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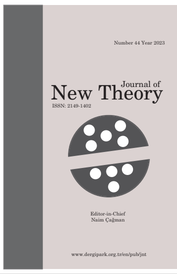

Inverse Problems for a Conformable Fractional Diffusion Operator

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Abstract — In this paper, we consider a diffusion operator with discrete boundary conditions, which include the conformable fractional derivatives of order α such that $0 < \alpha \leq 1$ instead of the ordinary derivatives in the classical diffusion operator. We prove that the coefficients of the given operator are uniquely determined by the Weyl function and spectral data, which consist of a spectrum and normalizing numbers. Moreover, using the well-known Hadamard's factorization theorem, we prove that the characteristic function $\Delta_\alpha(\rho)$ is determined by the specification of its zeros for each fixed α. The obtained results in this paper can be regarded as partial α-generalizations of similar findings obtained for the classical diffusion operator.

Keywords *Inverse problem, diffusion operator, conformable fractional derivative, Weyl function, spectral data*

Mathematics Subject Classification (2020) 34A55, 26A33

1. Introduction

Inverse spectral problems aim to reconstruct the coefficients of an operator from given data such as the Weyl function, nodal points, and spectral data (two spectra or a spectrum and normalizing numbers). For the last century, these kinds of problems for various classical Sturm-Liouville, diffusion, and Dirac operators have been extensively investigated; for more details, see [1–7].

The beginning of the fractional derivative dates back to 1695, and many fractional derivative concepts have been proposed until today, such as the Riemann-Liouville fractional derivative, the Caputo fractional derivative, and the Atangana fractional derivative. In 2014, Khalil et al. [8] introduced the conformable fractional derivative. Then, many researchers identified important and fundamental properties of this derivative in [9–14]. In 2017, Jarad et al. [15] showed that this derivative is necessary and useful for generating new types of fractional operators. In recent years, numerous significant studies [16–20] have been conducted on inverse problems related to various conformable fractional operators, including the diffusion operator.

We consider a conformable fractional diffusion operator with discrete boundary conditions, denoted as $L_\alpha = L_\alpha(p(x), q(x), h, H)$. The form of this operator is as follows:

$$\ell_\alpha y := -T_x^\alpha T_x^\alpha y + \left[2\rho p(x) + q(x)\right] y = \rho^2 y, \quad 0 < x < \pi \tag{1}$$

$$U_\alpha(y) := T_x^\alpha y(0) - hy(0) = 0$$

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and

$$V_\alpha(y) := T_x^\alpha y(\pi) + Hy(\pi) = 0$$

where ρ is the spectral parameter, $h, H \in \mathbb{R}$, $q(x) \in W_{2,\alpha}^1[0,\pi]$ and $p(x) \in W_{2,\alpha}^2[0,\pi]$ are real-valued functions such that $p(x) \neq const$, $T_x^\alpha y$ is a conformable fractional derivative of order $\alpha \in (0,1]$ of y at x,

$$T_x^\alpha y(x) = \lim_{h\to 0} \frac{y(x + hx^{1-\alpha}) - y(x)}{h}, \quad \text{for all } x > 0$$

$$W_{2,\alpha}^1[0,\pi] = \{f(x) \mid f(x) \text{ is absolutely continuous on } [0,\pi] \text{ and } T_x^\alpha f(x) \in L_{2,\alpha}(0,\pi)\}$$

$$W_{2,\alpha}^2[0,\pi] = \{f(x) \mid f(x) \text{ and } T_x^\alpha f(x) \text{ are absolutely continuous on } [0,\pi] \text{ and } T_x^\alpha T_x^\alpha f(x) \in L_{2,\alpha}(0,\pi)\}$$

and the space $L_{2,\alpha}(0,\pi)$ consists of all the functions $f : [0,\pi] \to \mathbb{R}$ satisfying the condition

$$\left(\int_0^\pi |f(x)|^2 d_\alpha x\right)^{1/2} = \left(\int_0^\pi |f(x)|^2 x^{\alpha-1} dx\right)^{1/2} < \infty$$

This operator is referred to as the Conformable Fractional Diffusion Operator (CFDO).

In this paper, we have proved that the coefficients of the given operator can be uniquely determined by the Weyl function and spectral data, which consist of a spectrum and normalizing numbers.

2. Preliminaries

This section provides some basic notions to be needed in the following sections. Let the functions $\varphi = \varphi(x,\rho;\alpha)$, $\psi = \psi(x,\rho;\alpha)$, and $S = S(x,\rho;\alpha)$ be the solutions of Equation 1 satisfying the following initial conditions

$$\varphi(0,\rho;\alpha) = 1 \quad \text{and} \quad T_x^\alpha \varphi(0,\rho;\alpha) = h \tag{2}$$

$$\psi(\pi,\rho;\alpha) = 1 \quad \text{and} \quad T_x^\alpha \psi(\pi,\rho;\alpha) = -H \tag{3}$$

$$S(0,\rho;\alpha) = 0 \quad \text{and} \quad T_x^\alpha S(0,\rho;\alpha) = 1 \tag{4}$$

respectively. From [19, 21, 22], these solutions satisfy the following asymptotic formulas, for $|\rho| \to \infty$ and each fixed α,

$$\varphi = \cos\left(\frac{\rho x^\alpha}{\alpha} - \theta(x)\right) + O\left(\frac{1}{|\rho|}\exp\left(\frac{|\mathrm{Im}\rho|\, x^\alpha}{\alpha}\right)\right) \tag{5}$$

$$T_x^\alpha \varphi = -\rho \sin\left(\frac{\rho x^\alpha}{\alpha} - \theta(x)\right) + O\left(\exp\left(\frac{|\mathrm{Im}\rho|\, x^\alpha}{\alpha}\right)\right) \tag{6}$$

$$\psi = \cos\left(\frac{\rho(\pi^\alpha - x^\alpha)}{\alpha} - \theta(\pi) + \theta(x)\right) + O\left(\frac{1}{|\rho|}\exp\left(\frac{|\mathrm{Im}\rho|(\pi^\alpha - x^\alpha)}{\alpha}\right)\right) \tag{7}$$

$$T_x^\alpha \psi = \rho \sin\left(\frac{\rho(\pi^\alpha - x^\alpha)}{\alpha} - \theta(\pi) + \theta(x)\right) + O\left(\exp\left(\frac{|\mathrm{Im}\rho|(\pi^\alpha - x^\alpha)}{\alpha}\right)\right) \tag{8}$$

$$S = \frac{1}{\rho}\sin\left(\frac{\rho x^\alpha}{\alpha} - \theta(x)\right) + O\left(\frac{1}{\rho^2}\exp\left(\frac{|\mathrm{Im}\rho|\, x^\alpha}{\alpha}\right)\right) \tag{9}$$

$$T_x^\alpha S = \cos\left(\frac{\rho x^\alpha}{\alpha} - \theta(x)\right) + O\left(\frac{1}{\rho}\exp\left(\frac{|\mathrm{Im}\rho|\, x^\alpha}{\alpha}\right)\right) \tag{10}$$

where

$$\theta(x) = \int_0^x p(t)\, d_\alpha t$$

We denote

$$\Delta_\alpha(\rho) = W_\alpha[\psi,\varphi] = \begin{vmatrix} \psi & \varphi \\ T_x^\alpha \psi & T_x^\alpha \varphi \end{vmatrix} = \psi T_x^\alpha \varphi - \varphi T_x^\alpha \psi \tag{11}$$

where $W_\alpha[\psi,\varphi]$ is the fractional Wronskian of the functions ψ and φ. Furthermore, the $\Delta_\alpha(\rho)$ is called as the characteristic function of the operator L_α and is entire function in ρ for each fixed α.

Lemma 2.1. [23] For each fixed α, $\Delta_\alpha(\rho)$ does not depend on x and can be written as

$$\Delta_\alpha(\rho) = V_\alpha(\varphi) = -U_\alpha(\psi) \tag{12}$$

Lemma 2.2. [23] The zeros $\{\rho_n\}$ of the function $\Delta_\alpha(\rho)$ are coincide with the eigenvalues of the operator L_α, and for eigenfunctions $\psi(x,\rho_n;\alpha)$ and $\varphi(x,\rho_n;\alpha)$, there exists a sequence $\{\beta_n\}$ such that

$$\psi(x,\rho_n;\alpha) = \beta_n \varphi(x,\rho_n;\alpha) \quad \text{and} \quad \beta_n \neq 0 \tag{13}$$

are satisfied for each fixed α.

It is clear from Equations 2, 3, and 13 that $\beta_n = \psi(0,\rho_n;\alpha) = \frac{1}{\varphi(\pi,\rho_n;\alpha)}$.

Lemma 2.3. [23] The equality $\dot{\Delta}_\alpha(\rho_n) = -2\rho_n\beta_n\alpha_n$ is valid where $\dot{\Delta}_\alpha(\rho) = \dfrac{d\Delta_\alpha(\rho)}{d\rho}$ and the normalizing numbers are

$$\alpha_n = \int_0^\pi \varphi^2(x,\rho_n;\alpha)\,d_\alpha x - \frac{1}{\rho_n}\int_0^\pi p(x)\,\varphi^2(x,\rho_n;\alpha)\,d_\alpha x$$

Definition 2.4. The data $\{\rho_n,\alpha_n\}_{n\geq 1}$ are called the spectral data of the operator L_α.

Let $\{\rho_n\}$ be the eigenvalues set of the operator L_α. From [23], the numbers ρ_n hold the following estimate:

$$\rho_n = \frac{n\alpha}{\pi^{\alpha-1}} + c_{\alpha,0} + \frac{c_{\alpha,1}}{n} + o\left(\frac{1}{n}\right), \quad n\to\infty$$

where

$$c_{\alpha,0} = \frac{\alpha}{\pi^\alpha}\int_0^\pi p(x)d_\alpha x$$

and

$$c_{\alpha,1} = \frac{1}{\pi}\left[h + H + \frac{1}{2}\int_0^\pi \left(q(x) + p^2(x)\right)d_\alpha x\right]$$

Let $G_\delta = \left\{\rho \mid \left|\rho - \frac{n\alpha}{\pi^{\alpha-1}}\right| \geq \delta,\ n\in\{1,2,\ldots\}\right\}$ where δ is a sufficiently small positive number. It is obvious from Equations 5, 6, and 12 that the function $\Delta_\alpha(\rho)$ satisfies the inequality

$$|\Delta_\alpha(\rho)| \geq c_\delta |\rho| \exp\left(\frac{|\mathrm{Im}\rho|}{\alpha}\pi^\alpha\right), \quad \rho\in G_\delta \tag{14}$$

3. Main Results

This section proves uniqueness theorems for the solution of inverse problems according to the Weyl function and spectral data, which consist of a spectrum and normalizing numbers. Together with L_α, we consider a second operator $\widetilde{L}_\alpha = \widetilde{L}_\alpha(p(x),\widetilde{q}(x),\widetilde{h},\widetilde{H})$ of the following form

$$\widetilde{\ell}_\alpha y := -T_x^\alpha T_x^\alpha y + [2\rho p(x) + \widetilde{q}(x)]\,y = \rho^2 y, \quad 0<x<\pi$$

$$\widetilde{U}_\alpha(y) := T_x^\alpha y(0) - \widetilde{h}y(0) = 0$$

and

$$\widetilde{V}_\alpha(y) := T_x^\alpha y(\pi) + \widetilde{H} y(\pi) = 0$$

We note that if a certain symbol σ denotes an object related to L_α, then $\widetilde{\sigma}$ will denote an analogous object related to $\widetilde{L}_\alpha$.

It can be observed that $W_\alpha\left[\varphi, S\right]\big|_{x=0} = 1 \neq 0$. Thus, the functions φ and S are linearly independent, and the function ψ can be written as

$$\psi\left(x,\rho;\alpha\right) = c_1\left(\rho;\alpha\right)\varphi\left(x,\rho;\alpha\right) + c_2\left(\rho;\alpha\right) S\left(x,\rho;\alpha\right) \tag{15}$$

where $c_1\left(\rho;\alpha\right)$ and $c_2\left(\rho;\alpha\right)$ are arbitrary constant for each fixed α. It is clear from Equation 15 that

$$\psi\left(0,\rho;\alpha\right) = c_1\left(\rho;\alpha\right)\varphi\left(0,\rho;\alpha\right) + c_2\left(\rho;\alpha\right) S\left(0,\rho;\alpha\right)$$

and

$$T_x^\alpha \psi\left(0,\rho;\alpha\right) = c_1\left(\rho;\alpha\right) T_x^\alpha \varphi\left(0,\rho;\alpha\right) + c_2\left(\rho;\alpha\right) T_x^\alpha S\left(0,\rho;\alpha\right)$$

From Equations 2, 4, and 12,

$$c_1\left(\rho;\alpha\right) = \psi\left(0,\rho;\alpha\right)$$

and

$$c_2\left(\rho;\alpha\right) = T_x^\alpha \psi\left(0,\rho;\alpha\right) - h\psi\left(0,\rho;\alpha\right) = -\Delta_\alpha\left(\rho\right)$$

Consequently, Equation 15 is rewritten as

$$\psi\left(x,\rho;\alpha\right) = \psi\left(0,\rho;\alpha\right)\varphi\left(x,\rho;\alpha\right) - \Delta_\alpha\left(\rho\right) S\left(x,\rho;\alpha\right)$$

or

$$-\frac{\psi\left(x,\rho;\alpha\right)}{\Delta_\alpha\left(\rho\right)} = -\frac{\psi\left(0,\rho;\alpha\right)}{\Delta_\alpha\left(\rho\right)}\varphi\left(x,\rho;\alpha\right) + S\left(x,\rho;\alpha\right) \tag{16}$$

If we denote

$$\Phi\left(x,\rho;\alpha\right) := -\frac{\psi\left(x,\rho;\alpha\right)}{\Delta_\alpha\left(\rho\right)} \quad \text{and} \quad M_\alpha\left(\rho\right) := \Phi\left(0,\rho;\alpha\right) = -\frac{\psi\left(0,\rho;\alpha\right)}{\Delta_\alpha\left(\rho\right)} \tag{17}$$

then, from Equation 16,

$$\Phi\left(x,\rho;\alpha\right) = S\left(x,\rho;\alpha\right) + M_\alpha\left(\rho\right)\varphi\left(x,\rho;\alpha\right) \tag{18}$$

The functions $\Phi\left(x,\rho;\alpha\right)$ and $M_\alpha\left(\rho\right)$ are called as the Weyl solution and the Weyl function of the operator L_α, respectively. It is obvious that $\Phi\left(x,\rho;\alpha\right)$ is the solution of Equation 1 under the conditions $U_\alpha\left(\Phi\right) = 1$, $V_\alpha\left(\Phi\right) = 0$, and $M_\alpha\left(\rho\right)$ is a meromorphic function with poles in $\{\rho_n\}$.

Theorem 3.1. If $M_\alpha\left(\rho\right) = \widetilde{M}_\alpha\left(\rho\right)$ for each fixed α, then $q(x) = \widetilde{q}\,(x)$, almost everywhere in $[0,\pi]$, $h = \widetilde{h}$, and $H = \widetilde{H}$. Thus, the Weyl function uniquely determines the operator L_α.

PROOF.
Consider the functions $P_1\left(x,\rho;\alpha\right)$ and $P_2\left(x,\rho;\alpha\right)$ defined by

$$P_1\left(x,\rho;\alpha\right) = \varphi\left(x,\rho;\alpha\right) T_x^\alpha \widetilde{\Phi}\left(x,\rho;\alpha\right) - \Phi\left(x,\rho;\alpha\right) T_x^\alpha \widetilde{\varphi}\left(x,\rho;\alpha\right) \tag{19}$$

and

$$P_2\left(x,\rho;\alpha\right) = \Phi\left(x,\rho;\alpha\right)\widetilde{\varphi}\left(x,\rho;\alpha\right) - \varphi\left(x,\rho;\alpha\right)\widetilde{\Phi}\left(x,\rho;\alpha\right) \tag{20}$$

From Equation 18,

$$P_1\left(x,\rho;\alpha\right) = \varphi\left(x,\rho;\alpha\right) T_x^\alpha \widetilde{S}\left(x,\rho;\alpha\right) - S\left(x,\rho;\alpha\right) T_x^\alpha \widetilde{\varphi}\left(x,\rho;\alpha\right) + \left[\widetilde{M}_\alpha\left(\rho\right) - M_\alpha\left(\rho\right)\right]\varphi\left(x,\rho;\alpha\right) T_x^\alpha \widetilde{\varphi}\left(x,\rho;\alpha\right)$$

and

$$P_2(x,\rho;\alpha)=S(x,\rho;\alpha)\,\widetilde{\varphi}(x,\rho;\alpha)-\varphi(x,\rho;\alpha)\,\widetilde{S}(x,\rho;\alpha)+\left[M_\alpha(\rho)-\widetilde{M}_\alpha(\rho)\right]\varphi(x,\rho;\alpha)\,\widetilde{\varphi}(x,\rho;\alpha)$$

Since $M_\alpha(\rho)=\widetilde{M}_\alpha(\rho)$, the functions $P_1(x,\rho;\alpha)$ and $P_2(x,\rho;\alpha)$ are entire in ρ, for each fixed α. Moreover, from Equations 11 and 17,

$$W_\alpha[\varphi(x,\rho;\alpha),\Phi(x,\rho;\alpha)]=-\frac{W_\alpha[\varphi(x,\rho;\alpha),\psi(x,\rho;\alpha)]}{\Delta_\alpha(\rho)}=1$$

and similarly,

$$W_\alpha\left[\widetilde{\varphi}(x,\rho;\alpha),\widetilde{\Phi}(x,\rho;\alpha)\right]=1$$

Thus, Equation 19 can be rewritten as

$$P_1(x,\rho;\alpha)=1+\varphi(x,\rho;\alpha)\left[T_x^\alpha\widetilde{\Phi}(x,\rho;\alpha)-T_x^\alpha\Phi(x,\rho;\alpha)\right]+\Phi(x,\rho;\alpha)\left[T_x^\alpha\varphi(x,\rho;\alpha)-T_x^\alpha\widetilde{\varphi}(x,\rho;\alpha)\right]$$

It follows from the asymptotic formulas of Equations 5-10 and Equality 14 that

$$|P_1(x,\rho;\alpha)-1|\le\frac{C_\delta}{|\rho|}\quad\text{and}\quad|P_2(x,\rho;\alpha)|\le\frac{C_\delta}{|\rho|},\quad x\in[0,\pi],\ \ |\rho|\in G_\delta$$

Therefore, since $\lim\limits_{|\rho|\to\infty}|P_1(x,\rho;\alpha)-1|=\lim\limits_{|\rho|\to\infty}|P_2(x,\rho;\alpha)|=0$ by the well-known Liouville's theorem, we obtain for $x\in[0,\pi]$ and each fixed α that

$$P_1(x,\rho;\alpha)=1\quad\text{and}\quad P_2(x,\rho;\alpha)=0\tag{21}$$

Hence, by using Equations 19-21, we get the following system

$$\begin{cases}\varphi(x,\rho;\alpha)\,T_x^\alpha\widetilde{\Phi}(x,\rho;\alpha)-\Phi(x,\rho;\alpha)\,T_x^\alpha\widetilde{\varphi}(x,\rho;\alpha)=1\\ \Phi(x,\rho;\alpha)\,\widetilde{\varphi}(x,\rho;\alpha)-\varphi(x,\rho;\alpha)\,\widetilde{\Phi}(x,\rho;\alpha)=0\end{cases}\tag{22}$$

If System 22 is solved according to functions $\varphi(x,\rho;\alpha)$ and $\Phi(x,\rho;\alpha)$, then

$$\varphi(x,\rho;\alpha)=\widetilde{\varphi}(x,\rho;\alpha)$$

and

$$\Phi(x,\rho;\alpha)=\widetilde{\Phi}(x,\rho;\alpha)$$

is obtained, for all x and ρ and each fixed α. Thus, $q(x)=\widetilde{q}(x)$, almost everywhere in $[0,\pi]$, $h=\widetilde{h}$, and $H=\widetilde{H}$. Consequently, $L_\alpha=\widetilde{L}_\alpha$. □

Lemma 3.2. For each fixed α, the characteristic function $\Delta_\alpha(\rho)$ is determined by the specification of its zeros as:

$$\Delta_\alpha(\rho)=C\rho\exp(C_1\rho)\prod_{n=1}^{\infty}\left(1-\frac{\rho}{\rho_n}\right)\exp\left(\frac{\rho}{\rho_n}\right)$$

where

$$C=\sin\theta(\pi)\prod_{n=1}^{\infty}\frac{\rho}{\rho_n^0},\quad C_1=-\frac{\pi^\alpha}{\alpha}\cot\theta(\pi)+\sum_{n=1}^{\infty}\left(\frac{1}{\rho_n^0}-\frac{1}{\rho_n}\right),\quad \rho_n^0=\left(n+\frac{\theta(\pi)}{\pi}\right)\frac{\alpha}{\pi^{\alpha-1}},\quad n\in\{1,2,\cdots\}$$

Proof.

It is clear from Equations 5, 6, and 12 that the characteristic function $\Delta_\alpha(\rho)$ holds the following asymptotic representation:

$$\Delta_\alpha(\rho)=-\rho\sin\left(\frac{\rho\pi^\alpha}{\alpha}-\theta(\pi)\right)+O\left(\exp\left(\frac{|\mathrm{Im}\rho|}{\alpha}\pi^\alpha\right)\right)\tag{23}$$

Consider the function

$$\Delta_\alpha^0(\rho) = -\rho \sin\left(\frac{\rho\pi^\alpha}{\alpha} - \theta(\pi)\right) \tag{24}$$

The zeros of the function $\Delta_\alpha^0(\rho)$ are $\rho = 0$ and $\rho_n^0 = \left(n + \frac{\theta(\pi)}{\pi}\right)\frac{\alpha}{\pi^{\alpha-1}}$ such that $n \in \{1, 2, 3, \cdots\}$. Since $\Delta_\alpha^0(\rho)$ is an entire function, according to the Hadamard's factorization theorem,

$$\Delta_\alpha^0(\rho) = -\rho^m \exp(g(\rho)) \prod_{n=1}^{\infty} E_p\left(\frac{\rho}{\rho_n^0}\right) \tag{25}$$

where $m \geq 0$ is the multiplicity of the zero eigenvalue, $g(\rho)$ is a polynomial with $\operatorname{der}(g(\rho)) = p$, and

$$E_p(\xi) = \begin{cases} (1-\xi), & n = 0 \\ (1-\xi)\exp\left(\frac{\xi}{1} + \frac{\xi^2}{2} + \ldots + \frac{\xi^n}{n}\right), & \text{otherwise} \end{cases}$$

Since the multiplicity of the zero is 1, $m = 1$. Besides, for every $r > 0$ and for $p = 1$, the series $\sum\limits_{n=1}^{\infty} \frac{r^{1+p}}{|\rho_n^0|^{1+p}}$ converges. Therefore, Equation 25 can rewrite as

$$\Delta_\alpha^0(\rho) = -\rho \exp(a\rho + b)) \prod_{n=1}^{\infty}\left(1 - \frac{\rho}{\rho_n^0}\right)\exp\left(\frac{\rho}{\rho_n^0}\right)$$

If we consider the following equalities to find the constants a and b,

$$\lim_{\rho\to 0} \sin\left(\frac{\rho\pi^\alpha}{\alpha} - \theta(\pi)\right) = \lim_{\rho\to 0} \exp(a\rho + b)) \prod_{n=1}^{\infty}\left(1 - \frac{\rho}{\rho_n^0}\right)\exp\left(\frac{\rho}{\rho_n^0}\right)$$

and

$$\lim_{\rho\to 0}\frac{d}{d\rho}\ln\left[\sin\left(\frac{\rho\pi^\alpha}{\alpha} - \theta(\pi)\right)\right] = \lim_{\rho\to 0}\frac{d}{d\rho}\ln\left[C^0\exp(a\rho)) \prod_{n=1}^{\infty}\left(1 - \frac{\rho}{\rho_n^0}\right)\exp\left(\frac{\rho}{\rho_n^0}\right)\right]$$

then

$$C^0 = \exp(b) = -\sin\theta(\pi)$$

and

$$C_1^0 = a = -\frac{\pi^\alpha}{\alpha}\cot\theta(\pi)$$

respectively. Thus,

$$\Delta_\alpha^0(\rho) = -\rho C^0 \exp\left(C_1^0\rho\right) \prod_{n=1}^{\infty}\left(1 - \frac{\rho}{\rho_n^0}\right)\exp\left(\frac{\rho}{\rho_n^0}\right) \tag{26}$$

Moreover,

$$\Delta_\alpha(\rho) = C\exp(C_1\rho))\rho^m \prod_{n=1}^{\infty}\left(1 - \frac{\rho}{\rho_n}\right)\exp\left(\frac{\rho}{\rho_n}\right) \tag{27}$$

where C and C_1 are constants and $m \geq 0$. According to Equations 23 and 24,

$$\frac{\Delta_\alpha(\rho)}{\Delta_\alpha^0(\rho)} = 1 + O\left(\frac{1}{\rho}\right), \quad |\rho| \to \infty$$

Then, together with Equations 26 and 27,

$$\frac{\Delta_\alpha(\rho)}{\Delta_\alpha^0(\rho)} = -\frac{C}{C^0}\rho^{m-1}\prod_{n=1}^{\infty}\frac{\rho_n^0}{\rho_n}\prod_{n=1}^{\infty}\left(1 + \frac{\rho_n - \rho_n^0 - \rho}{\rho_n^0 - \rho}\right)\exp\left(\sum_{n=1}^{\infty}\frac{\rho_n^0 - \rho_n}{\rho_n\rho_n^0} + C_1 - C_1^0\right)\rho$$

Consequently,

$$m = 1, \quad C = -C^0\prod_{n=1}^{\infty}\frac{\rho_n}{\rho_n^0}, \quad \text{and} \quad C_1 = C_1^0 + \sum_{n=1}^{\infty}\left(\frac{1}{\rho_n^0} - \frac{1}{\rho_n}\right)$$

□

Theorem 3.3. If $\{\rho_n, \alpha_n\}_{n\geq 1} = \{\widetilde{\rho}_n, \widetilde{\alpha}_n\}_{n\geq 1}$ for each fixed α, then $q(x) = \widetilde{q}(x)$, almost everywhere in $[0,\pi]$, $h = \widetilde{h}$, and $H = \widetilde{H}$. Thus, the spectral data $\{\rho_n, \alpha_n\}_{n\geq 1}$ uniquely determines the operator L_α.

PROOF.
Since $\rho_n = \widetilde{\rho}_n$, according to Lemma 3.2, $\Delta_\alpha(\rho) = \widetilde{\Delta}_\alpha(\rho)$. Using Lemma 2.3 and $\alpha_n = \widetilde{\alpha}_n$, $\beta_n = \widetilde{\beta}_n$ and thus $\psi(0, \rho_n; \alpha) = \widetilde{\psi}(0, \rho_n; \alpha)$. For each fixed α, let

$$H_\alpha(\rho) := \frac{\psi(0,\rho;\alpha) - \widetilde{\psi}(0,\rho;\alpha)}{\Delta_\alpha(\rho)}$$

It is clear that $H_\alpha(\rho)$ is entire on ρ. Moreover, by using Equations 7 and 14,

$$H_\alpha(\rho) := O\left(\frac{1}{\rho^2}\right), \quad |\rho| \to \infty$$

Hence, $H_\alpha(\rho) \equiv 0$ and $\psi(0,\rho;\alpha) = \widetilde{\psi}(0,\rho;\alpha)$. Consequently, from Equation 17, $M_\alpha(\rho) \equiv \widetilde{M}_\alpha(\rho)$. Thus, the proof is completed by Theorem 3.1. □

4. Conclusion

The Weyl function and spectral data are very natural and useful spectral characteristics in inverse problem theory. Until today, by using these concepts, many inverse problems have been studied for various classes of operators, such as regular or singular Sturm-Liouville operators, diffusion operators, and Dirac operators, including the classical derivatives. In [18], some inverse problems for the Sturm-Liouville operator, including conformable fractional derivatives, are investigated.

In this study, the diffusion operator, which includes conformable fractional derivatives, is considered, and the inverse problems are investigated for this operator for the first time according to both the Weyl function and spectral data. This study can be considered as a partial α-generalization of similar findings for the classical diffusion operator.

Considering this study's results, some inverse problems can be investigated in the future for various conformable operators with jump conditions, parameter-dependent boundary conditions, or non-local boundary conditions.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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