$\|f-(\mathcal{I}^{v}f)^{\alpha}\|_{\infty}\leq(1+\mathcal{K})\|f\|_{\infty}+\frac{2\mathcal{K}|\alpha|_{\infty}}{1-|\alpha|_{\infty}}\|f\|_{\infty}.$$

Thus, the required result follows immediately. □

Remark 2. *In (Priyanka and Gowrisankar 2021b), a fractal operator $\mathcal{F}^{v}$ has been proposed to associate the given function $f\in C(I)$ to the Riemann-Liouville fractional integral of its fractal version namely, $\mathcal{I}^{v}(f^{\alpha})$ and discussed some of its elementary properties. Whereas, here the fractal operator $\mathcal{F}^{\alpha,v}$ is defined on $C(I)$ to associate the fractional integral of $f\in C(I)$ to its fractal version, namely $(\mathcal{I}^{v}f)^{\alpha}$.*

COMPOSITE FRACTAL FUNCTIONS

This section discusses the composition of fractal functions and demonstrate that the compositions are again fractal functions.

Composition of α-fractal Function

Let $J=[y_1,y_{N+1}]\subset\mathbb{R}$ and $l_{1,j}:I\to I_j$ be the homeomorphic maps defined by $l_{1,j}(x)=a_{1,j}x+b_{1,j}$ satisfying

$$d(l_{1,j}(a),l_{1,j}(b))\leq r_1 d(a,b),\ 0\leq r_1<1,\ a,b\in I,$$

where d is a Euclidean metric or its equivalent metric and

$$l_{1,j}(x_1)=x_j,\ l_{1,j}(x_{N+1})=x_{j+1},\ j\in\mathbb{N}_N. \tag{11}$$

Let $K_1:=I\times J$. Define the continuous functions $F_{1,j}:K_1\to\mathbb{R}$ to be contraction with respect to second variable satisfying

$$F_{1,j}(x_1,y_1)=y_j,\ F_{1,j}(x_{N+1},y_{N+1})=y_{j+1},\ j\in\mathbb{N}_N, \tag{12}$$

The general form of the maps $F_{1,j}$ is given by

$$F_{1,j}(x,y)=\alpha_j y+q_j(x),$$

where $\alpha_j=(\alpha_1,\alpha_2,\ldots,\alpha_{N+1})$ is the free parameter chosen in the interval $[0,1)$, which scales the graph vertically and referred as vertical scaling factor, q_j is a suitable continuous function satisfying

$$q_j(x_1)=y_j-\alpha_j y_1,\ q_j(x_{N+1})=y_{j+1}-\alpha_j y_{N+1}.$$

The system $\{K_1;(l_{1,i},F_{1,j}):j\in\mathbb{N}_N\}$ is a IFS and its attractor G_f is the graph of fractal interpolation function $h:I\to\mathbb{R}$ interpolating the data set $\{(x_j,y_j)\in I\times\mathbb{R}:j\in\mathbb{N}_{N+1}\}$ such that $h(x_j)=y_j$, for $j\in\mathbb{N}_{N+1}$. In (Dai and Liu 2023), the functional equation of h is provided by

$$h(x)=F_{1,j}(l_{1,j}^{-1}(x),F(l_{1,j}^{-1}(x))),$$

(or)

$$h(l_{1,j}(x))=\alpha_j h(x)+q_j(x),\ x\in I,\ j\in\mathbb{N}_N.$$

On the other hand, if the data set $\{(x_j,f(x_j)):j\in\mathbb{N}_{N+1}\}$ is given to approximate, where f is a continuous function, the following choice of $q_j(x)=f\circ l_{1,j}(x)-\alpha_j b(x)$ generates an α-fractal function satisfying

$$f^{\alpha}(l_{1,j}(x))=\alpha_j f^{\alpha}(x)+f\circ l_{1,j}(x)-\alpha_j b(x)$$

and $f^{\alpha}(x_j)=f(x_j)$, $\forall\, j\in\mathbb{N}_{N+1}$, here b is the base function obeying the conditions provided in (4). Let $\mathcal{N}=[f^{\alpha}(x_1),f^{\alpha}(x_{N+1})]$ and $\mathcal{N}_j=[f^{\alpha}(x_j),f^{\alpha}(x_{j+1})]$, $j\in\mathbb{N}_N$. Now, to interpolate the data set $\{(f^{\alpha}(x_j),z_j):j\in\mathbb{N}_{N+1}\}$, $z_j\in\mathbb{R}$ for all $j\in\mathbb{N}_{N+1}$, a new fractal interpolation function $h:\mathcal{N}\to\mathbb{R}$ is constructed with the maps $m_{1,j}$ and $G_{1,j}$ defined below which respectively obey the conditions of $l_{1,j}$ and $F_{1,j}$,

$$\begin{aligned}m_{1,j}(x)&=c_{1,j}(x)+d_{1,j},\\G_{1,j}(f^{\alpha}(x),z)&=\alpha_j z+p_j(f^{\alpha}(x)),\ j\in\mathbb{N}_N,\end{aligned}$$

where p_j is a linear polynomial of x satisfying $p_j(f^{\alpha}(x_0))=z_j$, $p_j(f^{\alpha}(x_{N+1}))=z_{j+1}$. Note that the domain of h agrees with $f^{\alpha}(I)$, thus it is possible to composite g with f^{α}. Similar to the composite fractal interpolation function discussed in (Dai and Liu 2023), the *composite α-fractal function* $h(f^{\alpha})$ can be defined such that $h(f^{\alpha}(x_j))=z_j$ and its associated functional equation is expressed by

$$h(f^{\alpha}(x))=G_{1,j}(m_{1,j}^{-1}(f^{\alpha}(x)),h(m_{1,j}^{-1}(f^{\alpha}(x)))),\ f^{\alpha}(x)\in\mathcal{N}_j,\ j\in\mathbb{N}_N.$$

From the above equation, it is seen that the composite function $h(f^{\alpha})$ interpolates $\{(x_j,z_j):j\in\mathbb{N}_{N+1}\}$. For instance, consider the α-fractal function f_1^{α} corresponding to the germ function $f_1(x)=x^2+2x$ and base function $b_1(x)=3x$ with $\alpha=(0.5,-0.5,0.5)$. Its graphical illustration is provided in Fig. 1(a). The linear fractal interpolation function h_1 corresponding to the data set $\{(f_1^{\alpha}(x_j),z_j)=\{(0,0),(0.25,0.2),(0.56,0.5),(1,0.25)\}\}$ is represented in Fig. 1(b). The composite α-fractal function $h_1(f_1^{\alpha})$ is provided in Fig. 1(c). Considering the height function $f_2(x)=2x^3$ and base function $b_2(x)=x$ with the scalings $\alpha=(0.7,-0.7,0.7)$. The graph of another α-fractal function f_2^{α} approximating f_2 is provided in Fig. 2(a). The data set $\{(f_2^{\alpha}(x_j),z_j)=\{(0,0),(0.25,0.2),(0.84,0.5),(2,0.25)\}\}$ is approximated using the linear FIF h_2 and it is graphically illustrated in Fig. 2(b). Fig. 2(c) represents the graph of the composite α-fractal function $h_2(f_2^{\alpha})$.

Composition of Fractal Spline

In (Barnsley and Harrington 1989), Barnsely has extended the continuity of q_j to be differentiable in order to achieve differentiable fractal functions as narrated below. Consider $l_{1,j}$ and $F_{1,j}$ as defined above satisfying Eqns.(11) and (12). For $n>0$, suppose

$$|\alpha_j|<a_j^n,$$

and $q_j\in\mathcal{C}^n(I)$, then

$$F_{1,jk}(x,y)=\frac{\alpha_j y+q_j^{(k)}(x)}{a_j^k},$$

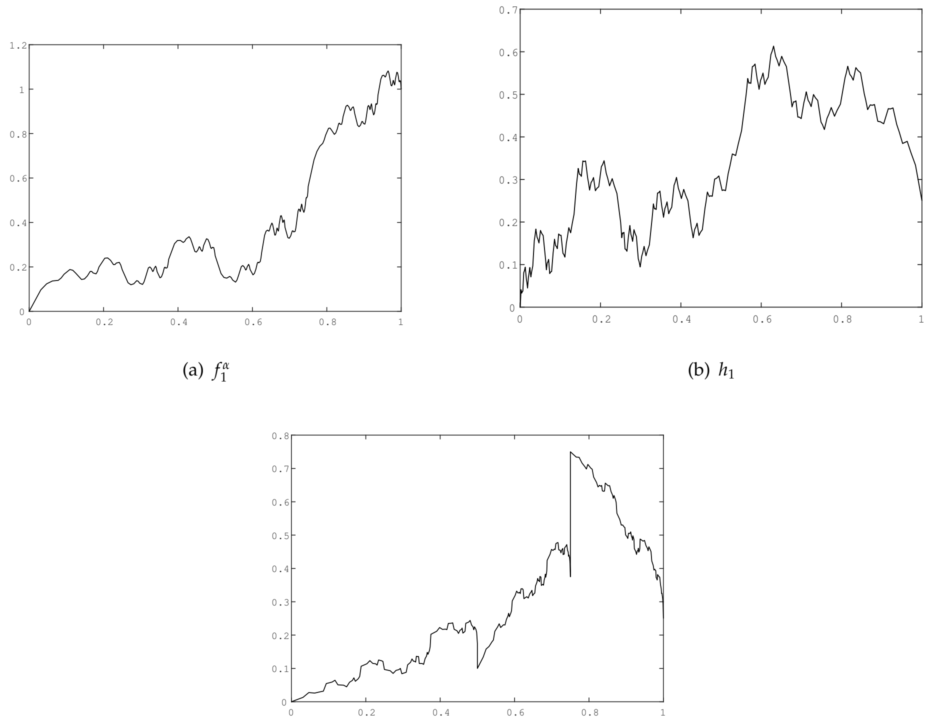
$$y_{1,k}=\frac{q_1^k(x_1)}{a_1^k-\alpha_1},\ y_{N+1,k}=\frac{q_{n-1}^k(x_{N+1})}{a_N^k-\alpha_N},\ \text{for } k=1,2,\ldots,n.$$

Moreover, if

$$F_{1,(j-1)k}(x_{N+1},y_{N+1,k})=F_{1,jk}(x_1,y_{1,k}),\ j=2,3,\ldots,N,\ k=1,2,\ldots,n,$$

then the IFS $\{K_1;(l_{1,j},F_{1,j}):j\in\mathbb{N}_N\}$ generates $h\in\mathcal{C}^k(I)$ and $h^{(k)}$ is the FIF generated by the IFS

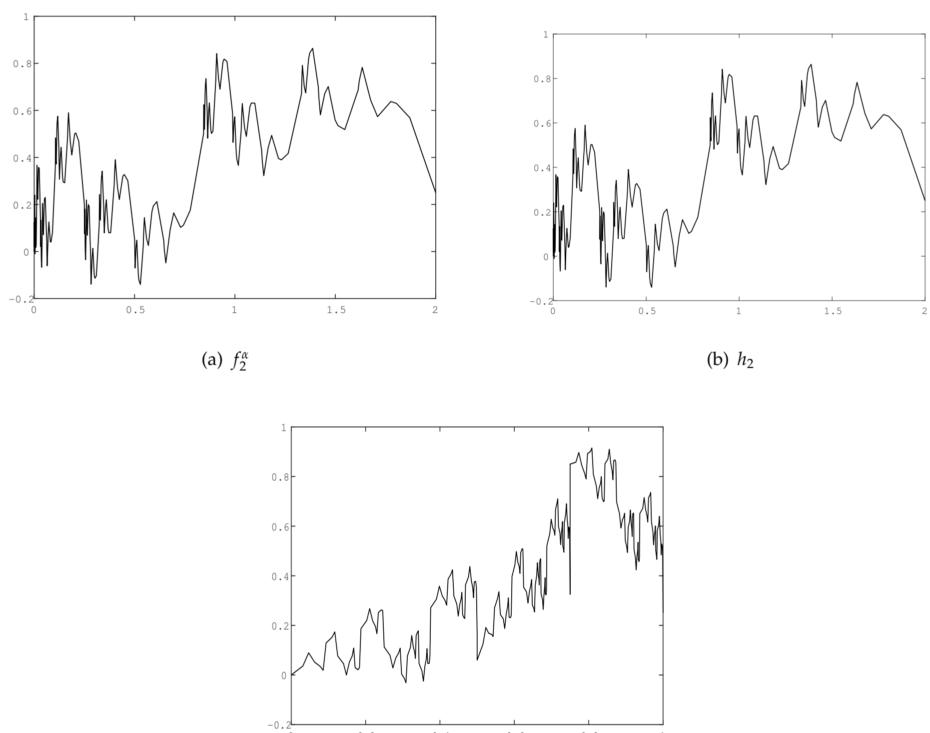
$$\{K_1;(l_{1,j},F_{1,jk}):j\in\mathbb{N}_N,k=1,2,\ldots,n\}. \tag{13}$$



(a) f_1^α (b) h_1

(c) $h_1(f_1^\alpha)$

Figure 1 Graphical illustration of (a) α-fractal function f_1^α, (b) linear FIF h_1 and (c) its composition $h_1(f_1^\alpha)$



(a) f_2^α (b) h_2

(c) $h_2(f_2^\alpha)$

Figure 2 Graphical illustration of (a) α-fractal function f_2^α, (b) linear FIF h_2 and (c) its composition $h_2(f_2^\alpha)$

Remark 3. *In addition to the differentiability of q_j, for the existence of a differentiable fractal interpolation function, it is important to make sure the scaling parameter α_j obeys Eqn.(). Then, for each $k = 1, 2, \ldots, n$, the fractal spline $h^{(k)} : I \to \mathbb{R}$ interpolates a new data set $\{(x_j, y_{jk}) \in I \times \mathbb{R} : j \in \mathbb{N}_{N+1}\}$ and its functional equation is given by*

$$h^{(k)}(x) = F_{1,jk}(l_{1,jk}^{-1}(x), h^{(k)}(l_{1,jk}^{-1}(x))),$$

(or)

$$h^{(k)}(l_{1,j}(x)) = \frac{1}{a_j^k}(\alpha_j y + q_j^{(k)}(x)),\ x \in I,\ j \in \mathbb{N}_N,\ k = 1, 2, \ldots, n. \tag{14}$$

For each $k = 1, 2, \ldots, n$, let $\{(y_{jk}, z_{jk}) : j \in \mathbb{N}_{N+1}\}$ be the new set of interpolation data, where $y_{1,k} < y_{2,k} < \ldots < y_{N+1,k}$ is a partition of $J_1 = [y_{1,k}, y_{N+1,k}]$ and $z_{jk} \in R_1 = [z_{1,k}, z_{N+1,k}] \subset \mathbb{R}$. Let $J_{1j} = [y_{j,k}, y_{j+1,k}]$, $R_{1j} = [z_{j,k}, z_{j+1,k}]$ for $j \in \mathbb{N}_N$. To interpolate the data set

$$\{(y_{jk}, z_{jk}) \in J_1 \times R_1 : j \in \mathbb{N}_{N+1}\},\ \text{for each } k = 1, 2, \ldots, n,$$

an another fractal interpolation function g is constructed similar to the FIF h. Set $K_2 = J_1 \times R_1$. Let $l_{2,jk} : J_1 \to J_{j,k}$ and $F_{2,jk} : K_2 \to \mathbb{R}$, for each $k = 1, 2, \ldots, n$, obeying

$$l_{2,jk} = a_{2,jk} y + b_{2,jk},$$
$$l_{2,jk}(y_{1,k}) = y_{j,k},\ l_{2,jk}(y_{N+1,k}) = y_{j+1,k},$$
$$d(F_{2,jk}(s, t_1), F_{2,jk}(s, t_2)) \le r_{2,j} d(t_1, t_2),\ 0 \le r_2 < 1,\ s \in J_1,\ t_1, t_2 \in R_1,$$
$$F_{2,jk}(y_{1,k}, z_{1,k}) = z_{j,k},\ F_{2,jk}(y_{N+1,k}, z_{N+1,k}) = z_{j+1,k},\ j \in \mathbb{N}_N.$$

The attractor G_g of the hyperbolic IFS

$$\{K_2; (l_{2,jk}, F_{2,jk}) : j \in \mathbb{N}_N\} \tag{15}$$

is the graph of $g : J_1 \to \mathbb{R}$ such that $g(y_{jk}) = z_{jk}$, for $j \in \mathbb{N}_{N+1}$ and for each $k = 1, 2, \ldots, n$. Note that

$$g(y) = F_{2,jk}(l_{2,jk}^{-1}(y), g(l_{2,jk}^{-1}(y))),\ y \in J_1,\ j \in \mathbb{N}_N,\ k = 1, 2, \ldots, n \tag{16}$$

is the functional equation of FIF g.

Since $J_1 \subseteq h^{(k)}(I)$, assuming $h^{(k)}(x) \in J_1$, for $x \in I$, ensures the continuity of $g(h^{(k)}(x))$ on I. An IFS is constructed to illustrate the composition of fractal function and fractal spline $g(h^{(k)})$ is again a fractal interpolation function interpolating the data set $\{(x_j, z_{jk}) \in I \times \mathbb{R} : j \in \mathbb{N}_{N+1},\ k = 1, 2, \ldots, n\}$. Let $h^{(k)}(I) = J_1$. From Eqn.(16),

$$g(h^{(k)}(x)) = F_{2,jk}(l_{2,jk}^{-1}(h^{(k)}(x)), g(l_{2,jk}^{-1}(h^{(k)}(x)))),\ h^{(k)}(x) \in J_{1j},$$

for $j \in \mathbb{N}_N$, $k = 1, 2, \ldots, n$. Let $l_j : I \to I_j$ be the function agreeing with $l_{1,j}(x)$ for all $x \in I$. And $F_{jk} : K \to \mathbb{R}$ be the continuous maps defined by

$$\begin{aligned} F_{jk}(x, z) &= F_{jk}(l_j^{-1}(x_1), g^*(h^{(k)}(l_j^{-1}(x_1)))) \\ &= F_{2,jk}(l_{2,jk}^{-1}(h^{(k)}(x_1)), g^*(l_{2,jk}^{-1}(h^{(k)}(x_1)))),\ j \in \mathbb{N}_N, \end{aligned} \tag{17}$$

where $x_1 \in I_j$, $g^* \in \mathcal{C}_1 = \{g(y) + t, z_{1,k} - z_{N+1,k} \le t \le z_{N+1,k} - z_{1,k},\ y \in J_1\}$, $k = 1, 2, \ldots, n$, the set of continuous translation maps and $h(h^{(k)}(l_j^{-1}(x_1))) = z$.

For all $x \in I$, $z, z^* \in R_1$, there exists $x^* \in I_j$, $h_1, h_2 \in \mathcal{C}$ such that

$$l_j^{-1}(x^*) = x,\ h_1(h^{(k)}(l_j^{-1}(x^*))) = z,\ h_2(h^{(k)}(l_j^{-1}(x^*))) = z^*.$$

Then

$$\begin{aligned} &d(F_{jk}(x, z), F_{jk}(x, z^*)) \\ &\quad= d(F_{jk}(l_j^{-1}(x^*)), h_1(h^{(k)}(l_j^{-1}(x^*))), F_{jk}(l_j^{-1}(x^*)), \\ &\qquad h_2(h^{(k)}(l_j^{-1}(x^*)))) \\ &\quad= d(F_{2,jk}(l_{2,jk}^{-1}(h^{(k)}(x^*)), h_1(l_{2,jk}^{-1}(h^{(k)}(x^*)))), \\ &\qquad F_{2,jk}(l_{2,jk}^{-1}(h^{(k)}(x^*)), h_2(l_{2,jk}^{-1}(h^{(k)}(x^*))))). \end{aligned}$$

From the contractivity of $F_{2,jk}$ with respect to second argument, it follows that

$$\begin{aligned} d(F_{jk}(x, z), F_{jk}(x, z^*)) &\le r_{2,j} d(h_1(l_{2,jk}^{-1}(h^{(k)}(x^*))), h_2(l_{2,jk}^{-1}(h^{(k)}(x^*)))) \\ &\le r_2 d(z, z^*), \end{aligned}$$

where $r_2 = \max\{r_{2,j} : j \in \mathbb{N}_N\}$. Therefore, the map F_{jk} satisfies the contractivity condition with contraction ratio r_2. Now, it is necessary to verify the join-up conditions. From Eqn.(17), for $h^{(k)}(x_j) = y_{jk} \in J_{jk}$,

$$\begin{aligned} F_{jk}(x_1, z_1) &= F_{jk}(l_j^{-1}(x_j), g(h^{(k)}(l_j^{-1}(x_j)))) \\ &= F_{2,jk}(l_{2,jk}^{-1}(h^{(k)}(x_j)), g(l_{2,jk}^{-1}(h^{(k)}(x_j)))) \\ &= F_{2,jk}(y_{1k}, z_{1k}) \\ &= z_{jk}. \end{aligned}$$

Meanwhile, for $h^{(k)}(x_{j+1}) = y_{(j+1)k} \in J_{(j+1)k}$,

$$\begin{aligned} F_{jk}(x_{N+1}, z_{N+1}) &= F_{jk}(l_j^{-1}(x_{j+1}), g(h^{(k)}(l_j^{-1}(x_{j+1})))) \\ &= F_{2,jk}(l_{2,jk}^{-1}(h^{(k)}(x_{j+1})), g(l_{2,jk}^{-1}(h^{(k)}(x_{j+1})))) \\ &= F_{2,jk}(y_{(N+1)k}, z_{(N+1)k}) \\ &= z_{(j+1)k}. \end{aligned}$$

The above contractivity maps and the join-up conditions determine an IFS

$$\{I \times \mathbb{R}; (l_j, F_{jk}) : j \in \mathbb{N}_N,\ k = 1, 2, \ldots, n\} \tag{18}$$

which corresponds to the composite fractal interpolation function $g(h^{(k)})$.

Theorem 5. *Let $h^{(k)}$ be the differentiable fractal function generated by the IFS (13). Then the IFS defined in (18) determines a FIF $g(h^{(k)})$ satisfying*

$$g(h^{(k)}(x_j)) = z_{jk},\ \text{for } j \in \mathbb{N}_N,\ k = 1, 2, \ldots, n.$$

Proof. Let h^* be the FIF generated by the IFS (18) such that

$$h^*(x) = F_{jk}(l_j^{-1}(x), h^*(l_j^{-1}(x))),\ x \in I_j.$$

From (),

$$g(h^{(k)}(l_j(x)) = F_{2,jk}(l_{2,jk}^{-1}(h^{(k)}(l_j(x))), g(l_{2,jk}^{-1}(h^{(k)}(l_j(x))))).$$

Meanwhile, from (17),

$$g(h^{(k)}(l_j(x))) = F_{2,jk}(l_{2,jk}(h^{(k)}(x_1)), g(l_{2,jk}^{-1}(h^{(k)}(x)))).$$

Uniqueness of FIF yields

$$h^*(x) = g(h^{(k)}(x))$$

such that $g(h^{(k)}(x_j)) = z_{jk}$, $j \in \mathbb{N}_N$, $k = 1, 2, \ldots, n$. □

Remark 4. *Theorem 5 has illustrated that composite of fractal spline with a non-differentiable fractal function provided a fractal function of non-differentiable nature. Similar to this construction, one can generate the composite α-fractal spline and explore its corresponding fractal operator.*

Remark 5. *Encompassing the recent trend of fractional calculus, one can investigate the fractional integral and fractional derivative of composite fractal functions as well as verify for the resultant functions to be again attractors of new IFS.*

CONCLUSION

As the fractional integral of a continuous function $(\mathcal{I}^v f)$ enjoys the continuity, a new family of fractal functions $(\mathcal{I}^v f)^\alpha$ is generated in the present paper. In this regard, a fractal operator is also proposed and its bound is estimated as $\left(1 + \frac{2|\alpha|_\infty}{1-|\alpha|_\infty}\right) \mathcal{K}\|f\|_\infty$, where $\mathcal{K} = \frac{x_{N+1}-x_1}{v\Gamma(v)}$, with the proper choice of bounded linear base function. In addition, the composition of α-fractal function is discussed. The concept of composition operation is studied to the case of differentiable fractal function $h^{(k)}$. The composition of differentiable fractal function $h^{(k)}$ with a non-differentiable fractal function g yielded a non-differentiable fractal function $g(h^{(k)})$ satisfying the necessary end point conditions. The composite fractal functions can be employed for approximating complex real data generated from multiple functions. For instance, in engineering the composite functions can establish a concrete relationship between different physical quantities, especially in unit conversions.

Availability of data and material

Not applicable.

Conflicts of interest

The author declares that there is no conflict of interest regarding the publication of this paper.

Ethical standard

The author has no relevant financial or non-financial interests to disclose.

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