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A NOTE ON STONE SPACES

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ABSTRACT. The goal of this paper is to characterize each of compact, totally disconnected, Stone relation spaces, and Stone reflexive spaces as well as examine the relationships between them. Finally, we investigate some properties of them and compare our results.

1. INTRODUCTION

If a topological space X is Hausdorff, totally disconnected, and compact, then X is called a Stone space [14]. Stone spaces are used in algebra, topology, functional analysis, the representation theory of rings, algebraic geometry, and mathematical logic [13, 14, 16, 17].
Categorical setting of compact Hausdorff spaces are studied by several authors[5, 9, 12, 15].

The notion of closedness which is being used in defining the Hausdorffness, openness, compactness, total disconnectedness was introduced in [3].

The category **Rel** of relation spaces where objects are sets with a binary relation and where morphisms $f : (A_1, R) \to (B_1, S)$ are functions with $f(a)Sf(b)$ if aRb for all $a, b \in A_1$ [10].
The category **RRel** of reflexive relation spaces is the full subcategory of **Rel** and they are topological categories [10].

Let $B \neq \emptyset$ and let $B^2 \bigvee_{\Delta} B^2$ be taking two distinct copies of B^2 identified along Δ.
The map $S : B^2 \vee_{\Delta} B^2 \to B^2$ is given by $S(a,b)_1 = (a,b,b)$ and $S(a,b)_2 = (a,a,b)$ and the map $A : B^2 \vee_{\Delta} B^2 \to B^3$ is given by $A(a,b)_1 = (a,b,a)$ and $A(a,b)_2 = (a,a,b)$.
The map $\nabla : B^2 \bigvee_{\Delta} B^2 \to B^2$ is given by $\bigtriangledown((a,b)_j) = (a,b)$ for $j = 1, 2$ [3].

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Let $X \in Ob(\mathbf{E})$ with $U(X) = B$, where $\mathbf{E}$ is a set based topological category.
Let S_W (resp. A_W) be the initial lift of the U-source S (resp. A) : $B^2 \bigvee_\Delta B^2 \to U(X^3) = B^3$.

Definition 1.1. *(cf.* [3, 4]*).*
(1) *If the initial lift of the U-source $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ and $A : B^2 \vee_\Delta B^2 \to U(X^3)$ is discrete, then X is said to be a $\overline{T_0}$ object, where D is the discrete functor.*
(2) *If the initial lift of the U-source $\nabla : B^2 \vee_\Delta B^2 \to U(D(B^2))$ and $id : B^2 \vee_\Delta B^2 \to U(B^2 \vee_\Delta B^2)'$ is discrete, then X is said to be a T_0' object.*
(3) *If $S_W = A_W$, then X is said to be a $Pre\overline{T}_2$ object.*
(4) *If X is $Pre\overline{T}_2$ and T_0' (resp. $\overline{T}_0$), then X is said to be a KT_2 (resp. $\overline{T}_2$) object.*

Let $\bigvee_x^\infty B$ be taking countably many disjoint copies of B and identifying them at the point $x \in B$. The map $A_x^\infty : \bigvee_x^\infty B \to B^\infty$ (resp. $\bigtriangledown_x^\infty : \bigvee_x^\infty B \longrightarrow B$) is given by $A_x^\infty(a_i) = (x, ..., x, a, x, x, ...)$ (resp. $\bigtriangledown_x^\infty(a_i) = a$ for all $i \in I$), where a_i is in the i-th component of $\bigvee_x^\infty B$ and B^∞ is the countable product of B [3].

Definition 1.2. *(cf.* [3, 5]*).*
(1) *If the initial lift of the U-source $\bigtriangledown_x^\infty : \vee_x^\infty B \to UD(B)$ and $A_x^\infty : \vee_x^\infty B \to U(X^\infty)$ is discrete, then $\{x\}$ is said to be closed.*
(2) *If $\{*\}$, the image of N, is closed in X/N or $N = \emptyset$, then N is said to be closed, where X/N is the final lift of the epi U-sink $Q : U(X) \to B/N = (B \backslash N) \cup \{*\}$, identifying N with a point *.*
(3) *If N^C, the complement of N, is closed, then N is said to be open.*
(4) *If the projection map $\pi_2 : X \times Z \longrightarrow Z$ is closed for each object Z in $\mathbf{E}$, then X is said to be a compact object.*

In **Top** (the category of topological spaces and continuous functions), $\overline{T}_0$ and T_0' (resp. $\overline{T}_2$ and KT_2) reduce to T_0 (resp. T_2) axiom [3]. Also, compactness (resp. openness and closedness) coincides with the usual compactness (resp. openness and closedness) [5].

Theorem 1.1. ***(1)*** *Every subset of a relation space is closed.*
(2) *Every relation space is compact.*

Proof. **(1)** Let (B, R)be a relation space and $N \subset B$. If $N = \emptyset$, then by Definition 1.2, N is closed. If $N = \{x\}$ for some $x \in B$, then let R_1 be the initial structure on $\vee_x^\infty B$ induced by $\bigtriangledown_x^\infty : \vee_x^\infty B \to (B, \emptyset)$ and $A_x^\infty : \vee_x^\infty B \to (B^\infty, R^\infty)$, where $\emptyset$ is the discrete relation on B and R^∞ is the product relation on B^∞. Since $\bigtriangledown_x^\infty : \vee_x^\infty B \to (B, \emptyset)$ is a relation preserving map and $(B, \emptyset)$ is discrete, we have $R_1 = \emptyset$ and so, $\{x\}$ is closed in (B, R).
If N has cardinality at least 2, then $\{*\}$ is closed in B/N and by Definition 1.2, N is closed.
(2) follows from Part (1) and Definition 1.2. □

Theorem 1.2. *A reflexive space (B, R) is compact iff for every $x \in A$ there exist $a, b \in B$ with xRa and bRx.*

Proof. It is proved in [8]. □

2. Stone Spaces

We introduce two new Stone objects in a topological category and find relationships between them. Moreover, we characterize each of Stone relation spaces and

Stone reflexive spaces and compare our results.
Let $X \in Ob(\mathbf{E})$ and $N \subset X$. Recall, in [7], that the quasi-component closure $Q_X(N)$ of N is the intersection of all open and closed subsets of X containing N.

Definition 2.1. ***(1)*** *If every quasi-component of X contains only one point, then X is said to be totally disconnected.*
(2) *If X is KT_2 (resp. $\overline{T}_2$), compact, and totally disconnected, then X is called a TKT_2 (resp. $T\overline{T}_2$) object.*
An object satisfying the condition (2) will be called a Stone object.

In **Top**, the notion of total disconnectedness coincide with the usual total disconnectedness [2, 7, 11]. Moreover, TKT_2 and $T\overline{T}_2$ Stone spaces reduce to the usual Stone spaces [14].

Theorem 2.1. *Every $T\overline{T}_2$ Stone object is TKT_2.*

Proof. Let $X \in Ob(\mathbf{E})$, where $U : \mathbf{E} \to \mathbf{Set}$ is topological.
If X is a $T\overline{T}_2$ Stone object, then, X is $\overline{T}_2$ and by Definition 1.1, X is $Pre\overline{T_2}$ and $\overline{T}_0$. Since X is $\overline{T}_0$, by Theorem 2.7 of [4], X is T_0' and so, X is KT_2. Hence, X is TKT_2. □

Theorem 2.2. ***(1)*** *Every relation space is totally disconnected.*
(2) *For a relation space (B, R), the following are equivalent:*
(i) *(B, R) is $T\overline{T}_2$.*
(ii) *(B, R) is TKT_2.*
(iii) *For each $x, y \in B$ there exists $z \in B$ with xRz and yRz, then for any $w \in B$, xRw iff yRw.*

Proof. **(1)** Since by Theorem 1.3, $Q(s) = \{s\}$ for all $s \in B$, then (B, R) is totally disconnected.
(2) By Theorem 1.3 and Part (1), a relation space (B, R) is compact and totally disconnected. By Theorem 3.5 of [8], we get the result. □

Theorem 2.3. *A reflexive space(B, R) is TKT_2 iff it is $T\overline{T}_2$.*

Proof. By Theorems 3.2 and 5.2 of [7], a reflexive space (B, R) is $\overline{T}_2$ iff it is KT_2 and totally disconnected, and by Definition 2.1, one has the result. □

Let $\mathbf{TKT_2Rel}$ and $\mathbf{T\overline{T}_2Rel}$ be the full subcategory of **Rel** whose objects are the TKT_2 or $T\overline{T}_2$ Stone relation spaces.

Theorem 2.4. *The categories $\mathbf{TKT_2Rel}$ and $\mathbf{T\overline{T}_2Rel}$ are isomorphic topological categories.*

Proof. By Theorem 2.2 and Theorem 3.4 of [6], one has the result. □

Recall, in [8], that if X is KT_2 (resp. $\overline{T}_2$), compact, and extremally disconnected, then X is called a EKT_2 (resp. $E\overline{T}_2$) Stone object.

We can infer the foolowing results:
(1) In **Rel**, by Theorem 2.2 and Theorem 4.5 of [8], all TKT_2, EKT_2, $E\overline{T}_2$, and $T\overline{T}_2$ Stone relation spaces are equivalent and by Theorem 2.4, the subcategories $\mathbf{TKT_2Rel}$, $\mathbf{T\overline{T}_2Rel}$, $\mathbf{EKT_2Rel}$, and $\mathbf{E\overline{T}_2Rel}$ have all limits and colimits. By Theorem 4.5 of [8] and Theorems 1.3 and 2.2, a relation space is totally disconnected

iff it is extremally disconnected.
(2) In **RRel**, by Theorems 3.2 and 5.2 of [7] and Theorem 4.6 of [8], $\overline{T}_2$ implies each of KT_2, extremally disconnected, and totally disconnected. The indiscrete reflexive space $(\{m,n\},\{m,n\}^2)$ is KT_2 and extremally disconnected but it is neither $\overline{T}_2$ nor totally disconnected. $(\{m,n\},\{(m,m),(n,n),(n,m)\})$ is totally disconnected but it is neither KT_2 nor $\overline{T}_2$. By Theorem 4.6 of [8] and Theorem 2.3, $TKT_2 = T\overline{T}_2 = E\overline{T}_2 \Rightarrow EKT_2$ but $(\{m,n\},\{m,n\}^2)$ is EKT_2 but it is neither TKT_2 nor $T\overline{T}_2$ nor a $E\overline{T}_2$ Stone reflexive space.
(3) In arbitrary topological category, by Theorem 2.1, every $T\overline{T}_2$ Stone object is TKT_2.

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