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ON $k-$CONFORMABLE FRACTIONAL OPERATORS

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Abstract. In this study, we define the left and right fractional $k-$conformable integrals and derivatives. Furthermore, we obtained the fractional $k-$conformable derivatives of functions associated with some spaces and express their properties.

1. Introduction

Fractional calculus was born in 1695. Moreover, the significant of fractional calculus gained more and more over the years. This field is substantial not only in the field of mathematics, but especially in terms of applied sciences. Application of fractional calculus were by way of majority utilized in numerous fields of science and engineering. The most widely utilized were Caputo and Riemann-Liouville derivatives. The most common use fields of Riemann-Liouville are physics, mechanics, electronics, chemistry, biology, engineering and other fields$[3-7]$. The Riemann-Liouville approach base upon iterating $n-$times the integral operator and is the fractional integral of noninteger order. The core of the standard fractional calculus can not be enough us for the required kernel . Furthermore, we need required kernel in order to obtain unification of fractional derivatives in their studies $[8-9]$. Additionally, Differentiation operator is the most appropriate operator for a starting point for the iteration method. In this circumstances, Abdeljawad described the left and right generalized conformable derivatives, respectively [10] ,

$$_aT^{\alpha}f(x)=(x-a)^{1-\alpha}f'(x),$$

$$T_b^{\alpha}f(x)=(b-x)^{1-\alpha}f'(x).$$

In here, let f is a differentiable function, we have left and right integrals the following forms [1] ,

$$_a^{\beta}J^{\alpha}f(t)=\tfrac{1}{\Gamma(\beta)}\int_a^x\left(\tfrac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}f(t)\tfrac{dt}{(t-a)^{1-\alpha}}$$

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and

$$^{\beta}J_{b}^{\alpha}f\left(x\right)=\frac{1}{\Gamma(\beta)}\int_{a}^{x}\left(\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}f\left(t\right)\frac{dt}{(b-a)^{1-\alpha}}.$$

Respectively. In this point, Authors in [1] defined new fractional operators which have two parameters and also these operators have kernels different from usual kernels. In this article, we pay attention studies of depending on [1] and also we obtained new $k-$conformable fractional integrals and derivatives by way of new fractional operators in this paper. Additionally, we will give some basic definitions and tools related to classical fractional calculus.

Definition 1.1. $[16],[8]$ A real valued function $f(t),\ t>0$ is said to be in the space $C_{\mu},\ \mu\in\mathbb{R}$ if there exists a complex number $p>\mu$ such that $f(t)=t^{p}f_{1}\left(t\right),$ where $f_{1}(t)\in C\left[0,\infty\right].$

Definition 1.2. $[16],[8]$ A function $f(t)\in C_{\mu},\ t>0$ is said to be in the $L_{p,k}\left(a,b\right)$ space if

$$L_{p,k}\left(a,b\right)=\left\{f:\left\Vert f\right\Vert_{L_{p,k}(a,b)}=\left(\int_{a}^{b}\left|f(t)^{p}\right|t^{k}dt\right)^{\frac{1}{p}}<\infty,\ 1\leq p<\infty,\ k\geq0\right\}.$$

Definition 1.3. [16] Consider the space $X_{c}^{p}\left(a,b\right)\ (c\in\mathbb{R},\ 1\leq p<\infty)$ of those real-valued Lebesgue measurable functions f on $[a,b]$ for which

$$\left\Vert f\right\Vert=\left(\int_{a}^{b}\left|t^{c}f(t)\right|^{p}\frac{dt}{t}\right)^{\frac{1}{p}}<\infty,\ (1\leq p<\infty,\ cp\geq1)$$

and for the case $p=\infty$

$$\left\Vert f\right\Vert_{X_{c}^{\infty}}=ess\sup_{a\leq t<b}\left[t^{c}f\left(t\right)\right],\ c\geq0.$$

Additionally, If we take $c=\frac{k+1}{p}\ (1\leq p<\infty,\ k\geq0)$ the space $X_{c}^{p}\left(a,b\right),$ we have the $L_{p,k}\left(a,b\right)-$space. Moreover, If we take $c=\frac{1}{p}\ (1\leq p<\infty)$ the space $X_{c}^{p}\left(a,b\right)$,we have the $L^{p}\left(a,b\right)-$space[16].

Katugampola obtained the generalized left and right fractional integrals for $\beta\in\mathbb{C}$ and $\mathrm{Re}\left(\beta\right)>0$ in [8] :

$$\left({}_{a}I^{\beta,\alpha}f\right)(t)=\frac{1}{\Gamma(\beta)}\int_{a}^{t}\left(\frac{t^{\alpha}-y^{\alpha}}{\alpha}\right)^{\beta-1}f\left(y\right)\frac{dy}{y^{1-\alpha}}\tag{1.1}$$

and

$$\left(I_{b}^{\beta,\alpha}f\right)(t)=\frac{1}{\Gamma(\beta)}\int_{t}^{b}\left(\frac{y^{\alpha}-t^{\alpha}}{\alpha}\right)^{\beta-1}f\left(y\right)\frac{dy}{y^{1-\alpha}},\tag{1.2}$$

respectively.

The following forms are left and right generalized fractional derivatives for $\beta\in\mathbb{C}$ and $\mathrm{Re}\left(\beta\right)\geq0$ in [9] :

$$\begin{aligned}\left({}_{a}D^{\beta,\alpha}f\right)(t)&=\xi^{n}\left({}_{a}I^{n-\beta,\alpha}f\right)(t)\\&=\frac{\xi^{n}}{\Gamma(n-\beta)}\int_{a}^{t}\left(\frac{t^{\alpha}-y^{\alpha}}{\alpha}\right)^{n-\beta-1}f\left(y\right)\frac{dy}{y^{1-\alpha}}\end{aligned}\tag{1.3}$$

and

$$\begin{aligned}\left(D_b^{\beta,\alpha} f\right)(t) &= (-\xi)^n \left({}_aI^{n-\beta,\alpha} f\right)(t) \\ &= \frac{(-\xi)^n}{\Gamma(n-\beta)} \int_t^b \left(\frac{y^\alpha - t^\alpha}{\alpha}\right)^{n-\beta-1} f(y) \frac{dy}{y^{1-\alpha}},\end{aligned} \tag{1.4}$$

respectively, where $\alpha > 0$ and where $\xi = t^{1-\alpha} \frac{d}{dt}$.

The following forms are the left and right generalized Caputo fractional derivatives which defined by the authors in [15] by using [9] ,

$$\begin{aligned}\left({}_a^C D^{\beta,\alpha} f\right)(t) &= \left({}_aI^{n-\beta,\alpha} (\xi)^n f\right)(t) \\ &= \frac{1}{\Gamma(n-\beta)} \int_a^t \left(\frac{t^\alpha - u^\alpha}{\alpha}\right)^{n-\beta-1} \frac{\xi^n f(y) dy}{y^{1-\alpha}}\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}\left({}^C D_b^{\beta,\alpha} f\right)(t) &= \left({}_aI^{n-\beta,\alpha} (-\xi)^n f\right)(t) \\ &= \frac{1}{\Gamma(n-\beta)} \int_t^b \left(\frac{y^\alpha - t^\alpha}{\alpha}\right)^{n-\beta-1} \frac{(-\xi)^n f(y) dy}{y^{1-\alpha}}.\end{aligned} \tag{1.6}$$

Respectively.

Now, after giving $k-$ conformable fractional integral and derivatives, respectively, we will demonstrate important consequences and some basic properties for these operators. Furthermore, we will obtain the properties of the defined $k-$conformable derivative and also we will acquire the $k-$conformable fractional derivatives on the Caputo setting. In conclusion, we will develop the previously obtained results for the generalized conformable derivatives and integrals.

2. THE $k-$CONFORMABLE FRACTIONAL OPERATORS

In this part, Abdeljawad defined the conformable integrals and we expanded to higher order in [10] . Furthermore, Jarad and et al. defined fractional integrals in [1] . Now, by considering these studies, we should give the following $k-$conformable derivative by using definitions of conformable derivative,

$${}_a^hT^\alpha f(t) = \lim_{\varepsilon\to 0} \frac{f\left(t + \varepsilon \frac{\left(t^{k+1} - a^{k+1}\right)^{1-\alpha}}{t^k}\right) - f(t)}{\varepsilon}. \tag{2.1}$$

We should consider (2.1). In here,

$$\Delta t = \varepsilon \frac{\left(t^{k+1} - a^{k+1}\right)^{1-\alpha}}{t^k} \Rightarrow \varepsilon = \frac{\Delta t . t^k}{\left(t^{k+1} - a^{k+1}\right)^{1-\alpha}}. \tag{2.2}$$

We choose Δt in the form. Then,

$$\begin{aligned}{}_a^hT^\alpha f(t) &= \frac{\left(t^{k+1} - a^{k+1}\right)^{1-\alpha}}{t^k} \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \\ &= \frac{\left(t^{k+1} - a^{k+1}\right)^{1-\alpha}}{t^k} f'(t).\end{aligned} \tag{2.3}$$

We can state the left and right $k-$conformable derivatives, respectively, as

$$\begin{aligned}{}_a^hT^\alpha f(x) &= \left(\frac{t^{k+1} - a^{k+1}}{t^k}\right)^{1-\alpha} f'(x), \\ {}^hT_b^\alpha f(x) &= \left(\frac{b^{k+1} - t^{k+1}}{t^k}\right)^{1-\alpha} f'(x).\end{aligned} \tag{2.4}$$

Moreover, we obtain $k-$conformable integral operator. For this,

$$\int_a^x \frac{t_1^k dt_1}{\left(t_1^{k+1}-a^{k+1}\right)^{1-\alpha}} \int_a^{t_1} \frac{t_2^k dt_2}{\left(t_2^{k+1}-a^{k+1}\right)^{1-\alpha}} \cdots \int_a^{t_{n-1}} \frac{t_n^k f(t_n) dt_n}{\left(t_n^{k+1}-a^{k+1}\right)^{1-\alpha}}, \tag{2.5}$$

we should get $n-$times repeated integral of the forms. In addition, if we apply a method as in classic fractional integral techniques,

$$ {}_a^k J^{n,\alpha} f(x) = \frac{1}{\Gamma(n)} \int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha - \left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{n-1} \frac{t^k f(t) dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}. \tag{2.6}$$

We can write the equality. Furthermore, we can obtain definition of the following for $k-$conformable integrals with the help of this equality[2] .

Definition 2.1. Let $f \in X_c$. The left and right $k-$conformable fractional integrals of order $n \in \mathbb{C}$, $\mathrm{Re}(n) \geq 0$ and $\alpha > 0$,

$$ {}_a^k J^{n,\alpha} f(x) = \frac{1}{\Gamma(n)} \int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha - \left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{n-1} \frac{t^k f(t) dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \tag{2.7}$$

and

$$ {}^k J_b^{n,\alpha} f(x) = \frac{1}{\Gamma(n)} \int_x^b \left[\frac{\left(b^{k+1}-x^{k+1}\right)^\alpha - \left(b^{k+1}-t^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{n-1} \frac{t^k f(t) dt}{\left(b^{k+1}-t^{k+1}\right)^{1-\alpha}}, \tag{2.8}$$

respectively.

In here, we will give the following new definition by considering the $k-$conformable derivative and integral operators.

Definition 2.2. Let $f \in X_c$. The left and right $k-$conformable fractional derivatives of order $\beta \in \mathbb{C}$ and $\mathrm{Re}(\beta) \geq 0$,

$$\begin{aligned} {}_a^k D^{\beta,\alpha} f(x) &= {}_a^k T^{n,\alpha} \left({}_a^k J^{n-\beta,\alpha}\right) f(x) \\ &= \frac{{}_a^k T^{n,\alpha}}{\Gamma(n-\beta)} \int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha - \left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{n-\beta-1} \frac{t^k f(t) dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} {}^k D_b^{\beta,\alpha} f(x) &= {}^k T_b^{n,\alpha} \left({}^k J_b^{n-\beta,\alpha}\right) f(x) \\ &= \frac{{}^k T_b^{n,\alpha} (-1)^n}{\Gamma(n-\beta)} \int_x^b \left[\frac{\left(b^{k+1}-x^{k+1}\right)^\alpha - \left(b^{k+1}-t^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{n-\beta-1} \frac{t^k f(t) dt}{\left(b^{k+1}-t^{k+1}\right)^{1-\alpha}}, \end{aligned} \tag{2.10}$$

where $n = [\mathrm{Re}(\beta)] + 1$,

$$\begin{aligned} {}_a^k T^{n,\alpha} &= \underbrace{{}_a^k T^{\alpha} \; {}_a^k T^{\alpha} \cdots {}_a^k T^{\alpha}}_{n-times}, \\ {}^k T_b^{n,\alpha} &= \underbrace{{}^k T_b^{\alpha} \; {}^k T_b^{\alpha} \cdots {}^k T_b^{\alpha}}_{n-times}, \end{aligned} \tag{2.11}$$

and ${}_a^k T^{\alpha}$ and ${}^k T_b^{\alpha}$ are the left and right fractional $k-$conformable differential operators.

Theorem 2.3. *Let $f \in X_c$. Then, we get for fractional integrals for $Re(\beta) > 0$ and $Re(\gamma) > 0$,*

$$\begin{aligned} {}_a^kJ^{\beta,\alpha}\left({}_a^kJ^{\gamma,\alpha}\right)f(x) &= {}_a^kJ^{(\beta+\gamma),\alpha}f(x), \\ {}^kJ_b^{\beta,\alpha}\left({}^kJ_b^{\gamma,\alpha}\right)f(x) &= {}^kJ_b^{(\beta+\gamma),\alpha}f(x). \end{aligned} \tag{2.12}$$

Proof. We have with the aid of (2.7),

(2.13)

$$\begin{aligned} &{}_a^kJ^{\beta,\alpha}\left({}_a^kJ^{\gamma,\alpha}\right)f(x) \\ &= \frac{1}{\Gamma(\beta)}\int_a^x\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha-\left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{\beta-1}\frac{\left({}_a^kJ^{\gamma,\alpha}\right)t^kdt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ &= \frac{1}{\Gamma(\beta)\Gamma(\gamma)}\int_a^x\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha-\left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{\beta-1} \\ &\times\left(\int_a^t\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha-\left(u^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{\gamma-1}\frac{u^kf(u)du}{\left(u^{k+1}-a^{k+1}\right)^{1-\alpha}}\right)\frac{t^kf(t)dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ &= \frac{1}{\Gamma(\beta)\Gamma(\gamma)}\int_a^x\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha-\left(u^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{\beta+\gamma-1}\left(\int_0^1(1-z)^{\beta-1}z^{\gamma+1}dz\right)\frac{u^kf(u)du}{\left(u^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ &= \frac{1}{\Gamma(\beta+\gamma)}\int_a^x\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha-\left(u^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{\beta+\gamma-1}\frac{u^kf(u)du}{\left(u^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ &= {}_a^kJ^{(\beta+\gamma),\alpha}f(x). \end{aligned}$$

In here, we have used the change of variable,

$$\left(t^{k+1}-a^{k+1}\right)^\alpha = \left(u^{k+1}-a^{k+1}\right)^\alpha + z\left[\left(x^{k+1}-a^{k+1}\right)^\alpha-\left(u^{k+1}-a^{k+1}\right)^\alpha\right].$$

The second formula can be demonstrated in the same manner. □

Lemma 2.4. *Let $f \in X_c$. We have for $Re(v) > 0$,*

$$\begin{aligned} {}_a^kJ^{\beta,\alpha}\left(t^{k+1}-a^{k+1}\right)^{\alpha(v-1)}(x) &= \frac{\Gamma(v)}{\Gamma(\beta+v)}\frac{\left[\left(x^{k+1}-a^{k+1}\right)^\alpha\right]^{\beta+v-1}}{[\alpha(k+1)]^\beta}, \\ {}^kJ_b^{\beta,\alpha}\left(b^{k+1}-t^{k+1}\right)^{\alpha(v-1)}(x) &= \frac{\Gamma(v)}{\Gamma(\beta+v)}\frac{\left[\left(b^{k+1}-x^{k+1}\right)^\alpha\right]^{\beta+v-1}}{[\alpha(k+1)]^\beta}. \end{aligned} \tag{2.14}$$

Proof. We have with the aid of (2.7),

$$\begin{aligned} &{}_a^kJ^{\beta,\alpha}\left(t^{k+1}-a^{k+1}\right)^{\alpha(v-1)}(x) \\ &= \frac{1}{\Gamma(\beta)}\int_a^x\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha-\left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{\beta-1}\frac{\left[\left(t^{k+1}-a^{k+1}\right)^\alpha\right]^{v-1}t^kdt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ &= \frac{\left[\left(x^{k+1}-a^{k+1}\right)^\alpha\right]^{\beta+v-1}}{\Gamma(\beta)[\alpha(k+1)]^{\beta-1}}\int_0^1(1-z)^{\beta-1}z^{v-1}dz \\ &= \frac{\Gamma(v)}{\Gamma(\beta+v)}\frac{\left[\left(x^{k+1}-a^{k+1}\right)^\alpha\right]^{\beta+v-1}}{[\alpha(k+1)]^\beta}. \end{aligned} \tag{2.15}$$

In here, we have used the change of variable,

$$\left(t^{k+1}-a^{k+1}\right)^\alpha = z\left(x^{k+1}-a^{k+1}\right)^\alpha.$$

The second formula can be demonstrated in the same manner. □

Lemma 2.5. *Let $f \in X_c$. We have for $Re\,(n-\alpha) > 0$,*

$$\left[{}_{a}^{k}D^{\beta,\alpha}\left(t^{k+1}-a^{k+1}\right)^{\alpha(v-1)}\right](x) = \frac{[\alpha(k+1)]^{\beta}\Gamma(v)}{\Gamma(v-\beta)}\left[\left(x^{k+1}-a^{k+1}\right)^{\alpha}\right]^{v-\beta-1}, \tag{2.16}$$

$$\left[{}^{k}D_{b}^{\beta,\alpha}\left(b^{k+1}-t^{k+1}\right)^{\alpha(v-1)}\right](x) = \frac{[\alpha(k+1)]^{\beta}\Gamma(v)}{\Gamma(v-\beta)}\left[\left(b^{k+1}-x^{k+1}\right)^{\alpha}\right]^{v-\beta-1}.$$

Proof. We have with the aid of (2.9) ,

$$\begin{aligned} &\left[{}_{a}^{k}D^{\beta,\alpha}\left(t^{k+1}-a^{k+1}\right)^{\alpha(v-1)}\right](x) \\ &= \frac{{}_{a}^{k}T^{n,\alpha}}{\Gamma(n-\beta)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta-1}\frac{\left[\left(t^{k+1}-a^{k+1}\right)^{\alpha}\right]^{v-1}t^{k}dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ &= \frac{{}_{a}^{k}T^{n,\alpha}\left[\left(x^{k+1}-a^{k+1}\right)^{\alpha}\right]^{n+v-\beta-1}}{\Gamma(n-\beta)[\alpha(k+1)]^{n-\beta}}\int_{0}^{1}(1-z)^{n-\beta-1}z^{v-1}dz \\ &= \frac{[\alpha(k+1)]^{\beta}\Gamma(v)}{\Gamma(v-\beta)}\left[\left(x^{k+1}-a^{k+1}\right)^{\alpha}\right]^{v-\beta-1}. \end{aligned} \tag{2.17}$$

In here, we have used the change of variable

$$\left(t^{k+1}-a^{k+1}\right)^{\alpha} = z\left(x^{k+1}-a^{k+1}\right)^{\alpha}.$$

The second formula can be demonstrated in the same manner. □

Remark 2.6. It can be shown that

$$\begin{aligned} {}_{a}^{k}D^{\beta,\alpha}f &= {}_{a}^{k}J^{\beta,-\alpha}, \\ {}^{k}D_{b}^{\beta,\alpha}f &= {}^{k}J_{b}^{\beta,-\alpha}. \end{aligned} \tag{2.18}$$

3. $k-$CONFORMABLE FRACTIONAL DERIVATIVES ON THE CERTAIN SPACES

In this part, we will give some definitions with related to lemma and theorem. Moreover, we will demonstrate the substantial results of the $k-$conformable fractional derivatives on the space $C_{\alpha,a}^{n}$ and $C_{\alpha,b}^{n}$.

Definition 3.1. [10] For $0 < \alpha \leq 1$ and an interval $[a,b]$ define,

$$\begin{aligned} {}^{k}I_{\alpha}([a,b]) &= \Big\{f:[a,b]\to\mathbb{R}: f(x) = \left({}_{a}^{k}I^{\beta,\alpha}\varphi\right)(x) + f(a) \\ &\qquad \text{for some } \varphi \in {}^{k}L_{\alpha}(a)\Big\} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} {}_{\alpha}^{k}I([a,b]) &= \Big\{g:[a,b]\to\mathbb{R}: g(x) = \left({}^{k}I_{b}^{\beta,\alpha}\varphi\right)(x) + g(b) \\ &\qquad \text{for some } \varphi \in {}^{k}L_{\alpha}(b)\Big\}. \end{aligned} \tag{3.2}$$

Where

$$ {}^{k}L_{\alpha}(a) = \left\{\varphi:[a,b]\to\mathbb{R},\ \left({}_{a}^{k}I^{\beta,\alpha}\varphi\right)(x) \text{ exists } \forall x\in[a,b]\right\} \tag{3.3}$$

and

$$ {}^{k}L_{\alpha}(b) = \left\{\varphi:[a,b]\to\mathbb{R},\ \left({}^{k}I_{b}^{\beta,\alpha}\varphi\right)(x) \text{ exists } \forall x\in[a,b]\right\}. \tag{3.4}$$

Definition 3.2. We can clearly describe for $\alpha \in (0,1]$ and $n = 1,2,3,...,$

$$(3.5)\qquad \begin{aligned} C^n_{\alpha,a}([a,b]) &= \left\{ f : [a,b] \to \mathbb{R} \ such \ \ that \ {}^k_aT^{n-1,\alpha} f \in^k_a I^{\beta,\alpha}([a,b]) \right\}, \\ C^n_{\alpha,b}([a,b]) &= \left\{ f : [a,b] \to \mathbb{R} \ such \ \ that \ {}^kT^{n-1,\alpha}_b f \in^k I^{\beta,\alpha}_b([a,b]) \right\}. \end{aligned}$$

Lemma 3.3. *Let $f \in C^n_{\alpha,a}([a,b])$ and $\alpha > 0$. Then, f is presented in form,*

$$(3.6)\qquad \begin{aligned} f(x) = \tfrac{1}{(n-1)!} \int_a^x & \left[\tfrac{\left(x^{k+1}-a^{k+1}\right)^\alpha - \left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)} \right]^{n-1} \tfrac{\varphi(t) t^k dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ & + \textstyle\sum_{s=0}^{n-1} \left[\tfrac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)} \right]^s \tfrac{1}{s!} {}^k_aT^{s,\alpha} f(a). \end{aligned}$$

In this place is $\varphi(t) = \left({}^k_aT^{s,\alpha} f\right)(t)$.

Proof. Since $f \in C^n_{\alpha,a}([a,b])$, ${}^k_aT^{n-1,\alpha} f \in {}^kI_\alpha([a,b])$ and φ is continuous function, we have,

$$(3.7)\qquad \begin{aligned} {}^k_aT^{n-1,\alpha} f(x) &= \int_a^x \tfrac{\varphi(t) t^k dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} +^k_a T^{n-1,\alpha} f(a), \\ \tfrac{\left(x^{k+1}-a^{k+1}\right)^{1-\alpha}}{x^k} \tfrac{d}{dx} \ {}^k_aT^{n-2,\alpha} f(x) &= \int_a^x \tfrac{\varphi(t) t^k dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} +^k_a T^{n-1,\alpha} f(a) \\ \tfrac{d}{dx} \ {}^k_aT^{n-2,\alpha} f(x) &= \Bigg[\tfrac{x^k}{\left(x^{k+1}-a^{k+1}\right)^{1-\alpha}} \int_a^x \tfrac{\varphi(t) t^k dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ &= + \tfrac{x^k}{\left(x^{k+1}-a^{k+1}\right)^{1-\alpha}} \cdot {}^k_aT^{n-1,\alpha} f(a) \Bigg]. \end{aligned}$$

We integrate the both of side (3.7) from a to x by replacing $x \to t$ and $t \to s$ on the both side of the equation, then,

$$(3.8)\qquad \begin{aligned} {}^k_aT^{n-2,\alpha} f(x) &= \int_a^x \left[\tfrac{\left(x^{k+1}-a^{k+1}\right)^\alpha - \left(s^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)} \right] \tfrac{\varphi(s) s^k ds}{\left(s^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ &+ \tfrac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)} \cdot {}^k_aT^{n-1,\alpha} f(a) +^k_a T^{n-2,\alpha} f(a). \end{aligned}$$

By applying the equality same method once more, we get,

$$(3.9)\qquad \begin{aligned} {}^k_aT^{n-3,\alpha} f(x) = \int_a^x & \tfrac{1}{2} \left[\tfrac{\left(x^{k+1}-a^{k+1}\right)^\alpha - \left(s^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)} \right]^2 \tfrac{\varphi(s) s^k ds}{\left(s^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ & + \tfrac{1}{2} \left[\tfrac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)} \right]^2 \cdot {}^k_aT^{n-1,\alpha} f(a) \\ & + \tfrac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)} \cdot {}^k_aT^{n-2,\alpha} f(a) +^k_a T^{n-3,\alpha} f(a). \end{aligned}$$

If the same method is applied $n-3$ times, we have,

$$(3.10)\qquad \begin{aligned} f(x) &= \tfrac{1}{(n-1)!} \int_a^x \left[\tfrac{\left(x^{k+1}-a^{k+1}\right)^\alpha - \left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)} \right]^{n-1} \tfrac{\varphi(t) t^k dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ & + \textstyle\sum_{s=0}^{n-1} \left[\tfrac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)} \right]^s \tfrac{1}{s!} \cdot {}^k_aT^{s,\alpha} f(a). \end{aligned}$$

For $\varphi(t) =^k_a T^{n,\alpha} f(t)$. It is clear that a similar lemma for right $k-$conformable fractional derivative. ∎

Lemma 3.4. *Let $f \in C^n_{\alpha,b}([a,b])$ for $\alpha > 0$. Then, f is presented in form,*

$$f(x) = \frac{1}{(n-1)!}\int_x^b \left[\frac{\left(b^{k+1}-x^{k+1}\right)^\alpha - \left(b^{k+1}-t^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{n-1} \frac{\varphi(t)t^k dt}{\left(b^{k+1}-t^{k+1}\right)^{1-\alpha}} + \sum_{s=0}^{n-1}\left[\frac{\left(b^{k+1}-x^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^s \frac{(-1)^s}{s!} \cdot {}^kT_b^{s,\alpha} f(a). \tag{3.11}$$

For $\varphi(t) = \left({}^kT_b^{s,\alpha} f\right)(t)$.

Proof. The proof is likewise as *Lemma* 3. □

Now we will give $k-$conformable fractional derivatives on $C^n_{\alpha,a}$ and $C^n_{\alpha,b}$ in the theorem 2.

Theorem 3.5. *Let $\beta \in \mathbb{C}$, $Re(\beta) > 0$ and $n = [\beta] + 1$. The left and right $k-$conformable fractional derivatives are demonstrated in the form for $f \in C^n_{\alpha,a}$ and $f \in C^n_{\alpha,b}$. Then,*

$${}^k_aD^{\beta,\alpha} f(x) = \left({}^k_aJ^{n-\beta}\left({}^k_aT^{n,\alpha} f\right)\right)(x) + \sum_{m=0}^{n-1} \frac{{}^k_aT^{n,\alpha} f(a)}{\Gamma(m-\beta+1)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{m-\beta} \tag{3.12}$$

and

$${}^kD_b^{\beta,\alpha} f(x) = \left({}^kJ_b^{n-\beta}\left({}^kT_b^{n,\alpha} f\right)\right)(x) + \sum_{m=0}^{n-1} \frac{(-1)^m \cdot {}^kT_b^{n,\alpha} f(b)}{\Gamma(m-\beta+1)}\left[\frac{\left(b^{k+1}-t^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{m-\beta}. \tag{3.13}$$

Proof. By using $f \in C^n_{\alpha,a}([a,b])$, we should choose $f(x)$ in the *Lemma* 3 by replacing $x \to t$ and $t \to s$ that is as following form,

$$f(x) = \frac{1}{(n-1)!}\int_a^t \left[\frac{\left(t^{k+1}-a^{k+1}\right)^\alpha - \left(s^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{n-1} \frac{{}^k_aT^{n,\alpha} f(s)s^k ds}{\left(s^{k+1}-a^{k+1}\right)^{1-\alpha}} + \sum_{m=0}^{n-1}\left[\frac{\left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^m \frac{1}{m!} \cdot {}^k_aT^{m,\alpha} f(a). \tag{3.14}$$

In here, we can state the following equality by using (2.9) for (3.14),

(3.15)

$$\begin{aligned}
{}^k_aD^{\beta,\alpha} f(x) &= \frac{{}^k_aT^{n,\alpha}}{\Gamma(n-\beta)}\int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha - \left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{n-\beta-1} \frac{t^k f(t)dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\
&= \frac{{}^k_aT^{n,\alpha}}{\Gamma(n-\beta)}\int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha - \left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{n-\beta-1} \\
&\times \left(\frac{1}{(n-1)!}\int_a^t \left[\frac{\left(t^{k+1}-a^{k+1}\right)^\alpha - \left(s^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{n-1} \frac{{}^k_aT^{n,\alpha} f(s)s^k ds}{\left(s^{k+1}-a^{k+1}\right)^{1-\alpha}}\right) \frac{t^k f(t)dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\
&+ \frac{{}^k_aT^{n,\alpha}}{\Gamma(n-\beta)}\int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha - \left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{n-\beta-1} \\
&\times \left(\sum_{m=0}^{n-1}\left[\frac{\left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^m \frac{1}{m!} \cdot {}^k_aT^{m,\alpha} f(a)\right) \frac{t^k f(t)dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}.
\end{aligned}$$

We used changing the order of integration and gamma and beta functions. Additionally, we use the following the equations,

(3.16)

$$\left(t^{k+1}-a^{k+1}\right)^\alpha = \left(s^{k+1}-a^{k+1}\right)^\alpha + z\left[\left(x^{k+1}-a^{k+1}\right)^\alpha - \left(s^{k+1}-a^{k+1}\right)^\alpha\right]$$

and

$$\left(t^{k+1}-a^{k+1}\right)^{\alpha}=u\left(x^{k+1}-a^{k+1}\right)^{\alpha}.$$

We obtained following form,

(3.17)

$$\begin{aligned}{}_{a}^{k}D^{\beta,\alpha}f\left(x\right) &= \frac{{}_{a}^{k}T^{n,\alpha}}{\Gamma(n-\beta)(n-1)!}\int_{a}^{x}\frac{{}_{a}^{k}T^{n,\alpha}f(s)s^{k}ds}{(s^{k+1}-a^{k+1})^{1-\alpha}}\\ &\times\left(\int_{0}^{1}\left(1-z\right)^{n-\beta-1}\left(z\right)^{n-1}dz\right)\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{2n-\beta-1}\\ &+\sum_{m=0}^{n-1}\frac{{}_{a}^{k}T^{n,\alpha}.T^{n,\alpha}f(a)}{\Gamma(n-\beta).m!}\\ &\times\left(\int_{0}^{1}\left(1-u\right)^{n-\beta-1}\left(u\right)^{m}du\right)\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta+m}.\end{aligned}$$

In here, we obtain by means of operator ${}_{a}^{k}T^{n,\alpha}$,

(3.18)

$$\begin{aligned}{}_{a}^{k}D^{\beta,\alpha}f\left(x\right) &= \frac{1}{\Gamma(n-\beta)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(s^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta-1}\frac{{}_{a}^{k}T^{n,\alpha}f(s).s^{k}ds}{(s^{k+1}-a^{k+1})^{1-\alpha}}\\ &\quad+\sum_{m=0}^{n-1}\frac{{}_{a}^{k}T^{n,\alpha}f(a)}{\Gamma(m-\beta+1)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{m-\beta}.\end{aligned}$$

We completed the proof. The proof of right $k-$conformable fractional derivative can be done by same way. □

Theorem 3.6. *We suppose that is* $Re\left(\beta\right)>m>0$ *for* $m\in\mathbb{N}$. *Then,*

(3.19)

$$\begin{aligned}{}_{a}^{k}T^{m,\alpha}\left({}_{a}^{k}J^{\beta,\alpha}f\left(x\right)\right) &= {}_{a}^{k}J^{\beta-m,\alpha}f\left(x\right),\\ {}^{k}T_{b}^{m,\alpha}\left({}^{k}J_{b}^{\beta,\alpha}f\left(x\right)\right) &= {}^{k}J_{b}^{\beta-m,\alpha}f\left(x\right).\end{aligned}$$

Proof. We have by using (2.7),

(3.20)

$${}_{a}^{k}T^{m,\alpha}\left({}_{a}^{k}J^{\beta,\alpha}f\left(x\right)\right) = {}_{a}^{k}T^{m,\alpha}\left[\frac{1}{\Gamma(\beta)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-1}\frac{t^{k}f(t)dt}{(t^{k+1}-a^{k+1})^{1-\alpha}}\right].$$

By using Leibniz rule for integrals,

(3.21)

$$\begin{aligned}&{}_{a}^{k}T^{m,\alpha}\left({}_{a}^{k}J^{\beta,\alpha}f\left(x\right)\right)\\ &= {}_{a}^{k}T^{m-1,\alpha}\left[\frac{1}{\Gamma(\beta-1)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-2}\frac{t^{k}f(t)dt}{(t^{k+1}-a^{k+1})^{1-\alpha}}\right]\\ &= {}_{a}^{k}T^{m-2,\alpha}\left[\frac{1}{\Gamma(\beta-2)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-3}\frac{t^{k}f(t)dt}{(t^{k+1}-a^{k+1})^{1-\alpha}}\right]\\ &\vdots\\ &= \left[\frac{1}{\Gamma(\beta-m)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-m-1}\frac{t^{k}f(t)dt}{(t^{k+1}-a^{k+1})^{1-\alpha}}\right]\\ &= {}_{a}^{k}J^{\beta-m,\alpha}f\left(x\right).\end{aligned}$$

The poof is done. The second formula can be demonstrated similarly. □

Corollary 3.6.1. *If we take $Re(\gamma) < Re(\beta)$, Then,*

$$
\begin{aligned}
&{}_{a}^{k}D^{\gamma,\alpha}\left({}_{a}^{k}J^{\beta,\alpha}f(x)\right) = {}_{a}^{k}J^{\beta-\gamma,\alpha}f(x),\\
&{}^{k}D_{b}^{\gamma,\alpha}\left({}^{k}J_{b}^{\beta,\alpha}f(x)\right) = {}^{k}J_{b}^{\beta-\gamma,\alpha}f(x).
\end{aligned}
\tag{3.22}
$$

Proof. By using *Theorem* 1 and *Theorem* 3, we obtain,

$$
\begin{aligned}
{}_{a}^{k}D^{\gamma,\alpha}\left({}_{a}^{k}J^{\beta,\alpha}f(x)\right) &= {}_{a}^{k}T^{m,\alpha}\left({}_{a}^{k}J^{m-\gamma,\alpha}\left({}_{a}^{k}J^{\beta,\alpha}f(x)\right)\right)\\
&= {}_{a}^{k}T^{m,\alpha}\left({}_{a}^{k}J^{\beta+m-\gamma,\alpha}f(x)\right)\\
&= {}_{a}^{k}J^{\beta-\gamma,\alpha}f(x).
\end{aligned}
\tag{3.23}
$$

The proof is done. The second formula can be demonstrated likewise. □

Theorem 3.7. *Let $\beta > 0$ and $f \in C_{\alpha,a}^{n}[a,b]\left(f \in C_{\alpha,b}^{n}[a,b]\right)$. Then,*

$$
\begin{aligned}
&{}_{a}^{k}D^{\beta,\alpha}\left({}_{a}^{k}J^{\beta,\alpha}f(x)\right) = f(x),\\
&{}^{k}D_{b}^{\beta,\alpha}\left({}^{k}J_{b}^{\beta,\alpha}f(x)\right) = f(x).
\end{aligned}
\tag{3.24}
$$

Proof. If we possess by using (2.7) and (2.9),

$$
\begin{aligned}
&{}_{a}^{k}D^{\beta,\alpha}\left({}_{a}^{k}J^{\beta,\alpha}f(x)\right)\\
&= \frac{{}_{a}^{k}T^{n,\alpha}}{\Gamma(n-\beta)\Gamma(\beta)}\int_{a}^{x}\int_{a}^{t}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta-1}\\
&\quad\times\left[\frac{\left(t^{k+1}-a^{k+1}\right)^{\alpha}-\left(u^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-1}\frac{u^{k}f(u)du}{\left(u^{k+1}-a^{k+1}\right)^{1-\alpha}}\frac{t^{k}f(t)dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}\\
&= \frac{{}_{a}^{k}T^{n,\alpha}}{\Gamma(n-\beta)\Gamma(\beta)}\int_{a}^{x}\frac{u^{k}f(u)du}{\left(u^{k+1}-a^{k+1}\right)^{1-\alpha}}\\
&\quad\times\left(\int_{0}^{1}(1-y)^{n-\beta-1}(y)^{\beta-1}dy\right)\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(u^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-1}\\
&= \frac{{}_{a}^{k}T^{n,\alpha}}{\Gamma(n-\beta)\Gamma(\beta)}\frac{\Gamma(n-\beta)\Gamma(\beta)}{\Gamma(n)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(u^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-1}\frac{f(u)u^{k}du}{\left(u^{k+1}-a^{k+1}\right)^{1-\alpha}}\\
&= {}_{a}^{k}T^{n,\alpha}\left({}_{a}^{k}J^{n,\alpha}f\right)(x)\\
&= f(x).
\end{aligned}
\tag{3.25}
$$

The proof is completed. □

Theorem 3.8. *Let $Re(\beta) > 0$, $n = Re(\beta)$, $f \in X_c$ and ${}_{a}^{k}J^{\beta,\alpha}f \in C_{\alpha,a}^{n}[a,b]\left({}^{k}J_{b}^{\beta,\alpha}f \in C_{\alpha,b}^{n}[a,b]\right)$. Then, we have,*

$$
{}_{a}^{k}J^{\beta,\alpha}\left({}_{a}^{k}D^{\beta,\alpha}f(x)\right) = f(x) - \sum_{j=1}^{n}\frac{{}_{a}^{k}D^{\beta-j,\alpha}.f(a)}{\Gamma(\beta-j+1)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-j}
\tag{3.26}
$$

and

$$
{}^{k}J_{b}^{\beta,\alpha}\left({}^{k}D_{b}^{\beta,\alpha}f(x)\right) = f(x) - \sum_{j=1}^{n}\frac{(-1)^{n}.{}^{k}D_{b}^{\beta-j,\alpha}f(b)}{\Gamma(\beta-j+1)}\left[\frac{\left(b^{k+1}-x^{k+1}\right)^{\alpha}}{\alpha}\right]^{\beta-j}.
\tag{3.27}
$$

Proof. We can write by using (2.7) and (2.9),

$$
{}_{a}^{k}J^{\beta,\alpha}\left({}_{a}^{k}D^{\beta,\alpha}f(x)\right) = \frac{1}{\Gamma(\beta)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-1}\frac{{}_{a}^{k}T^{n,\alpha}\left({}_{a}^{k}J^{n-\beta,\alpha}f(t)\right)t^{k}dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}.
\tag{3.28}
$$

Using the integration by parts once, we have,
(3.29)

$$\begin{aligned} {}_a^kJ^{\beta,\alpha}\left({}_a^kD^{\beta,\alpha}f(x)\right) &= \frac{{}_a^kT^{1,\alpha}}{\Gamma(\beta+1)}\int_a^x\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha-\left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^\beta\frac{{}_a^kT^{n,\alpha}\left({}_a^kJ^{n-\beta,\alpha}f(t)\right)t^kdt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ &\quad -\frac{1}{\Gamma(\beta+1)}.{}_a^kT^{n,\alpha}\left({}_a^kJ^{n-\beta,\alpha}f(t)\right).\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^\beta. \end{aligned}$$

Using the integration by parts $n-$times, we have,
(3.30)

$$\begin{aligned} &{}_a^kJ^{\beta,\alpha}\left({}_a^kD^{\beta,\alpha}f(x)\right) \\ &= \frac{{}_a^kT^{1,\alpha}}{\Gamma(\beta-n+1)}\int_a^x\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha-\left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{\beta-n}\frac{\left({}_a^kJ^{n-\beta,\alpha}f(t)\right)t^kdt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ &\quad -\sum_{j=1}^n\frac{{}_a^kT^{n-j,\alpha}\left({}_a^kJ^{n-\beta,\alpha}f(a)\right)}{\Gamma(\beta+2-j)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{\beta-j+1} \\ &= {}_a^kT^{1,\alpha}\left[{}_a^kJ^{\beta-n+1,\alpha}\left({}_a^kJ^{n-\beta,\alpha}f(x)\right)-\sum_{j=1}^n\frac{{}_a^kT^{n-j,\alpha}\left({}_a^kJ^{n-\beta,\alpha}f(a)\right)}{\Gamma(\beta+2-j)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{\beta-j+1}\right] \\ &= {}_a^kT^{1,\alpha}\left[\left({}_a^kJ^{1,\alpha}f(x)\right)-\sum_{j=1}^n\frac{{}_a^kT^{n-j,\alpha}\left({}_a^kJ^{n-\beta,\alpha}f(a)\right)}{\Gamma(\beta+2-j)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{\beta-j+1}\right] \\ &= f(x)-\sum_{j=1}^n\frac{{}_a^kD^{\beta-j,\alpha}f(a)}{\Gamma(\beta+1-j)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right]^{\beta-j}. \end{aligned}$$

Proof is done. The second formula can be demonstrated the same way. □

4. $k-$CONFORMABLE FRACTIONAL DERIVATIVES IN CAPUTO SETTING

At this stage, we will give some definitions concerned with the theorem and we will demonstrate some properties of the $k-$conformable derivative on Caputo setting.

Definition 4.1. Let $\alpha > 0$, $Re(\beta) \geq 0$ and $n = [Re(\beta)] + 1$. If we take $f \in C^n_{\alpha,a}\left(f \in C^n_{\alpha,b}\right)$,
(4.1)

$$\left({}_a^{k,C}D^{\beta,\alpha}f(x)\right) = {}_a^kD^{\beta,\alpha}\left[f(t)-\sum_{m=0}^{n-1}\frac{{}_a^kT^{m,\alpha}f(a)}{m!}\left(\frac{\left(t^{k+1}-a^{k+1}\right)^\alpha}{\alpha(k+1)}\right)^m\right](x)$$

and
(4.2)

$$\left({}_a^{k,C}D^{\beta,\alpha}f(x)\right) = {}^kD_b^{\beta,\alpha}\left[f(t)-\sum_{m=0}^{n-1}\frac{(-1)^m.{}^kT_b^{m,\alpha}f(b)}{m!}\left(\frac{\left(b^{k+1}-t^{k+1}\right)^\alpha}{\alpha(k+1)}\right)^m\right](x).$$

We acquire the left and right Caputo $k-$conformable fractional derivatives, respectively.

Theorem 4.2. *Let $Re(\beta) \geq 0$ and $n = [Re(\beta)]+1$. If we take $f \in C^n_{\alpha,a}\left(f \in C^n_{\alpha,b}\right)$,*

$$ {}_a^{k,C}D^{\beta,\alpha}f(x) = {}_a^kJ^{n-\beta,\alpha}\left({}_a^kT^{n,\alpha}\right) \tag{4.3}$$

and

$$ {}^{k,C}D_b^{\beta,\alpha}f(x) = {}^kJ_b^{n-\beta,\alpha}\left({}^kT_b^{n,\alpha}\right). \tag{4.4}$$

We acquire the left and right Caputo $k-$conformable fractional derivatives in Caputo setting, respectively.

Proof. By considering $Definition$ 5, we have,

(4.5)
$$\begin{aligned}
&\left({}_{a}^{k,C}D^{\beta,\alpha}f\left(x\right)\right)\\
&={}_{a}^{k}D^{\beta,\alpha}\left[f\left(t\right)-\sum_{m=0}^{n-1}\frac{{}_{a}^{k}T^{m,\alpha}f(a)}{m!}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{m}\right](x)\\
&={}_{a}^{k}D^{\beta,\alpha}f\left(x\right)-\sum_{m=0}^{n-1}\frac{{}_{a}^{k}T^{m,\alpha}f(a)}{m!}\frac{{}_{a}^{h}T^{n,\alpha}}{\Gamma(n-\beta)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta+m}\frac{\Gamma(n-\beta)\Gamma(m+1)}{\Gamma(n-\beta+m+1)}\\
&={}_{a}^{k}D^{\beta,\alpha}f\left(x\right)-\sum_{m=0}^{n-1}\frac{{}_{a}^{k}T^{m,\alpha}f(a)}{\Gamma(m-\beta+1)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{m-\beta}.
\end{aligned}$$

Proof is done. □

Lemma 4.3. *Let $\alpha>0$, $Re\left(\beta\right)\geq 0$, $n=\left[Re\left(\beta\right)\right]+1$ and $\mathrm{Re}\left(\beta\right)\notin\mathbb{N}$. If $f\in C_{\alpha,a}^{n}\left[a,b\right]\left(f\in C_{\alpha,b}^{n}\left[a,b\right]\right)$, we have,*

(4.6)
$$\left.\begin{aligned}{}_{a}^{k}J^{\beta-s,\alpha}f\left(a\right)=0,\\ {}_{a}^{k}J^{\beta-s,\alpha}f\left(b\right)=0\end{aligned}\right\}\ \textit{for}\ s=0,1,...,n-1.$$

Proof. We obtain,

(4.7)
$$\begin{aligned}
{}_{a}^{k}J^{\beta-s,\alpha}f\left(x\right)&={}_{a}^{k}D^{s,\alpha}\left({}_{a}^{k}J^{\beta,\alpha}f\left(x\right)\right)\\
&=\frac{1}{\Gamma(\beta-s)}\int_{a}^{x}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-s-1}\frac{f(t)t^{k}dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}.
\end{aligned}$$

In here, we can state via Hölder's inequality,

(4.8)
$$\begin{aligned}
&\left|{}_{a}^{k}J^{\beta-s,\alpha}f\left(x\right)\right|\\
&\leq\frac{1}{\Gamma(\beta-s)}\left(\int_{a}^{x}\left|f\left(t\right)\right|^{p}t^{k}\right)^{\frac{1}{p}}\left(\int_{a}^{x}\left(\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-s-1}\frac{f(t)t^{k}dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}\right)^{q}\right)^{\frac{1}{q}}\\
&\leq\frac{\left\|f\right\|_{X_{c}}}{(re(\beta)-s)\Gamma(\beta-s)}\left(\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right)^{(re(\beta)-s)}.
\end{aligned}$$

For $x=a$, we can say that

$$ {}_{a}^{k}J^{\beta-s,\alpha}f\left(a\right)=0. \tag{4.9}$$

Proof is done. □

Lemma 4.4. *Let $\alpha>0$, $Re\left(\beta\right)\geq 0$ and $n=\left[Re\left(\beta\right)\right]+1$. If we take ${}_{a}^{k}T^{n,\alpha}\in C\left[a,b\right]\left({}^{k}T_{b}^{n,\alpha}\in C_{\alpha,b}^{n}\right)$, we obtain,*

(4.10)
$$\begin{aligned}{}_{a}^{k,C}D^{\beta,\alpha}f\left(a\right)=0,\\ {}^{k,C}D_{b}^{\beta,\alpha}f\left(b\right)=0.\end{aligned}$$

Proof. It is clearly seen that

$$\left|{}_{a}^{k,C}D^{\beta,\alpha}f\left(x\right)\right|\leq\frac{\left\|{}_{a}^{k}T^{n,\alpha}\right\|_{X_{c}}}{(n-re(\beta))\Gamma(n-\beta)}\left(\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right)^{(n-re(\beta))} \tag{4.11}$$

and

$$\left|{}^{k,C}D_b^{\beta,\alpha} f(x)\right| \leq \frac{\left\|{}^{k}T_b^{n,\alpha}\right\|_{X_c}}{(n-re(\beta))\Gamma(n-\beta)} \left(\frac{\left(b^{k+1}-x^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right)^{(n-re(\beta))}. \tag{4.12}$$

Proof is done. □

Theorem 4.5. *Let $Re(\beta) \geq 0$, $n = [Re(\beta)]+1$ and $f \in C^n_{\alpha,a}[a,b]$ $\left(f \in C^n_{\alpha,b}[a,b]\right)$,*

(1) If we get $Re(\beta) \notin \mathbb{N}$ or $\beta \in \mathbb{N}$, then,

$$\begin{aligned} {}_a^{k,C}D^{\beta,\alpha}\left({}_a^{k}J^{\beta,\alpha} f(x)\right) &= f(x), \\ {}^{k,C}D_b^{\beta,\alpha}\left({}^{k}J_b^{\beta,\alpha} f(x)\right) &= f(x). \end{aligned} \tag{4.13}$$

(2) If we take $Re(\beta) \neq 0$ or $Re(\alpha) \in \mathbb{N}$, then,

$$\begin{aligned} {}_a^{k,C}D^{\beta,\alpha}\left({}_a^{k}J^{\beta,\alpha} f(x)\right) &= f(x) - \frac{{}_a^{k}J^{\beta-n+1,\alpha}f(a)}{\Gamma(n-\beta)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta}, \\ {}^{k,C}D_b^{\beta,\alpha}\left({}^{k}J_b^{\beta,\alpha} f(x)\right) &= f(x) - \frac{{}^{k}J_b^{\beta-n+1,\alpha}f(a)}{\Gamma(n-\beta)}\left[\frac{\left(b^{k+1}-x^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta}. \end{aligned} \tag{4.14}$$

Proof. By using *Definition* 6, we have,

$$\begin{aligned} &{}_a^{k,C}D^{\beta,\alpha}\left({}_a^{k}J^{\beta,\alpha} f(x)\right) \\ &\quad = f(x) - \frac{{}_a^{k}T^{n,\alpha}}{\Gamma(n-\beta)}\int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta-1} \\ &\qquad \times \left(\sum_{m=0}^{n-1}\frac{{}_a^{k}J^{n+m-\beta,\alpha}f(a)}{m!}\left(\frac{\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right)^{m}\right)\frac{t^k.dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}} \\ &\quad = f(x) - \frac{{}_a^{h}T^{n,\alpha}}{\Gamma(n-\beta)}\left(\sum_{m=0}^{n-1}\frac{{}_a^{h}J^{n+m-\beta,\alpha}f(a)}{m!}\right) \\ &\qquad \times \int_a^x \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}-\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{n-\beta-1}\left[\frac{\left(t^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{m}\frac{t^k.dt}{\left(t^{k+1}-a^{k+1}\right)^{1-\alpha}}. \end{aligned} \tag{4.15}$$

In here, by using the following the change of variable,

$$\left(t^{k+1}-a^{k+1}\right)^{\alpha} = z\left(x^{k+1}-a^{k+1}\right)^{\alpha}, \tag{4.16}$$

we can write,

$${}_a^{k,C}D^{\beta,\alpha}\left({}_a^{k}J^{\beta,\alpha} f(x)\right) = f(x) - \sum_{m=0}^{n-1}\frac{{}_a^{k}J^{m-\beta,\alpha}f(a)}{\Gamma(m-\beta+1)}\left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{m-\beta}. \tag{4.17}$$

In here, we have ${}_a^{k}J^{\beta-s,\alpha}f(a) = 0$ and ${}^{k}J_b^{\beta-s,\alpha}f(b) = 0$ for $Re(\beta) \notin \mathbb{N}$ by using *Lemma* 4. The case $\beta \in \mathbb{N}$ is unimportant. Additionaly, if $Re(\beta) \in \mathbb{N}$, we state ${}_a^{k}J^{\beta-s,\alpha}f(a) = 0$ and ${}^{k}J_b^{\beta-s,\alpha}f(b) = 0$ for $s = 0, 1, ..., n-2$ by using *Lemma* 4. □

Theorem 4.6. *Let $\beta \in \mathbb{C}$ and $f \in C^n_{\alpha,a}[a,b]$ $\left(f \in C^n_{\alpha,b}[a,b]\right)$. We have,*

$$
(4.18)\quad \begin{aligned}
{}^k_a J^{\beta,\alpha}\left({}^{k,C}_a D^{\beta,\alpha} f(x)\right) &= f(x) - \sum_{m=0}^{n-1} \frac{{}^h_a T^{m,\alpha} f(a)}{\Gamma(m+1)} \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^m, \\
{}^k J^{\beta,\alpha}_b\left({}^{k,C} D^{\beta,\alpha}_b f(x)\right) &= f(x) - \sum_{m=0}^{n-1} \frac{{}^h T^{m,\alpha}_b f(a)}{\Gamma(m+1)} \left[\frac{\left(b^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^m.
\end{aligned}
$$

Proof. In here, we can write the following as,

$$
(4.19)\quad \begin{aligned}
{}^k_a J^{\beta,\alpha}\left({}^{k,C}_a D^{\beta,\alpha} f(x)\right) &= {}^k_a J^{\beta,\alpha}\left({}^k_a J^{n-\beta,\alpha}\left({}^k_a T^{n,\alpha} f(x)\right)\right) \\
&= {}^k_a J^{n,\alpha}\left({}^k_a T^{n,\alpha} f(x)\right) \\
&= f(x) - \frac{{}^k_a D^{\beta-j,\alpha} f(a)}{\Gamma(\beta-j+1)} \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{\beta-j} \\
&= f(x) - \frac{{}^k_a D^{m,\alpha} f(a)}{\Gamma(m+1)} \left[\frac{\left(x^{k+1}-a^{k+1}\right)^{\alpha}}{\alpha(k+1)}\right]^{m}.
\end{aligned}
$$

Proof is done. □

Theorem 4.7. *Let $f \in C^{p+r}_{\alpha,a}[a,b]$ $\left(f \in C^{p+r}_{\alpha,b}[a,b]\right)$, $Re(\beta) \geq 0$, $Re(\mu) \geq 0$, $r-1 < [Re(\beta)] \leq r$ and $p-1 < [Re(\beta)] \leq p$. Then we get,*

$$
(4.20)\quad \begin{aligned}
{}^{k,C}_a D^{\beta,\alpha}\left({}^{k,C}_a D^{\mu,\alpha}(f(x))\right) &= {}^{k,C}_a D^{\beta+\mu,\alpha} f(x), \\
{}^{k,C} D^{\beta,\alpha}_b\left({}^{k,C} D^{\mu,\alpha}_b(f(x))\right) &= {}^{k,C} D^{\beta+\mu,\alpha}_b f(x).
\end{aligned}
$$

Proof. It is clear that the proof can complete by using $Theorem$ 1, $Theorem$ 4, $Theorem$ 6 and $Lemma$ 5. □

5. FRACTIONAL INTEGRALS AND DERIVATIVES CLASS

1. By considering $k=0$ in $Definition$ 2,

$$
{}^k_a J^{\beta,\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left[\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right]^{\beta-1} \frac{f(t)dt}{(t-a)^{1-\alpha}}.
$$

We obtain the left fractional conformable integrals in [1].

2. By considering $k=0$ and $\alpha=1$ in $Definition$ 2,

$$
{}^k_a J^{\beta,\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} f(t)\,dt.
$$

We obtain the left Riemann-Lioville fractional integrals.

3. By considering $k=0$, $\alpha=1$ and $a=-\infty$ in $Definition$ 2,

$$
{}^k_a J^{\beta,\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^x (x-t)^{\beta-1} f(t)\,dt.
$$

We obtain the left Liouville fractional integrals.

4. By considering $k=0$, $a=0$ and $\alpha=1$ in $Definition$ 2,

$$
{}^k_a J^{\beta,\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} f(t)\,dt.
$$

We obtain the left Riemann fractional integrals.

5. By considering $\alpha=1$, $k=0$ and $g(x) = E^{\gamma}_{\alpha,\beta}\left(\omega(x-t)^{\beta}\right) f(x)$ in $Definition$ 2,

$$
\Gamma(\beta)\, {}^k_a J^{\beta,\alpha} g(x) = \int_a^x (x-t)^{\beta-1} E^{\gamma}_{\alpha,\beta}\left(\omega(x-t)^{\beta}\right) f(t)\,dt.
$$

We obtain the left Prabhakar fractional integrals.

6. By considering $k=0$, $\alpha=1$ and $a=c$ in $Definition$ 2,

$$ {}_{c}^{k}J^{\beta,\alpha}f(x) \quad = \frac{1}{\Gamma(\beta)}\int_{c}^{x}(x-t)^{\beta-1}f(t)\,dt. $$

We obtain the left Chen fractional integrals.

7. By considering $k=0$ in $Definition$ 3, we have the left fractional conformable derivatives in [1],

$$ \begin{aligned} {}_{a}^{k}D^{\beta,\alpha}f(x) &= {}_{a}^{k}T^{n,\alpha}\left({}_{a}^{k}J^{n-\beta,\alpha}\right)f(x) \\ &= \frac{{}_{a}^{k}T^{n,\alpha}}{\Gamma(n-\beta)}\int_{a}^{x}\left[\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right]^{n-\beta-1}\frac{f(t)dt}{(t-a)^{1-\alpha}}. \end{aligned} $$

8. By considering $k=0$, $\alpha=1$ in $Definition$ 3, we have Riemann-Liouville fractional derivative,

$$ \begin{aligned} {}_{0}^{k}D^{\beta,\alpha}f(x) &= {}_{0}^{k}T^{n}\left({}_{0}^{k}J^{n-\beta,\alpha}\right)f(x) \\ &= \frac{{}_{0}^{k}T^{n}}{\Gamma(n-\beta)}\int_{0}^{x}[x-t]^{n-\beta-1}f(t)\,dt. \end{aligned} $$

9. Taking $k=0$, $\alpha=1$ in $Definition$ 3, we have the left Caputo fractional derivative,

$$ \begin{aligned} {}_{a}^{k}D^{\beta,\alpha}f(x) &= \left({}_{a}^{k}J^{n-\beta,\alpha}\left({}_{a}^{k}T^{n}\right)\right)f(x) \\ &= \frac{1}{\Gamma(n-\beta)}\int_{a}^{x}[x-t]^{n-\beta-1}\left({}_{a}^{k}T^{n}\right)f(t)\,dt. \end{aligned} $$

10. Taking $k=0$, $\alpha=1$ and $a=0$ in $Definition$ 3, we have the Riemann fractional derivative,

$$ {}_{0}^{k}D^{\beta,\alpha}f(x) \quad = \left(\frac{d}{dx}\right)^{n}\cdot{}_{0}^{k}J^{n-\beta}f(x). $$

12. Taking $k=0$, $\alpha=1$ and $a=c$ in $Definition$ 3, we have the Chen fractional derivative,

$$ {}_{c}^{k}D^{\beta,\alpha} \quad = \left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\beta)}\int_{c}^{x}(x-t)^{n-\beta-1}f(t)\,dt. $$

13. Taking $k=0$, $a=0$, $\alpha=1$ and $g(x)=f(x)-f(0)$ in $Definition$ 3, we have Jumarie fractional derivative,

$$ {}_{0}^{k}D^{\beta,\alpha} \quad = \left(\frac{d}{dx}\right)^{n}\cdot{}_{0}^{k}J^{n-\beta,\alpha}\left(f(x)-f(0)\right). $$

14. Taking $k=0$, $\alpha=1$,and $g(x)=E_{\rho,n-\beta}^{-\gamma}\left[\omega(x-t)^{\rho}\right]f(x)$ in $Definition$ 3, we have the Prabhakar fractional derivative,

$$ {}_{a}^{k}D^{\beta,\alpha}g(x) \quad = \left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\beta)}\int_{a}^{x}(x-t)^{n-\beta-1}E_{\rho,n-\beta}^{-\gamma}\left[\omega(x-t)^{\rho}\right]f(t)\,dt. $$

15. Taking $k=0$, $\alpha=1$, $a=-\infty$ in $Definition$ 3, we have the Liouville fractional derivative,

$$ {}_{-\infty}^{k}D^{\beta,\alpha}f(x) \quad = \left(-\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\beta)}\int_{-\infty}^{x}(x-t)^{n-\beta-1}f(t)\,dt. $$

16. Taking $k=0$, $\alpha=1$, $a=-\infty$ in $Definition$ 3, we have the Liouville-Caputo fractional derivative,

$$ {}_{-\infty}^{k}D^{\beta,\alpha}f(x) \quad = \frac{1}{\Gamma(n-\beta)}\int_{-\infty}^{x}(x-t)^{n-\beta-1}\left(-\frac{d}{dx}\right)^{n}f(t)\,dt. $$

17. Taking $k=0$, $\alpha=1$, $b=\infty$ in $Definition$ 3, we have the Weyl fractional derivative,

$$ {}^{k}D_{\infty}^{\beta,\alpha}f(x) \quad = (-1)^{n}\left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\beta)}\int_{x}^{\infty}(x-t)^{n-\beta-1}f(t)\,dt. $$

6. Conclusion

In this research, we defined the left and right $k-$ conformable fractional integral and derivatives, respectively, we demonstrated important consequences and some basic properties for these operators. Furthermore, we acquired the $k-$conformable fractional derivatives on the Caputo setting. In conclusion, we expressed the classical results for the generalized conformable derivatives and integrals.

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