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# Some $k$-Horn hypergeometric functions and their properties

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**Keywords**
*k-Gamma function, k-Hypergeometric function, k-Gauss hypergeometric function, Recursion formula*

**Abstract —** In the theory of special functions, the $k$-Pochhammer symbol is a generalization of the Pochhammer symbol. With the help of the $k$-Pochhammer symbol, we introduce and study a new generalization of the $k$-Horn hypergeometric functions such as, $G_1^k$, $G_2^k$ and $G_3^k$. Furthermore, several investigations have been carried out for some important recursion formulae for several one variable and two variables $k$-hypergeometric functions. In the light of these studies, we introduce some important recursion formulae for several newly generalized $k$-Horn hypergeometric functions. Finally, we present several relations between some $k$-Horn hypergeometric functions $G_1^k$, $G_2^k$ and $G_3^k$, and $k$-Gauss hypergeometric functions ${}_2F_1^k$.

**Subject Classification (2020):** 33C20, 60B99

## 1. Introduction

The theory of hypergeometric functions, a subfield of special functions, has developed in many ways and has been useful in the study of numerous useful properties in various fields such as mathematics, physics and engineering. It usually arises as a solution for many applied sciences such as electromagnetic, acoustic, thermal energy flow. Quite a number of studies have been published in the literature related to this field of study, for more details, see [1–6]. A very important achievement of the theory of hypergeometric functions of one variable motivated to further the development of the theory of multivariable hypergeometric functions. The study of multivariable hypergeometric functions has led to an increased interest in the study of multivariable hypergeometric functions in number theory, Lie theory algebras, group theory, representation theory, algebraic geometry and combinatorics, etc., and has gained popularity as a result of these advances [7–13].

Throughout this paper, we use the notations $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $\mathbb{N} = \{1, 2, 3, \dots\}$. The $k$-Gamma function can be defined [14] by

$$\Gamma_k(\omega) = \int_0^\infty t^{\omega-1} \exp(-\frac{t^k}{k})dt, \quad \Re(\omega) > 0 \tag{1.1}$$

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For $\nu \in \mathbb{C}$, the $k$-Pochhammer symbol $(\nu)_{m,k}$ is defined by [14,15]

$$(\nu)_{m,k} = \begin{cases} \prod_{r=0}^{m-1} (\nu + rk) = \dfrac{\Gamma_k(a+rk)}{\Gamma_k(\nu)}, & m \in \mathbb{N} \text{ and } k \geq 1 \\ 1, & m, k \leq 0 \end{cases} \tag{1.2}$$

and its well-known version is given as follow [15]:

$$(\nu)_{-m,k} = \frac{1}{\prod_{r=0}^{m-1}(\nu - rk)} = \frac{(-1)^n}{\prod_{r=0}^{m-1}(rk - \nu)}$$

Over the past decade, there have been several investigations on generalizations of one variable and multivariables hypergeometric functions (see, [16–20]). In addition, Horn hypergeometric functions, one of these hypergeometric functions, have recently been investigated extensively. For example, Şahin and Agha [21] introduced the recursion formalae for $G_1$ and $G_2$ Horn hypergeometric functions. Younis [22] establihed the Euler-type integral representations for Horn hypergeometric functions. Besides, Shehata and Moustafa [23] studied certain interesting applications for $G_1$, $G_2$, and $G_3$ Horn hypergeometric functions such as recursion relations, differential recursion formulae, integral operators, infinite summations. Shehata and Moustafa [24] demonstrated recursion formulas, differential and integral operators, integration formulas, and infinite summation for Horn hypergeometric functions $H_1$, $H_2$, $H_3$, $H_4$, $H_5$, $H_6$, and $H_7$.

The Horn functions $G_1$, $G_2$, and $G_3$ are introduced and investigated by Horn [25] as follow:

$$G_1(\mu, \nu, \kappa; z, w) = \sum_{x,y=0}^{\infty} (\mu)_{x+y}(\nu)_{x-y}(\kappa)_{y-x} \frac{z^x}{x!} \frac{w^y}{y!}, \quad |z| < r, \ |y| < s, \text{ and } r + s < 1 \tag{1.3}$$

$$G_2(\mu, \nu, \kappa, \sigma; z, w) = \sum_{x,y=0}^{\infty} (\mu)_x(\nu)_y(\kappa)_{x-y}(\sigma)_{y-x} \frac{z^x}{x!} \frac{w^y}{y!}, \quad |z| < 1 \text{ and } |y| < 1 \tag{1.4}$$

and

$$G_3(\mu, \nu; z, w) = \sum_{x,y=0}^{\infty} (\mu)_{2y-x}(\nu)_{2x-y} \frac{z^x}{x!} \frac{w^y}{y!}, \quad |z| < r, \ |y| < s, \text{ and } 27r^2s^2 + 18rs \pm 4(r - s) = 1 \tag{1.5}$$

Furthermore, with the help of the Horn functions expressed in the above equations and the $k$-Pochhammer symbol (1.2), we present the following newly defined $k$-Horn functions.

**Definition 1.1.** For $k \in \mathbb{R}^+$, $\mu, \nu, \kappa, \sigma \in \mathbb{C}$, the following $k$-Horn functions hold true:

$$G_1^k(\mu, \nu, \kappa; z, w) = \sum_{x,y=0}^{\infty} (\mu)_{x+y,k}(\nu)_{x-y,k}(\kappa)_{y-x,k} \frac{z^x}{x!} \frac{w^y}{y!}, \quad |z| < r, \ |y| < s, \text{ and } r + s < 1 \tag{1.6}$$

$$G_2^k(\mu, \nu, \kappa, \sigma; z, w) = \sum_{x,y=0}^{\infty} (\mu)_{x,k}(\nu)_{y,k}(\kappa)_{x-y,k}(\sigma)_{y-x,k} \frac{z^x}{x!} \frac{w^y}{y!}, \quad |z| < 1, \text{ and } |y| < 1 \tag{1.7}$$

and

$$G_3^k(\mu, \nu; z, w) = \sum_{x,y=0}^{\infty} (\mu)_{2y-x,k}(\nu)_{2x-y,k} \frac{z^x}{x!} \frac{w^y}{y!}, \quad |z| < r, \ |y| < s, \text{ and } 27r^2s^2 + 18rs \pm 4(r - s) = 1 \tag{1.8}$$

The main purpose of this paper is to introduce and investigate $k$-Horn hypergeometric functions with the help of the $k$-Pochhammer symbol (1.2), such as $G_1^k$, $G_2^k$, and $G_3^k$. Besides, we present some important recursion formulae for the some $k$-Horn hypergeometric functions. Moreover, we draw attention to the relation between some $k$-Horn hypergeometric functions and $k$-Gauss hypergeometric function.

## 2. Recursion Formulae of $G_1^k$ Function

This section presents some important recursion formulae of the $G_1^k$ hypergeometric function.

**Theorem 2.1.** The following formulae for the $G_1^k$ hold true:

$$\begin{aligned} G_1^k(a+rk,b,c;z,w) &= G_1^k(a,b,c;z,w) + wbk(c)_{-1,k}\sum_{t=1}^{r} G_1^k(a+tk,k+,c-;z,w) \\ &\quad + zck(b)_{-1,k}\sum_{t=1}^{r} G_1^k(a+tk,b-,c+;z,w) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} G_1^k(a-rk,b,c;z,w) &= G_1^k(a,b,c;z,w) - wbk(c)_{-1,k}\sum_{t=1}^{r} G_1^k(a-tk,b+,c-;z,w) \\ &\quad - zck(b)_{-1,k}\sum_{t=1}^{r} G_1^k(a-tk,b-,c+;z,w) \end{aligned} \tag{2.2}$$

where $r \in \mathbb{N}$.

**Proof.**

From the definition of the function $G_1^k$ and taking advantage of the following property

$$(\nu+k)_{m,k} = (1+\frac{mk}{\nu})(\nu)_{m,k} \tag{2.3}$$

we have

$$\begin{aligned} G_1^k(a+,b,c;z,w) &= \sum_{n,m=0}^{\infty} (a+k)_{m+n,k}(b)_{m-n,k}(c)_{n-m,k}\frac{z^n}{n!}\frac{w^m}{m!} \\ &= \sum_{n,m=0}^{\infty} \frac{(a+(m+n)k)}{a}(a)_{m+n,k}(b)_{m-n,k}(c)_{n-m,k}\frac{z^n}{n!}\frac{w^m}{m!} \\ &= \sum_{n,m=0}^{\infty} \left(1+\frac{mmk}{a}+\frac{nk}{a}\right)(a)_{m+n,k}(b)_{m-n,k}(c)_{n-m,k}\frac{z^n}{n!}\frac{w^m}{m!} \end{aligned} \tag{2.4}$$

By making the necessary index conversion, we obtain

$$\begin{aligned} G_1^k(a+,b,c;z,w) &= G_1^k(a,b,c;z,w) + \sum_{n,m=0}^{\infty}(m+1)k(a+k)_{m+n+1,k}(b)_{m-n+1,k}(c)_{n-m-1,k}\frac{zz^n}{n!}\frac{w^{m+1}}{(m+1)!} \\ &\quad + \sum_{n,m=0}^{\infty}(m+1)kk(a+k)_{m+n+1,k}(b)_{m-n-1,k}(c)_{n-m+1,k}\frac{z^{n+1}}{(n+1)!}\frac{w^m}{m!} \\ &= G_1^k(a,b,c;z,w) + bwk(c)_{-1,k}\sum_{n,m=0}^{\infty}(a+k)_{m+n,k}(b+k)_{m-n,k}(c-k)_{n-m,k}\frac{z^n}{n!}\frac{w^m}{m!} \\ &\quad + czk(b)_{-1,k}\sum_{n,m=0}^{\infty}(a+k)_{m+n,k}(b-k)_{m-n,k}(c+k)_{n-m,k}\frac{z^n}{n!}\frac{w^m}{m!} \\ &= G_1^k(a,b,c;z,w) + bwk(c)_{-1,k}G_1^k(a+,b+,c-;z,w) + czk(b)_{-1,k}G_1^k(a+,b-,c+;z,w) \end{aligned} \tag{2.5}$$

By utilising the relationship expressed above to the function $G_1^k$ with the parameter $a+2k$, we get

$$\begin{aligned} G_1^k(a+2k,b,c;z,w) &= G_1^k(a+k,b,c;z,w) + wbk(c)_{-1,k}G_1^k(a+2k,b+,c-;z,w) \\ &\quad + zck(b)_{-1,k}G_1^k(a+2k,b-,c+;z,w) \end{aligned} \tag{2.6}$$

Computing the function $G_1^k$ with the parameter $a+rk$ by (2.5) for $r$ times, we obtain the equality given by (2.1). Setting $a$ by $a-k$ in the (2.5), we have

$$G_1^k(a-k,b,c;z,w) = G_1^k(a,b,c;z,w) - wbk(c)_{-1,k}G_1^k(a,b+,c-;z,w) - zck(b)_{-1,k}G_1^k(a,b-,c+;z,w) \tag{2.7}$$

Using the above-mentioned equation to the function $G_1^k$ with the parameter $a-rk$ for $r$ times, we get (2.1). Since, the proof of (2.2) can be yielded similarly to the proof of (2.1), the details are omitted. □

The proofs of the following theorems Theorems 2.2 and 2.3 are easily obtained using a similar strategies as the proof of Theorem 2.1. Because of this, the details were omitted.

**Theorem 2.2.** The following formulae for the $G_1^k$ hold true:

$$\begin{aligned} G_1^k(a,b+rk,c;z,w) &= G_1^k(a,b,c;z,w) + awk(c)_{-1,k}\sum_{t=1}^{r} G_1^k(a+,b+tk,c-;z,w) \\ &\quad - aczk\sum_{t=1}^{r}\frac{(b+(t-1)k)_{-1,k}}{b+(t-1)k}G_1^k(a+,b+(t-2)k,c+;z,w) \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} G_1^k(a,b-rk,c;z,w) &= G_1^k(a,b,c;z,w) - awk(c)_{-1,k}\sum_{t=1}^{r} G_1^k(a+,b-(t-1)k,c-;z,w) \\ &\quad + aczk\sum_{t=1}^{r}\frac{(b-tk)_{-1,k}}{(b-tk)}G_1^k(a+,b-(t+1)k,c+;z,w) \end{aligned} \tag{2.9}$$

where $r \in \mathbb{N}$.

**Theorem 2.3.** The following formulae for the $G_1^k$ hold true:

$$\begin{aligned} G_1^k(a,b,c+rk;z,w) &= G_1^k(a,b,c;z,w) - abwk\sum_{t=1}^{r}\frac{(c+(t-1)k)_{-1,k}}{c+(t-1)k}G_1^k(a+,b+,c+(t-2)k;z,w) \\ &\quad + azk(b)_{-1,k}\sum_{t=1}^{r} G_1^k(a+,b-,c+tk;z,w) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} G_1^k(a,b,c-rk;z,w) &= G_1^k(a,b,c;z,w) - azk(b)_{-1,k}\sum_{t=1}^{r} G_1^k(a+,b-,c-(t-1)k;z,w) \\ &\quad + abwk\sum_{t=1}^{r}\frac{(c-tk)_{-1,k}}{(c-tk)}G_1^k(a+,b+,c-(t+1)k;z,w) \end{aligned} \tag{2.11}$$

where $r \in \mathbb{N}$.

**Theorem 2.4.** The $G_1^k$ satisfies the following formulae:

$$G_1^k(a+rk,b,c;z,w) = \sum_{t=0}^{r}\sum_{s=0}^{r-t}\binom{r}{t}\binom{r-t}{s}(zk)^s(wk)^t.(b)_{t-s,k}(c)_{s-t,k}G_1^k(a+(t+s)k,b+(t-s)k,c+(s-t)k;z,w) \tag{2.12}$$

**Proof.**

By using the induction method, we try to prove the proof of the given by (2.12). For $r=1$, (2.12) is satisfied. Suppose that the result (2.12) is true for $r=\varepsilon$. Show that the relation (2.12) is ensured for $r=\varepsilon+1$. Taking $r=\varepsilon$ in (2.12), we obtain

$$G_1^k(a+\varepsilon k,b,c;z,w) = \sum_{t=0}^{\varepsilon}\sum_{s=0}^{\varepsilon-t}\binom{\varepsilon}{t}\binom{\varepsilon-t}{s}(zk)^s(wk)^t.(b)_{t-s,k}(c)_{s-t,k}G_1^k(a+(t+s)k,b+(t-s)k,c+(s-t)k;z,w) \tag{2.13}$$

Applying $a$ by $a+k$ in the above relation, we have

$$G_1^k(a+(\varepsilon+1)k,b,c;z,w)=\sum_{t=0}^{\varepsilon}\sum_{s=0}^{\varepsilon-t}\binom{\varepsilon}{t}\binom{\varepsilon-t}{s}(zk)^s(wk)^t.(b)_{t-s,k}(c)_{s-t,k}G_1^k(a+(t+s+1)k,b+(t-s)k,c+(s-t)k;z,w) \quad (2.14)$$

In the aforementioned equation, applying (2.6) with the transformations $a\to a+(t+s)k$, $b\to b+(t-s)k$, and $c\to c+(s-t)k$. Using the relations

$$\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$$

and

$$\binom{n}{k}=0,\ \text{when } k>n \text{ or } k<0$$

and with some simplifications, we deduce the desired result (2.12). □

**Theorem 2.5.** The following formula holds for the function $G_1^k$:

$$G_1^k(a-rk,b,c;z,w)=\sum_{t=0}^{r}\sum_{s=0}^{r-t}\binom{r}{t}\binom{r-t}{s}(-zk)^s(-wk)^t.(b)_{t-s,k}(c)_{s-t,k}\,G_1^k(a+(t+s)k,b+(t-s)k,c+(s-t)k;z,w) \quad (2.15)$$

**Proof.**

Since the proof of Theorem 2.5 has a procedure similar to the induction method in the proof of Theorem 2.4, the proof is omitted. □

## 3. Recursion Formulae of $G_2^k$ and $G_3^k$

This section gives several recursion formulae for the function $G_2^k$ and $G_3^k$. We firstly represent the recursion formulae for the function $G_2^k$ and $G_3^k$ about the parameter $a$ and $\lambda$. The proof of the following theorems can be yielded same as the Theorems 2.1-2.4. Therefore, we omit the details.

**Theorem 3.1.** The following formulae hold true for $G_2^k$:

$$G_2^k(a+rk,\lambda,b,c;z,w)=G_2^k(a,\lambda,b,c;z,w)+bwk(c)_{-1,k}\sum_{t=1}^{r}G2^k(a+tk,\lambda,b+,c-;z,w) \quad (3.1)$$

and

$$G_2^k(a-rk,\lambda,b,c;z,w)=G_2^k(a,\lambda,b,c;z,w)-bwk(c)_{-1,k}\sum_{t=1}^{r}G_2^k(a-(t-1)k,\lambda,b+,c-;z,w) \quad (3.2)$$

**Theorem 3.2.** The following recursion formulae hold true for $G_2^k$ s:

$$G_2^k(a,\lambda+rk,b,c;z,w)=G_2^k(a,\lambda,b,c;z,w)+zck(b)_{-1,k}\sum_{t=1}^{r}G_2^k(a,\lambda+tk,b-,c+;z,w) \quad (3.3)$$

and

$$G_2^k(a,\lambda-rk,b,c;z,w)=G_2^k(a,\lambda,b,c;z,w)-zck(b)_{-1,k}\sum_{t=1}^{r}G_2^k(a,\lambda-(t-1)k,b-,c+;z,w) \quad (3.4)$$

We present the recursion formulae of the function $G_2^k$ about the parameter $b$ and $c$.

**Theorem 3.3.** Recursion formulae for the function $\mathrm{G}_2^k$ are as follow:

$$\begin{aligned} G_2^k(a,\lambda,b+rk,c;z,w) &= G_2^k(a,\lambda,b,c;z,w) + awk(c)_{-1,k}\sum_{t=1}^{r} G_2^k(a+,\lambda,b+tk,c-;z,w) \\ &\quad - \lambda czk\sum_{t=1}^{r}\frac{(b+(t-1)k)_{-1,k}}{(b+(t-1)k)} G_2^k(a,\lambda+,b-(t-2)k,c+;z,w) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} G_2^k(a,\lambda,b-rk,c;z,w) &= G_2^k(a,\lambda,b,c;z,w) - awk(c)_{-1,k}\sum_{t=1}^{r} G_2^k(a+,\lambda,b-(t-1)k,c-;z,w) \\ &\quad + \lambda czk\sum_{t=1}^{r}\frac{(b-tk)_{-1,k}}{(b-tk)} G_2^k(a,\lambda+,b-(t+1)k,c+;z,w) \end{aligned} \tag{3.6}$$

**Theorem 3.4.** The following recursion formulae hold true for $G_2^k$:

$$\begin{aligned} G_2^k(a,\lambda,b,c+rk;z,w) &= G_2^k(a,\lambda,b,c;z,w) + \lambda zk(b)_{-1,k}\sum_{t=1}^{r} G_2^k(a,\lambda+,b-,c+tk;z,w) \\ &\quad - abwk\sum_{t=1}^{r}\frac{(c+(t-1)k)_{-1,k}}{(c+(t-1)k)} G_2^k(a+,\lambda,b+,c+(t-2)k;z,w) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} G_2^k(a,\lambda,b,c-rk;z,w) &= G_2^k(a,\lambda,b,c;z,w) - \lambda zk(b)_{-1,k}\sum_{t=1}^{r} G_2^k(a,\lambda+,b-,c-(t-1)k;z,w) \\ &\quad + abwk\sum_{t=1}^{r}\frac{(c-tk)_{-1,k}}{(c-tk)} G_2^k(a+,\lambda,b+,c-(t+1)k;z,w) \end{aligned} \tag{3.8}$$

**Theorem 3.5.** The following recursion formulae hold true for $G_3^k$:

$$\begin{aligned} G_3^k(\alpha+rk,\lambda;z,w) &= G_3^k(\alpha,\lambda;z,w) + 2zk(\lambda)_{-1,k}\sum_{t=1}^{r}(\alpha+tk) G_3^k(\alpha+(t+1)k,\lambda-;z,w) \\ &\quad - wk\lambda(\lambda+k)\sum_{t=1}^{r}\frac{(\alpha+(t-1)k)_{-1,k}}{\alpha+(t-1)k} G_3^k(\alpha+(t-2)k,\lambda+2k;z,w) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} G_3^k(\alpha-rk,\lambda;z,w) &= G_3^k(\alpha,\lambda;z,w) - 2zk(\lambda)_{-1,k}\sum_{t=1}^{r}(\alpha-(t-1)k) G_3^k(\alpha-(t-2)k,\lambda-;z,w) \\ &\quad + wk\lambda(\lambda+k)\sum_{t=1}^{r}\frac{(\alpha-tk)_{-1,k}}{(\alpha-tk)} G_3^k(\alpha-(t+1)k,\lambda+2k;z,w) \end{aligned} \tag{3.10}$$

**Theorem 3.6.** The following recursion formulae hold true for $G_3^k$:

$$\begin{aligned} G_3^k(\alpha,\lambda+rk;z,w) &= G_3^k(\alpha,\lambda;z,w) + 2wk(\alpha)_{-1,k}\sum_{t=1}^{r}(\lambda+tk) G_3^k(\alpha-,\lambda+(t+1)k;z,w) \\ &\quad - zk\alpha(\alpha+k)\sum_{t=1}^{r}\frac{(\lambda+(t-1)k)_{-1,k}}{(\lambda+(t-1)k)} G_3^k(\alpha+2k,\lambda+(t-2)k;z,w) \end{aligned} \tag{3.11}$$

and

$$G_3^k(\alpha,\lambda-rk;z,w)=G_3^k(\alpha,\lambda;z,w)-\frac{2wk}{(\alpha-k)}\sum_{t=1}^{r}(\lambda-(t-1)k)G_3^k(\alpha-,\lambda-(t-2)k;z,w) + \sum_{t=1}^{r}\frac{\alpha(\alpha+k)zk}{(\lambda-tk)(\lambda-(t+1)k)}G_3^k(\alpha+2k,\lambda-(t+1)k;z,w) \tag{3.12}$$

## 4. Expression of $k$-Horn Functions in terms of $k$-Gauss Hypergeometric Function

This section gives the several important equations $k$-Horn functions which can be represented as $k$-Gauss hypergeometric function ${}_2F_1^k$.

**Theorem 4.1.** The connections between $k$-Gauss hypergeometric function of $G_1^k, G_2^k$, and $G_3^k$ are

$$G_1^k(\alpha,\beta,\gamma;z,w)=\sum_{x=0}^{\infty}(\alpha)_{x,k}\,(\beta)_{x,k}\,(\gamma)_{-x,k}\,\frac{w^x}{x!}\,{}_2F_1^k\Big[\alpha+xk,\gamma-xk;k-\beta-xk;-z\Big] \tag{4.1}$$

$$G_1^k(\alpha,\beta,\gamma;z,w)=\sum_{y=0}^{\infty}(\alpha)_{y,k}\,(\beta)_{-y,k}\,(\gamma)_{y,k}\,\frac{z^y}{y!}\,{}_2F_1^k\Big[\alpha+yk,\beta-yk;k-\gamma-yk;-w\Big] \tag{4.2}$$

$$G_2^k(\alpha,\lambda,\beta,\gamma;z,w)=\sum_{x=0}^{\infty}(\alpha)_{x,k}\,(\beta)_{x,k}\,(\gamma)_{-x,k}\,\frac{w^x}{x!}\,{}_2F_1^k\Big[\lambda,\gamma-xk;k-\beta-xk;-z\Big] \tag{4.3}$$

$$G_2^k(\alpha,\lambda,\beta,\gamma;z,w)=\sum_{y=0}^{\infty}(\lambda)_{y,k}\,(\beta)_{-y,k}\,(\gamma)_{y,k}\,\frac{z^y}{y!}\,{}_2F_1^k\Big[\alpha,\beta-yk;k-\gamma-yk;-w\Big] \tag{4.4}$$

$$G_3^k(\alpha,\lambda;z,w)=\sum_{x=0}^{\infty}(\alpha)_{-x,k}(\lambda)_{2x,k}\frac{w^x}{x!}\,{}_2F_1^k\left[\frac{\alpha-xk}{2},\frac{\alpha-xk+k}{2};k-\lambda-2xk;-4z\right] \tag{4.5}$$

and

$$G_3^k(\alpha,\lambda;z,w)=\sum_{y=0}^{\infty}(\alpha)_{2y,k}(\lambda)_{-y,k}\frac{z^y}{y!}\,{}_2F_1^k\left[\frac{\lambda-yk}{2},\frac{\lambda-yk+k}{2};k-\lambda-2yk;-4w\right] \tag{4.6}$$

**Proof.**

From the definition of the function $G_1^k$ (1.6) and using the following properties

$$(\varepsilon)_{x+y,k} = (\varepsilon)_{x,k}\,(\varepsilon+xk)_{y,k}$$

$$(\varepsilon)_{x-y,k} = (\varepsilon)_{x,k}\,(\varepsilon+xk)_{-y,k}$$

$$(\varepsilon)_{x-y,k} = \frac{(-1)^n\,(\varepsilon)_{x,k}}{(k-\varepsilon-xk)_{yk}}$$

$$(\varepsilon+k)_{x,k}=\left(1+\frac{xk}{\varepsilon}\right)(\varepsilon)_{x,k}$$

and

$$(\varepsilon)_{2x,k} = 2^{2x}\left(\frac{\varepsilon}{2}\right)_{x,k}\left(\frac{\varepsilon+k}{2}\right)_{x,k}$$

for $\varepsilon \in \mathbb{C}$, $x,\ y \in \mathbb{Z}^+$, and $k \in \mathbb{R}^+$, we have

$$\begin{aligned} G_1^k(\alpha,\beta,\gamma;z,w) &= \sum_{n,m=0}^{\infty} (\alpha)_{m,k}(\alpha+mk)_{n,k}\frac{(-1)^n(\beta)_{m,k}}{(k-\beta-mk)_{n,k}}(\gamma)_{-m,k}(\gamma-mk)_{n,k}\frac{z^n}{n!}\frac{w^m}{m!} \\ &= \sum_{m=0}^{\infty}(\alpha)_{m,k}(\beta)_{m,k}(\gamma)_{-m,k}\frac{w^m}{m!}\sum_{n=0}^{\infty}\frac{(\alpha+mk)_{n,k}(\gamma-mk)_{n,k}}{(k-\beta-mk)_{n,k}}\frac{(-z)^n}{n!} \\ &= \sum_{m=0}^{\infty}(\alpha)_{m,k}(\beta)_{m,k}(\gamma)_{-m,k}\frac{w^m}{m!}\,{}_2F_1^k\Big[\alpha+mk,\gamma-mk;k-\beta-mk-z\Big] \end{aligned} \tag{4.7}$$

Using same methodology, we obtain the relations (4.2)-(4.6). □

**Theorem 4.2.** The variable tranformation formulae for the functions $G_1^k$, $G_2^k$, and $G_3^k$ are:

$$\begin{aligned} G_1^k(\alpha,\beta,\gamma;z,w) &= \frac{\Gamma_k(k-\gamma)(1+kz)^{-(\frac{\alpha}{k})}}{\Gamma_k(\alpha)\Gamma_k(k-\gamma-\alpha)}\sum_{x=0}^{\infty}B_k(\alpha+xk,k-\gamma-\alpha)\frac{(\beta)_{x,k}}{x!}\left(\frac{-w}{1+kz}\right)^x \\ &\quad \cdot {}_2F_1^k\left[\alpha+xk,k-\gamma-\beta;k-\beta-xk;\frac{z}{1+kz}\right] \end{aligned} \tag{4.8}$$

$$\begin{aligned} G_1^k(\alpha,\beta,\gamma;z,w) &= \frac{\Gamma_k(k-\beta)(1+kw)^{-(\frac{\alpha}{k})}}{\Gamma_k(\alpha)\Gamma_k(k-\beta-\alpha)}\sum_{y=0}^{\infty}B_k(\alpha+yk,k-\beta-\alpha)\frac{(\beta)_{y,k}}{y!}\left(\frac{-z}{1+kw}\right)^y \\ &\quad \cdot {}_2F_1^k\left[\alpha+yk,k-\beta-\gamma;k-\gamma-yk;\frac{w}{1+kw}\right] \end{aligned} \tag{4.9}$$

$$\begin{aligned} G_2^k(\alpha,\lambda,\beta,\gamma;z,w) &= \frac{\Gamma_k(k-\gamma)(1+kz)^{-(\frac{\lambda}{k})}}{\Gamma_k(\alpha)\Gamma_k(k-\gamma-\alpha)}\sum_{x=0}^{\infty}B_k(\alpha+xk,k-\gamma-\alpha)\frac{(\beta)_{x,k}(-w)^x}{x!} \\ &\quad \cdot {}_2F_1^k\left[\lambda,k-\beta-\gamma;k-\beta-xk;\frac{-z}{1+kz}\right] \end{aligned} \tag{4.10}$$

$$\begin{aligned} G_2^k(\alpha,\lambda,\beta,\gamma;z,w) &= \frac{\Gamma_k(k-\beta)(1+kw)^{-(\frac{\alpha}{k})}}{\Gamma_k(\lambda)\Gamma_k(k-\gamma-\lambda)}\sum_{y=0}^{\infty}B_k(\lambda+yk,k-\beta-\lambda)\frac{(\gamma)_{y,k}(-z)^y}{y!} \\ &\quad \cdot {}_2F_1^k\left[\alpha,k-\beta-\gamma;k-\beta-yk;\frac{-w}{1+kw}\right] \end{aligned} \tag{4.11}$$

$$\begin{aligned} G_3^k(\alpha,\lambda;z,w) &= \sum_{x=0}^{\infty}\frac{B_k\left(\frac{\lambda}{2}+xk,k-\alpha-\frac{\lambda}{2}\right)}{B_k\left(\frac{\lambda}{2},k-\alpha-\frac{\lambda}{2}\right)}\frac{\left(\frac{\lambda+k}{2}\right)_{x,k}(-4w)^x}{x!(1+4z)^{\left(\frac{\alpha-xk+k}{2k}\right)}} \\ &\quad \cdot {}_2F_1^k\left[\frac{\alpha-xk+k}{2},k-\lambda-\frac{\alpha}{2}-\frac{3xk}{2};k-\lambda-2xk;\frac{4z}{1+4z}\right] \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} G_3^k(\alpha,\lambda;z,w) &= \sum_{y=0}^{\infty}\frac{B_k\left(\frac{\alpha}{2}+yk,k-\lambda-\frac{\alpha}{2}\right)}{B_k\left(\frac{\alpha}{2},k-\lambda-\frac{\alpha}{2}\right)}\frac{\left(\frac{\alpha+k}{2}\right)_{y,k}(-4z)^y}{y!(1+4w)^{\left(\frac{\lambda-yk+k}{2k}\right)}} \\ &\quad \cdot {}_2F_1^k\left[\frac{\lambda-yk+k}{2},k-\alpha-\frac{\lambda}{2}-\frac{3yk}{2};k-\alpha-2yk;\frac{4w}{1+4w}\right] \end{aligned} \tag{4.13}$$

**Proof.**

For convenience, denote the left-hand side of (4.8) by the expression $\mathcal{L}$ and using the following property

$$(\varepsilon + k)_{x,k} = \left(1 + \frac{xk}{\varepsilon}\right)(\varepsilon)_{x,k}$$

we get

$$\begin{aligned}
\mathcal{L} &= \sum_{x=0}^{\infty} (\alpha)_{x,k}(\beta)_{x,k}(\gamma)_{-x,k} \frac{w^x}{x!} \, {}_2F_1^k \left[\alpha + xk, \gamma - xk; k - \beta - xk - z\right] \\
&= \sum_{x=0}^{\infty} \frac{(\alpha)_{x,k}(\beta)_{x,k}(-w)^x}{(k-\gamma)_{x,k} x!} \sum_{y=0}^{\infty} \frac{(\alpha + xk)_{y,k}}{y!} \frac{\Gamma_k(\gamma - xk + yk)\Gamma_k(k - \beta - xk)}{\Gamma_k(\gamma - xk)\Gamma_k(k - \beta - xk + yk)} (-z)^y \qquad (4.14) \\
&= \sum_{x=0}^{\infty} \frac{(\alpha)_{x,k}(\beta)_{x,k}(-w)^x}{(k-\gamma)_{x,k} m!} \sum_{r=0}^{y} \frac{(\alpha + xk)_{r,k}(k - \gamma - \beta)_{r,k}(-1)^r}{(k - \beta - xk)_{r,k} r!} \sum_{y=0}^{\infty} \frac{(\alpha + xk + rk)_{y-r,k}}{(y-r)!}
\end{aligned}$$

Replacing $y$ by $y - r$ in contiguous relation, we obtain

$$\begin{aligned}
\mathcal{L} &= \sum_{x=0}^{\infty} \frac{(\alpha)_{x,k}(\beta)_{x,k}(-w)^x}{(k-\gamma)_{x,k} x!} (1 + kz)^{-(\frac{(\alpha+xk)}{k})} \, {}_2F_1^k \left[\alpha + xk, k - \gamma - \beta; k - \beta - xk; \frac{z}{1 + kz}\right] \\
&= \sum_{x=0}^{\infty} \frac{\Gamma_k(k-\gamma)B_k(\alpha + xk, k - \gamma - \alpha)}{\Gamma_k(\alpha)\Gamma_k(k - \gamma - \alpha)} \frac{(\beta)_{x,k}(-w)^x}{x!} . (1 + kz)^{-(\frac{(\alpha+xk)}{k})} \, {}_2F_1^k \left[\alpha + xk, k - \gamma - \beta; k - \beta - xk; \frac{z}{1 + kz}\right] \\
&= \frac{\Gamma_k(k-\gamma)}{\Gamma_k(\alpha)\Gamma_k(k - \gamma - \alpha)} \sum_{x=0}^{\infty} B_k(\alpha + xk, k - \gamma - \alpha) \frac{(\beta)_{x,k}(-w)^x}{x!} . (1 + kz)^{-(\frac{(\alpha+mk)}{k})} \, {}_2F_1^k \left[\alpha + xk, k - \gamma - \beta; k - \beta - xk; \frac{z}{1 + kz}\right] \\
&= \frac{\Gamma_k(k-\beta)(1 + kw)^{-(\frac{\alpha}{k})}}{\Gamma_k(\alpha)\Gamma_k(k - \beta - \alpha)} \sum_{x=0}^{\infty} B_k(\alpha + xk, k - \beta - \alpha) \frac{(\beta)_{x,k}}{x!} \left(\frac{-z}{1 + kw}\right)^x . \, {}_2F_1^k \left[\alpha + xk, k - \beta - \gamma; k - \gamma - xk \frac{w}{1 + kw}\right]
\end{aligned}$$

Therefore, the proof is completed. Using same methodology, we obtain the relations (4.9)-(4.13). □

## 5. Conclusion

In this work, $k$-Horn hypergeometric functions were first defined with the help of the $k$-Pochhammer symbol (1.2). Moreover, recursion formulae for $k$-Horn hypergeometric functions are obtained. In addition, some relations between some $k$-Horn functions and $k$-Gauss hypergeometric functions are analysed in detail.

Utilizing the arguments and definitions in this paper, the derivative formulae of the $k$-Horn functions defined in (1.6)-(1.8) can be obtained. Furthermore, the differential operator and integral operator can be applied to $k$-Horn functions in order to obtain innovative results in future work.

## Author Contributions

All the authors equally contributed to this work. This paper is derived from the first author's master's thesis supervised by the second and third authors. They all read and approved the final version of the paper.

## Conflicts of Interest

All the authors declare no conflict of interest.

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