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Research Article

Common fixed point theorems in multiplicative $\mathfrak{m}$-metric space with applications to the system of multiplicative integral equations and numerical results

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Abstract

This paper presents common fixed point results for a pair of self-mappings in multiplicative $\mathfrak{m}$-metric space. Also, we present the multiplicative partial metric structure as a specific caseăof a multiplicative m-metric space and demonstrate some common fixed point results. To support our conclusions, we present an illustrative example with discontinuous self-mappings. We also provide numerical iterations to approximate the common fixed point and graphs to visually substantiate the results. As the consequences of our results, we demonstrate several common fixed point results in $\mathfrak{m}$-metric space and partial metric space. Our findings generalize various fixed point results from the literature. Furthermore, we employ the results to demonstrate the existence and uniqueness of solutions to a system of multiplicative Fredholm integral equations.

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1. Introduction

Non-Newtonian calculus provides a new perspective to intricate systems and phenomena, with applications in diverse fields such as economics, engineering, and physics. This theory opens up new possibilities for analyzing and understanding these systems, offering a fresh perspective that can lead to innovative solutions and advancements in these fields. In 1972, Grossman and Katz [8] contributed significantly to non-Newtonian calculus, building on Robinson's foundational development of non-standard analysis [16] in the 1960s. Their work introduced a comprehensive framework based on ultrapowers and hyperreals, providing a rigorous structure for non-Newtonian calculus that aligns with conventional mathematics. Stanley [18] made substantial contributions to the discipline of "multiplicative calculus," commonly known as "geometric calculus." He was a pioneer in formulating

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the theory of alternative calculus, where the essential notions of differentiation and integration are defined using multiplicative algebraic operations. Multiplicative calculus has applications in various fields, including physics, biology, and economics, where multiplicative processes are more natural or relevant than additive ones. It provides a framework for dealing with such processes and offers insights that may not be easily obtained using traditional calculus.

The concept of a metric space is widely used in mathematics to explore distances and topology. In a typical metric space, the distance function is clearly defined and meets specific criteria, such as non-negativity, commutativity, and triangle inequality. However, in certain cases, these characteristics may be insufficient, necessitating the use of a generalized metric space. In such spaces, the distance function may not adhere to all the traditional prerequisites.

Alternative approaches always have a significant role in understanding and broadening a number of aspects in mathematics and related branches. The developments in multiplicative calculus and its applications to metric spaces offer new perspectives and tools for studying mathematical structures, particularly in areas where conventional calculus may not be suitable or applicable. These advancements have implications for various fields, including mathematics, physics, and computer science, providing alternative approaches to modeling and analyzing complex systems.

In 2008, Bashirov et al. [6] utilized multiplicative absolute values and established a new distance function in multiplicative calculus. This approach led to the establishment of a framework for a multiplicative metric space, which serves as an alternative to the conventional metric space. Subsequently, in 2012, Ozavsar and Cevikel [14] explored some fixed point results in multiplicative metric spaces for various types of contractions, including Banach-type, Kannan-type, and Chatterjea-type contractions (see [5, 7, 11]). These results have applications in the study of functional analysis and nonlinear analysis.

The generalization of metric space is always a key aspect of fixed point theory in order to extend the applicability to a broader class of spaces. In 1994, Matthews [12] gave insights to the scenario of non-zero self distance and conceptualized the notion of partial metric space. The space was further generalized in 2014 by Asadi et al. [4] by introducing the notion of an $\mathfrak{m}$-metric space. The generalized space possesses a rich topological structure as an extended version of partial metric space in several prerequisites. Later many researcher presented fixed point result using various contraction condition (for reference, see [1–3, 9, 10, 13, 15, 17, 19, 21]).

In this manuscript, we present some common fixed point results for a pair of self mapping in the framework of multiplicative $\mathfrak{m}$-metric space extending the fixed point results proved in [20]. We establish the multiplicative partial metric structure as a special case of multiplicative $\mathfrak{m}$-metric space and proved some common fixed point results. In order to prove the authenticity of the results we provide an illustrative example with discontinuous self mappings. We present some numerical iterations to approximate the common fixed point and graphs to provide the visual support to our findings. The proven results generalize a number of fixed point results in the existing literature. Additionally, we utilize the novel results to check the existence and uniqueness of the solution to a system of multiplicative integral equations.

2. Preliminaries

In this section, we discuss some definitions and results regarding the multiplicative $\mathfrak{m}$-metric space. Also, the symbols, $\mathbb{R}$ denote the set of real numbers, $\mathbb{N}$ denote natural numbers, $\mathbb{R}_0$ denote non-negative real numbers. Also, for non-empty set Ω, the multiplicative, usual and $\mathfrak{m}$-metric spaces are symbolize by the pairs (Ω, u), (Ω, d), $(\Omega, \mathfrak{m})$.

Definition 2.1 ([20]). A mapping $\delta : M \times M \to [1, \infty)$ is said to be multiplicative $\mathfrak{m}$-metric if :

(i) $\delta(\rho, \vartheta) = \delta(\vartheta, \vartheta) = \delta(\rho, \rho) \Leftrightarrow \rho = \vartheta$;

(ii) $\delta_{\rho\vartheta} \leq \delta(\rho, \vartheta)$;

(iii) $\delta(\rho, \vartheta) = \delta(\vartheta, \rho)$;

(iv) $\dfrac{\delta(\rho, \vartheta)}{\delta_{\rho\vartheta}} \leq \dfrac{\delta(\rho, \zeta)}{\delta_{\rho\zeta}} . \dfrac{\delta(\zeta, \vartheta)}{\delta_{\zeta\vartheta}}$,

where $\delta_{\rho\vartheta} = \min\Big\{\delta(\rho, \rho), \delta(\vartheta, \vartheta)\Big\}$, and $\delta^*_{\rho\vartheta} = \max\Big\{\delta(\rho, \rho), \delta(\vartheta, \vartheta)\Big\}$ for all $\rho, \vartheta, \zeta \in M$. Also, (M, δ) is said to be a multiplicative $\mathfrak{m}$-metric space (M.$\mathfrak{m}$-MS).

Example 2.2 ([20]). Let $M = [0, \infty)$ and $\delta(\rho, \vartheta) = e^{\frac{\rho+\vartheta}{2}}$, then (M, δ) is a M.$\mathfrak{m}$-MS. But (M, δ) is not a multiplicative metric space. As, for $\rho \neq 0$, $\delta(\rho, \rho) = e^{\rho} \neq 1$.

Remark 2.3 ([20]). Consider a M.$\mathfrak{m}$-MS (M, δ). Then, we have

(i) $1 \leq \delta^*_{\rho\vartheta} . \delta_{\rho\vartheta} = \delta(\rho, \rho) . \delta(\vartheta, \vartheta)$;

(ii) $1 \leq \dfrac{\delta^*_{\rho\vartheta}}{\delta_{\rho\vartheta}} = \left|\dfrac{\delta(\rho, \rho)}{\delta(\vartheta, \vartheta)}\right|_*$;

(iii) $\dfrac{\delta^*_{\rho\vartheta}}{\delta_{\rho\vartheta}} \leq \dfrac{\delta^*_{\rho\zeta}}{\delta_{\rho\zeta}} . \dfrac{\delta^*_{\zeta\vartheta}}{\delta_{\zeta\vartheta}}$,

for all $\rho, \vartheta, \zeta \in M$, where $\delta_{\rho\vartheta} = \min\Big\{\delta(\rho, \rho), \delta(\vartheta, \vartheta)\Big\}$, $\delta^*_{\rho\vartheta} = \max\Big\{\delta(\rho, \rho), \delta(\vartheta, \vartheta)\Big\}$ and

$$|a|_* = \begin{cases} a, a \geq 1; \\ \frac{1}{a}, a < 1. \end{cases} \quad \text{for } a \in \mathbb{R}_+.$$

Remark 2.4 ([20]). Consider a M.$\mathfrak{m}$-MS (M, δ). Then,

(i) (a) $\delta^w(\rho, \vartheta) = \dfrac{\delta(\rho, \vartheta) . \delta^*_{\rho\vartheta}}{\delta_{\rho\vartheta} . \delta_{\rho\vartheta}}$,

(b) $\delta_s(\rho, \vartheta) = \begin{cases} \frac{\delta(\rho, \vartheta)}{\delta_{\rho\vartheta}}, & \text{if } \rho \neq \vartheta \\ 1, & \text{if } \rho = \vartheta, \end{cases}$

are multiplicative metric on M.

(ii) (a) $\dfrac{\delta(\rho, \vartheta)}{\delta^*_{\rho\vartheta}} \leq \delta^w(\rho, \vartheta) \leq \delta(\rho, \vartheta); \delta^*_{\rho\vartheta}$.

(b) $\dfrac{\delta(\rho, \vartheta)}{\delta^*_{\rho\vartheta}} \leq \delta_s(\rho, \vartheta) \leq \delta(\rho, \vartheta)$,

for all $\rho, \vartheta \in M$.

Definition 2.5 ([20]). A sequence $\{\rho_\kappa\}$ in (M, δ) is said to be multiplicative

(i) **convergent** to ρ if

$$\lim_{\kappa\to\infty} \frac{\delta(\rho_\kappa, \rho)}{\delta_{\rho_\kappa\rho}} = 1.$$

(ii) **$\mathfrak{m}$-Cauchy** if

$$\lim_{\kappa,\ell\to\infty} \frac{\delta(\rho_\kappa, \rho_\ell)}{\delta_{\rho_\kappa\rho_\ell}} \quad \text{and} \quad \lim_{\kappa,\ell\to\infty} \frac{\delta^*_{\rho_\kappa\rho_\ell}}{\delta_{\rho_\kappa\rho_\ell}} \text{ exist finitely.}$$

Also, (M, δ) is said to be complete if for every multiplicative $\mathfrak{m}$-Cauchy sequence $\{\rho_\kappa\}$ in M, there exists a point $\rho \in M$ such that

$$\lim_{\kappa\to\infty} \frac{\delta(\rho_\kappa, \rho)}{\delta_{\rho_\kappa\rho}} = 1 \quad \text{and} \quad \lim_{\kappa\to\infty} \frac{\delta^*_{\rho_\kappa\rho}}{\delta_{\rho_\kappa\rho}} = 1.$$

Lemma 2.6 ([20]). *Let $\{\rho_\kappa\} \to \rho$ and $\{\vartheta_\kappa\} \to \rho$ be two sequences in (M, δ) such that $\rho_\kappa \to \rho$ and $\vartheta_\kappa \to \vartheta$. Then,*

$$\lim_{\kappa\to\infty} \frac{\delta(\rho_\kappa, \vartheta_\kappa)}{\delta_{\rho_\kappa\vartheta_\kappa}} = \frac{\delta(\rho, \vartheta)}{\delta_{\rho\vartheta}}.$$

Lemma 2.7 ([20]). *Let $\{\rho_\kappa\}$ be a sequence in (M, δ) such that $\rho_\kappa \to \rho$ and $\rho_\kappa \to \vartheta$. Then $\delta(\rho, \vartheta) = \delta_{\rho\vartheta}$. Also, in case $\delta(\rho, \rho) = \delta(\vartheta, \vartheta)$, then $\rho = \vartheta$.*

Lemma 2.8 ([20]). *Let $\{\rho_\kappa\}$ be a sequence in (M, δ). If there exists some $r \in [0, 1)$ such that*

$$\delta(\rho_{\kappa+1}, \rho_\kappa) \leq \delta(\rho_\kappa, \rho_{\kappa-1})^r, \text{ for all } \kappa \in \mathbb{N}. \tag{2.1}$$

Then,

(i) $\lim_{\kappa\to\infty} \delta(\rho_{\kappa+1}, \rho_\kappa) = 1$.
(ii) $\lim_{\kappa\to\infty} \delta(\rho_\kappa, \rho_\kappa) = 1$.
(iii) $\lim_{\kappa,\ell\to\infty} \delta_{\rho_\kappa\rho_\ell} = 1$.
(iv) *$\{\rho_\kappa\}$ is a multiplicative $\mathfrak{m}$-Cauchy sequence.*

3. Main result

In this section, we have established some common fixed point results using generalized contraction in the framework of $\mathfrak{m}$-metric space. Moreover, we have discussed the structure of multiplicative partial metric as a subcase of the M.$\mathfrak{m}$-MS and proved some common fixed point results.

Theorem 3.1. *Let $S, T : M \to M$ be self mappings defined on complete M.$\mathfrak{m}$-MS (M, δ). If there exist $a_1, a_2, a_3 \in [0, 1)$ with $a_1 + a_2 + a_3 < 1$ such that*

$$\delta(S\rho, T\vartheta) \leq (\delta(\rho, \vartheta))^{a_1} (\delta(\rho, S\rho))^{a_2} (\delta(\vartheta, T\vartheta))^{a_3}, \text{ for all } \rho, \vartheta \in M. \tag{3.1}$$

Then, either S or T has a fixed point say $\rho_0 \in M$. Moreover, if $\delta^*_{S\rho_0,T\rho_0} \leq \delta(S\rho_0, T\rho_0)$, then S, T have a unique common fixed point.

Proof. For $\rho_0 \in M$, we can easily construct a sequence in M defined as

$$\rho_{2\kappa+1} = S\rho_{2\kappa}, \text{ and } \rho_{2\kappa+2} = T\rho_{2\kappa+1} \text{ for } \kappa \in \mathbb{N}_0.$$

If for some $\kappa_0 \in \mathbb{N}_0$ we have $\rho_{2\kappa_0+1} = \rho_{2\kappa_0+2}$. Then,

$$\rho_{2\kappa_0+1} = S\rho_{2\kappa_0} = \rho_{2\kappa_0+2} = T\rho_{2\kappa_0+1}$$

implies that $\rho_{2\kappa_0+1} = \rho_{2\kappa_0+2}$ is the fixed point of mapping T.
Now, consider $\rho_{2\kappa+1} \neq \rho_{2\kappa+2}$ for $\kappa \in \mathbb{N}_0$. Then,

$$\begin{aligned}\delta(\rho_{2\kappa+1}, \rho_{2\kappa+2}) &= \delta(S\rho_{2\kappa}, T\rho_{2\kappa+1})\\ &\leq (\delta(\rho_{2\kappa}, \rho_{2\kappa+1}))^{a_1}(\delta(\rho_{2\kappa}, S\rho_{2\kappa}))^{a_2}(\delta(\rho_{2\kappa+1}, T\rho_{2\kappa+1}))^{a_3}\\ &= (\delta(\rho_{2\kappa}, \rho_{2\kappa+1}))^{a_1}(\delta(\rho_{2\kappa}, \rho_{2\kappa+1}))^{a_2}(\delta(\rho_{2\kappa+1}, \rho_{2\kappa+2}))^{a_3},\end{aligned}$$

or

$$\delta(\rho_{2\kappa+1}, \rho_{2\kappa+2})^{1-a_3} \leq (\delta(\rho_{2\kappa}, \rho_{2\kappa+1}))^{a_1+a_2} \Leftrightarrow \delta(\rho_{2\kappa+1}, \rho_{2\kappa+2}) \leq (\delta(\rho_{2\kappa}, \rho_{2\kappa+1}))^{\frac{a_1+a_2}{1-a_3}}.$$

Using similar arguments, we have

$$\delta(\rho_{2\kappa+2}, \rho_{2\kappa+3}) \leq (\delta(\rho_{2\kappa+1}, \rho_{2\kappa+2}))^{\frac{a_1+a_2}{1-a_3}}.$$

Therefore,

$$\delta(\rho_\kappa, \rho_{\kappa+1}) \leq \delta(\rho_{\kappa-1}, \rho_\kappa)^{\frac{a_1+a_2}{1-a_3}} \text{ or } \delta(\rho_{\kappa+1}, \rho_\kappa) \leq \delta(\rho_\kappa, \rho_{\kappa-1})^{\frac{a_1+a_2}{1-a_3}}, \text{ for all } \kappa \in \mathbb{N}.$$

As $\frac{a+b}{1-c} < 1$. Using Lemma 2.8, we have

$$\lim_{\kappa\to\infty} \delta(\rho_{\kappa+1}, \rho_\kappa) = 1, \tag{3.2}$$

$$\lim_{\kappa\to\infty} \delta(\rho_\kappa, \rho_\kappa) = 1, \tag{3.3}$$

$$\lim_{\kappa,\ell\to\infty} \delta_{\rho_\kappa,\rho_\ell} = 1 \tag{3.4}$$

and $\{\rho_\kappa\}$ is a multiplicative $\mathfrak{m}$-Cauchy sequence. As, (M, δ) is multiplicative $\mathfrak{m}$-complete, therefore there exist some $\rho \in M$ such that

$$\lim_{\kappa\to\infty} \frac{\delta(\rho_\kappa, \rho)}{\delta_{\rho_\kappa,\rho}} = 1 \text{ and } \lim_{\kappa\to\infty} \frac{\delta^*(\rho_\kappa, \rho)}{\delta^*_{\rho_\kappa,\rho}} = 1. \tag{3.5}$$

Moreover, using (3.3), we have

$$\lim_{\kappa\to\infty} \delta_{\rho_\kappa\rho} = \lim_{\kappa\to\infty} \min\{\delta(\rho_\kappa, \rho_\kappa), \delta(\rho, \rho)\} \leq \lim_{\kappa\to\infty} \delta(\rho_\kappa, \rho_\kappa) = 1. \tag{3.6}$$

Using (3.5), (3.6) and Remark , we have

$$\lim_{\kappa\to\infty} \delta(\rho_\kappa, \rho) = 1, \lim_{\kappa\to\infty} \delta^*_{\rho_\kappa\rho} = 1 \text{ and } \delta(\rho, \rho) = 1. \tag{3.7}$$

Also,

$$\delta_{\rho,S\rho} = \min\{\delta(\rho, \rho), \delta(S\rho, S\rho)\} \leq \delta(\rho, \rho) = 1, \tag{3.8}$$

and

$$\delta_{\rho,T\rho} = \min\{\delta(\rho, \rho), \delta(T\rho, T\rho)\} \leq \delta(\rho, \rho) = 1. \tag{3.9}$$

Further, using (3.6), (3.9) and the triangle inequality, we have

$$\begin{aligned}
\delta(\rho, T\rho) = \frac{\delta(\rho, T\rho)}{\delta_{\rho,T\rho}} &\leq \frac{\delta(\rho, \rho_{2\kappa+2})}{\delta_{\rho,\rho_{2\kappa+1}}}.\frac{\delta(\rho_{2\kappa+1}, T\rho)}{\delta_{\rho_{\rho_{2\kappa+1}},T\rho}} \\
&\leq \limsup_{\kappa\to\infty} \frac{\delta(\rho, \rho_{2\kappa+2})}{\delta_{\rho,\rho_{2\kappa+1}}}.\frac{\delta(\rho_{2\kappa+1}, T\rho)}{\delta_{\rho_{2\kappa+1},T\rho}} \\
&\leq \limsup_{\kappa\to\infty} \delta(\rho_{2\kappa+1}, T\rho),
\end{aligned}$$

or

$$\delta(\rho, T\rho) \leq \limsup_{\kappa\to\infty} \delta(S\rho_{2\kappa}, T\rho). \tag{3.10}$$

Using (3.1), (3.6), (3.9) and the triangle inequality in (3.10), we have

$$\begin{aligned}
\delta(\rho, T\rho) &\leq \limsup_{\kappa\to\infty} \delta(S\rho_{2\kappa}, T\rho) \\
&\leq \limsup_{\kappa\to\infty} (\delta(\rho_{2\kappa}, \rho))^{a_1} (\delta(\rho_{2\kappa}, S\rho_{2\kappa}))^{a_2} \delta(\rho, T\rho)^{a_3} \\
&\leq \limsup_{\kappa\to\infty} (\delta(\rho_{2\kappa}, \rho))^{a_1} (\delta(\rho_{2\kappa}, \rho_{2\kappa+1}))^{a_2} \delta(\rho, T\rho)^{a_3} \\
&= \delta(\rho, T\rho)^{a_3}.
\end{aligned}$$

Since, $a_3 < 1$. Hence,

$$\delta(\rho, T\rho) = 1. \tag{3.11}$$

Similarly, one can easily observe that

$$\delta(\rho, S\rho) = 1. \tag{3.12}$$

Using (3.1), (3.7), (3.11) and (3.12), we have

$$\delta(S\rho, T\rho) \leq (\delta(\rho, \rho))^{a_1}.(\delta(\rho, S\rho))^{a_2}.(\delta(\rho, T\rho))^{a_3} = 1,$$

or

$$\delta(S\rho, T\rho) = 1. \tag{3.13}$$

Also,

$$\delta_{S\rho,T\rho} = \min\{\delta(S\rho, S\rho), \delta(T\rho, T\rho)\} \leq \delta(S\rho, T\rho) = 1.$$

Suppose, $\delta(S\rho, S\rho) \le \delta(T\rho, T\rho)$. Then $\delta(S\rho, S\rho) = 1$. Hence, $\delta(S\rho, S\rho) = 1 = \delta(\rho,\rho) = \delta(\rho, S\rho)$ implies $S\rho = \rho$, i.e., ρ is the fixed point of S.
Further, suppose that $\delta^*_{S\rho,T\rho} \le \delta(S\rho, T\rho)$. Then,

$$\delta(T\rho, T\rho) = \max\{\delta(S\rho, S\rho), \delta(T\rho, T\rho)\} = \delta^*_{S\rho,T\rho} \le \delta(S\rho, T\rho) = 1,$$

or

$$\delta(T\rho, T\rho) = 1.$$

Therefore,

$$\delta(S\rho, T\rho) = \delta(S\rho, S\rho) = \delta(T\rho, T\rho) = 1,$$

implies $S\rho = T\rho = \rho$.
Uniqueness: Suppose that $\vartheta \ne \rho \in M$ is a common fixed point of S, T. Then, using (3.1), we have

$$\begin{aligned}\delta(\rho, \vartheta) &= \delta(S\rho, T\vartheta)\\ &\le \Big(\delta(\rho,\vartheta)\Big)^{a_1}\Big(\delta(\rho, S\rho)\Big)^{a_2}\Big(\delta(\vartheta, T\vartheta)\Big)^{a_3}\\ &= \Big(\delta(\rho,\vartheta)\Big)^{a_1}\Big(\delta(\rho,\rho)\Big)^{a_2}\Big(\delta(\vartheta,\vartheta)\Big)^{a_3}\\ &= \Big(\delta(\rho,\vartheta)\Big)^{a_1} < \delta(\rho,\vartheta),\end{aligned}$$

a contradiction. Hence, $\vartheta = \rho$. □

Definition 3.2. A mapping $\wp : M \times M \to [1, \infty)$ is said to be multiplicative partial-metric if :

(i) $\wp(\rho, \vartheta) = \wp(\vartheta, \vartheta) = \wp(\rho, \rho) \Leftrightarrow \rho = \vartheta$;
(ii) $\wp(\rho, \rho) \le \wp(\rho, \vartheta)$;
(iii) $\wp(\rho, \vartheta) = \wp(\vartheta, \rho)$;
(iv) $\wp(\rho, \vartheta) \le \frac{\wp(\rho,\zeta).\wp(\zeta,\vartheta)}{\wp(\zeta,\zeta)}$,

for all $\rho, \vartheta, \zeta \in M$. Also, $(M, \wp)$ is said to be a multiplicative partial-metric space.

Example 3.3. For $M = \mathbb{R}_0$ with $\wp(\rho, \vartheta) = e^{\max\{\rho,\vartheta\}}$, $(M, \wp)$ is a multiplicative partial-metric space(MPMS).

Remark 3.4. Every MPMS is a M.$\mathfrak{m}$-MS. But, converse is not true. For instance, $M = \mathbb{R}_0$ with $\delta(\rho, \vartheta) = e^{\frac{\rho+\vartheta}{2}}$ is a multiplicative $\mathfrak{m}$-metric. Clearly, it is not a multiplicative partial-metric as $\delta(1, 1) = e, \delta(2, 2) = e^2$ but $\delta(1, 2) = e^{\frac{3}{2}} < \delta(2, 2)$.

Theorem 3.5. *Let $S, T : M \to M$ be self mappings defined on complete MPMS $(M, \wp)$. If there exist $a_1, a_2, a_3 \in [0, 1)$ with $a_1 + a_2 + a_3 < 1$ such that*

$$\wp(S\rho, T\vartheta) \le (\wp(\rho, \vartheta))^{a_1}(\wp(\rho, S\rho))^{a_2}(\wp(\vartheta, T\vartheta))^{a_3}, \text{ for all } \rho, \vartheta \in M. \tag{3.14}$$

Then S, T have a unique common fixed point.

Proof. As every MPMS is a M.$\mathfrak{m}$-MS. Therefore, the result can be easily obtained using the approach discussed in Theorem 3.1 by taking in account of the fact that $\wp$ is a multiplicative partial metric and $\wp(S\rho, S\rho), \wp(T\rho, T\rho) \le \wp(S\rho, T\rho)$. □

Corollary 3.6. *Let $S, T : M \to M$ be self mappings defined on complete MPMS $(M, \wp)$. If there exists $a \in (0, 1)$ such that*

$$\wp(S\rho, T\vartheta) \le (\wp(\rho, \vartheta))^{a}, \text{ for all } \rho, \vartheta \in M.$$

Then S, T have a unique common fixed point.

Corollary 3.7. *Let $S, T : M \to M$ be self mappings defined on complete MPMS $(M, \wp)$. If there exists $k \in (0, \frac{1}{2})$ such that*

$$\wp(S\rho, T\vartheta) \leq (\wp(\rho, S\rho).\wp(\vartheta, T\vartheta))^k, \text{ for all } \rho, \vartheta \in M.$$

Then S, T have a unique common fixed point.

Example 3.8. Let $M = [0, \infty)$ equipped with multiplicative $\mathfrak{m}$-metric $\delta(\rho, \vartheta) = e^{\max\{\rho, \vartheta\}}$. Let S, T are self mapping defined on $M = [0, \infty)$ as

$$S\rho = \begin{cases} \frac{\rho}{5}, & \text{if } \rho \in [0,1) \\ \frac{1}{10}, & \text{otherwise.} \end{cases} \quad T\rho = \begin{cases} \frac{\rho}{7}, & \text{if } \rho \in [0,1) \\ \frac{1}{14}, & \text{otherwise.} \end{cases}.$$

The, (M, δ) is a complete M.$\mathfrak{m}$-MS. Also,

(i) for $\rho, \vartheta \in [0, 1)$, we have

$$\begin{aligned} \delta(S\rho, T\vartheta) &= e^{\max\{S\rho, T\vartheta\}} \\ &= e^{\max\{\frac{\rho}{5}, \frac{\rho}{7}\}} = e^{\frac{\rho}{5}} \\ &\leq \delta(\rho, \vartheta)^{\frac{1}{5}} \end{aligned}$$

(ii) for $\rho, \vartheta \geq 1$, we have

$$\begin{aligned} \delta(S\rho, T\vartheta) &= e^{\max\{S\rho, T\vartheta\}} \\ &= e^{\max\{\frac{1}{10}, \frac{1}{14}\}} = e^{\frac{1}{10}} \\ &\leq (e^{\max\{\rho, \vartheta\}})^{\frac{1}{5}} = \delta(\rho, \vartheta)^{\frac{1}{5}} \end{aligned}$$

(iii) for $\rho > 1, \vartheta \leq \frac{7}{10}$, we have

$$\begin{aligned} \delta(S\rho, T\vartheta) &= e^{\max\{S\rho, T\vartheta\}} \\ &= e^{\max\{\frac{1}{10}, \frac{\vartheta}{7}\}} = e^{\frac{1}{10}} \\ &\leq (e^{\max\{\rho, \vartheta\}})^{\frac{1}{5}} = \delta(\rho, \vartheta)^{\frac{1}{5}} \end{aligned}$$

(iv) for $\rho > 1, \frac{7}{10} < \vartheta < 1$, we have

$$\begin{aligned} \delta(S\rho, T\vartheta) &= e^{\max\{S\rho, T\vartheta\}} \\ &= e^{\max\{\frac{1}{10}, \frac{\vartheta}{7}\}} = e^{\frac{\vartheta}{7}} \\ &\leq (e^{\max\{\rho, \vartheta\}})^{\frac{1}{7}} \leq \delta(\rho, \vartheta)^{\frac{1}{5}} \end{aligned}$$

(v) for $\vartheta > 1, \rho \leq \frac{5}{14}$, we have

$$\begin{aligned} \delta(S\rho, T\vartheta) &= e^{\max\{S\rho, T\vartheta\}} \\ &= e^{\max\{\frac{\rho}{5}, \frac{1}{14}\}} = e^{\frac{1}{14}} \leq (e^{\vartheta})^{\frac{1}{7}} \\ &\leq (e^{\max\{\rho, \vartheta\}})^{\frac{1}{7}} \leq \delta(\rho, \vartheta)^{\frac{1}{5}} \end{aligned}$$

(vi) for $\vartheta > 1, \frac{5}{14} < \rho < 1$, we have

$$\begin{aligned} \delta(S\rho, T\vartheta) &= e^{\max\{S\rho, T\vartheta\}} \\ &= e^{\max\{\frac{\rho}{5}, \frac{1}{14}\}} = e^{\frac{\rho}{5}} \leq (e^{\rho})^{\frac{1}{5}} \\ &\leq (e^{\max\{\rho, \vartheta\}})^{\frac{1}{5}} \end{aligned}$$

and $\delta^*_{S\rho, T\rho} = \max\{\delta(S\rho, S\rho), \delta(T\rho, T\rho)\} = \max\left\{e^{S\rho}, e^{T\rho}\right\} \leq e^{\max\{S\rho, T\rho\}} = \delta(S\rho, T\rho)$. Therefore, S, T satisfies all the conditions of Theorem (3.1) with $a_1 = \frac{1}{5}, a_2 = 0 = a_3$. Hence, S, T have a unique common fixed point.

4. Comparison with existing literature

In this section, we have examined the potential consequences of the findings presented in the main section of the manuscript. The previous section's finding can be utilized to obtain some fixed point results in the framework of $\mathfrak{m}$-metric space [4] and partial metric space [12].

Remark 4.1. Let (M, δ) be a M.$\mathfrak{m}$-MS.Then, $\mathfrak{m}(\rho, \vartheta) = \ln(\delta(\rho, \vartheta))$ is an $\mathfrak{m}$-metric. Also, if $(M, \wp)$ be a MPMS, then $p(\rho, \vartheta) = \ln(\wp(\rho, \vartheta))$ is an partial-metric.

Theorem 4.2. *Let $S, T : M \to M$ be self mappings defined on complete $\mathfrak{m}$-metric space $(M, \mathfrak{m})$. If there exist $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$ such that*

$$\mathfrak{m}(S\rho, T\vartheta) \leq \alpha \left(\mathfrak{m}(\rho, \vartheta)\right) + \beta \left(\mathfrak{m}(\rho, S\rho)\right) + \gamma \left(\mathfrak{m}(\vartheta, T\vartheta)\right), \text{ for all } \rho, \vartheta \in M.$$

Then either S or T have a fixed point say $\rho_0 \in M$. Moreover, if $M_{S\rho_0, T\rho_0} \leq \mathfrak{m}(S\rho_0, T\rho_0)$, then S, T have a unique common fixed point.

Proof. Consider $\delta(\rho, \vartheta) = e^{\mathfrak{m}(\rho, \vartheta)}$. Then, S, T satisfies all the conditions of Theorem 3.1. Hence, S, T have a unique common fixed point. □

Theorem 4.3. *Let $T : M \to M$ be a self mapping defined on complete m-metric space $(M, \mathfrak{m})$. If there exist $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$ such that*

$$\mathfrak{m}(T\rho, T\vartheta) \leq \alpha \left(\mathfrak{m}(\rho, \vartheta)\right) + \beta \left(\mathfrak{m}(\rho, T\rho)\right) + \gamma \left(\mathfrak{m}(\vartheta, T\vartheta)\right), \text{ for all } \rho, \vartheta \in M.$$

Then T has a unique fixed point.

Proof. The result follows from Theorem 4.2 by substituting $S = T$. □

Theorem 4.4. *Let $S, T : M \to M$ be self mappings defined on complete partial-metric space (M, p).If there exist $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$ such that*

$$p(S\rho, T\vartheta) \leq \alpha \left(p(\rho, \vartheta)\right) + \beta \left(p(\rho, S\rho)\right) + \gamma \left(p(\vartheta, T\vartheta)\right), \text{ for all } \rho, \vartheta \in M.$$

Then, S, T have a unique common fixed point.

Proof. Consider $\wp(\rho, \vartheta) = e^{p(\rho, \vartheta)}$. Then, S, T satisfies all the conditions of Theorem 3.5. Hence, S, T have a unique common fixed point. □

Theorem 4.5. *Let $T : M \to M$ be a self mapping defined on complete partial-metric space (M, p). If there exist $\alpha, \beta, \gamma \in [0, 1)$ with $\alpha + \beta + \gamma < 1$ such that*

$$p(T\rho, T\vartheta) \leq \alpha \left(p(\rho, \vartheta)\right) + \beta \left(p(\rho, T\rho)\right) + \gamma \left(p(\vartheta, T\vartheta)\right), \text{ for all } \rho, \vartheta \in M.$$

Then T has a unique fixed point.

Proof. The result follows from Theorem 4.3 by substituting $S = T$. □

Remark 4.6. Several other consequences of the results can be seen as :

- (i) Theorem 1, Corollary 1 and Corollary 2 of [20], can be obtained using Theorem 3.1;
- (ii) Theorem 4.3 is an generalization of fixed point result proved in Theorem 3.1 and Theorem 3.2 of [4];
- (iii) Theorem 4.2 extends the results of [4] for a pair of self mapping.

5. Existence of the solution to the system of multiplicative Fredholm integral equation using multiplicative $\mathfrak{m}$-distance

In this section, we have presented the applicability of the proved results by establishing the existence of solution to a system of multiplicative integral equation.

Definition 5.1 ([8]). Consider a positive function $g : \mathbb{R} \to \mathbb{R}^+$. Then

$$\frac{d^*\, g(\rho)}{d\rho} = g^*(\rho) = \lim_{h\to 0}\left(\frac{g(\rho+h)}{g(\rho)}\right)^{\frac{1}{h}},$$

$$\int_a^b g(\rho)^{d\rho} = e^{\int_a^b ln(g(\rho))\, d\rho},$$

are respectively the multiplicative derivative and integral of g.

Theorem 5.2 ([8]). *Consider two multiplicative integral function f and g defined on $[a, b]$. Then*

(i) $\int_a^b (f(\rho).g(\rho))^{d\rho} = \int_a^b f(\rho)^{d\rho}. \int_a^b g(\rho)^{d\rho}$;

(ii) $\int_a^b \frac{f(\rho)}{g(\rho)}^{d\rho} = \frac{\int_a^b f(\rho)^{d\rho}}{\int_a^b g(\rho)^{d\rho}}$;

(iii) $\int_a^b ((g(\rho))^\kappa)^{d\rho} = (\int_a^b g(\rho)^{d\rho})^\kappa$;

(iv) $\left|\int_a^b g(\rho)^{d\rho}\right| \le \int_a^b |g(\rho)|^{d\rho}$.

Theorem 5.3. *Consider the following system of multiplicative integral equation of Fredholm type*

$$\begin{cases} \vartheta(z) = \left[\int_1^2 \left(\vartheta(s)^{K_1(s,z)}\right)^{ds}\right]^\alpha, \text{ where } s, z \in I = [1, 2] \\ \vartheta(z) = \left[\int_1^2 \left(\vartheta(s)^{K_2(s,z)}\right)^{ds}\right]^\alpha, \text{ where } s, z \in I = [1, 2], \end{cases} \tag{5.1}$$

where $K_1(s, z), K_2(s, z)$ are continuous function defined on $I \times I$ such that $|K_i(s, z)| \le \beta_i$ for $1 \le i \le 2$. If $\beta\alpha < 1$, where $\beta = \max\{\beta_1, \beta_2\}$, then we have a unique solution to the system of integral equations (5.1).

Proof. Consider the collection of all multiplicative continuous positive function on [1,2] denoted as $C^*[1, 2]$. Then the mapping $\delta : C^*[1, 2] \times C^*[1, 2] \to [1, \infty)$ defined as

$$\delta(\rho, \vartheta) = \sup_{z\in[1,2]}\left|\frac{\rho(z)}{\vartheta(z)}\right|_* . \min\left\{\sup_{z\in[1,2]}|\rho(z)|_*, \sup_{z\in[1,2]}|\vartheta(z)|_*\right\},$$

where $|a|_* = \begin{cases} a, a \ge 1; \\ \frac{1}{a}, a < 1. \end{cases}$ is a multiplicative $\mathfrak{m}$-metric. Moreover, $C^*[1, 2]$ is a complete M.$\mathfrak{m}$-MS.

Define the self mappings T_1 and T_2 on $C^*[1, 2]$ as

$$T_1(\vartheta(z)) = \left[\int_1^2 \left(\vartheta(s)^{K_1(s,z)}\right)^{ds}\right]^\alpha,$$

$$T_2(\vartheta(z)) = \left[\int_1^2 \left(\vartheta(s)^{K_2(s,z)}\right)^{ds}\right]^\alpha.$$

Consider

$$\delta(T_1(\vartheta_1), T_2(\vartheta_2)) = \sup_{z\in[1,2]}\left|\frac{T_1(\vartheta_1(z))}{T_2(\vartheta_2(z))}\right|_* . \min\left\{\sup_{z\in[1,2]}|T_1(\vartheta_1(z))|_*, \sup_{z\in[1,2]}|T_2(\vartheta_2(z))|_*\right\}$$

$$= \sup_{z\in[1,2]} \left| \left(\frac{\int_1^2 \left(\vartheta_1(s)^{K_1(s,z)}\right)^{ds}}{\int_1^2 \left(\vartheta_2(s)^{K_2(s,z)}\right)^{ds}} \right)^{\alpha} \right|_*$$

$$\times \min \left\{ \sup_{z\in[1,2]} \left| \left(\int_1^2 \left(\vartheta_1(s)^{K_1(s,z)}\right)^{ds} \right)^{\alpha} \right|_*, \sup_{z\in[1,2]} \left| \left(\int_1^2 \left(\vartheta_2(s)^{K_2(s,z)}\right)^{ds} \right)^{\alpha} \right|_* \right\}$$

$$\leq \sup_{z\in[1,2]} \left(\int_1^2 \left| \frac{\vartheta_1(s)}{\vartheta_2(s)} \right|_*^{ds} \right)^{\beta\alpha} . \min \left\{ \sup_{z\in[1,2]} \left(\int_1^2 |\vartheta_1(s)|_*^{ds} \right)^{\beta\alpha}, \sup_{z\in[1,2]} \left(\int_1^2 |\vartheta_2(s)|_*^{ds} \right)^{\beta\alpha} \right\}$$

$$\leq \left(\sup_{z\in[1,2]} \left| \frac{\vartheta_1(s)}{\vartheta_2(s)} \right|_* . \min \left\{ \sup_{z\in[1,2]} |\vartheta_1(z)|_*, \sup_{z\in[1,2]} |\vartheta_2(z)|_* \right\} \right)^{\beta\alpha}$$

$$= (\delta(\vartheta_1, \vartheta_2))^{\beta\alpha} .$$

Also, $\delta_{T_1(\vartheta),T_2(\vartheta)} \leq \delta(T_1(\vartheta), T_2(\vartheta))$. Therefore, T_1, T_2 satisfies all the conditions of Theorem 3.1 with $a_1 = \beta\alpha < 1$, $a_2 = a_3 = 0$. Hence, T_1, T_2 have a unique common fixed point i.e., system of equations (5.1) has a unique solution. □

6. Numerical approximation of common fixed point

In this section, we presented some iterations for approximating the common fixed point of S, T in Example 3.8. In addition, we graphically demonstrated the convergence of Iterative sequence and concluded that the fixed point of the mapping is independent of the iterative procedure's initial point (see Figures 1 and 2). The iteration scheme used for the approximation is given as

$$\text{For initial point } x_0,\ x_{2\kappa+1} = Sx_{2\kappa} \text{ and } x_{2\kappa+2} = Tx_{2\kappa+1}$$

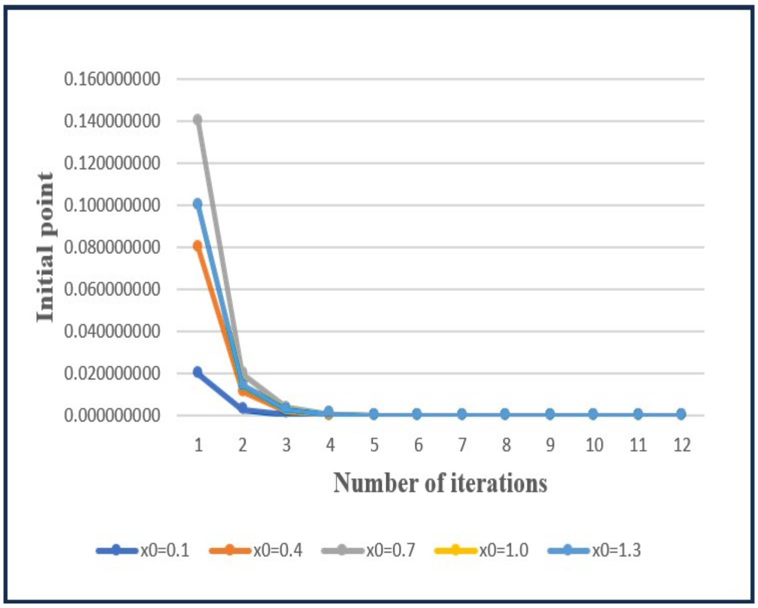

Figure 1. Convergence behaviour of iteration scheme at different initial points for Example 3.8

x0	0.10	0.40	0.70	1.00	1.30
x1	0.020000000	0.080000000	0.140000000	0.100000000	0.100000000
x2	0.002857143	0.011428571	0.020000000	0.014285714	0.014285714
x3	0.000571429	0.002285714	0.004000000	0.002857143	0.002857143
x4	0.000081633	0.000326531	0.000571429	0.000408163	0.000408163
x5	0.000016327	0.000065306	0.000114286	0.000081633	0.000081633
x6	0.000002332	0.000009329	0.000016327	0.000011662	0.000011662
x7	0.000000466	0.000001866	0.000003265	0.000002332	0.000002332
x8	0.000000067	0.000000267	0.000000466	0.000000333	0.000000333
x9	0.000000013	0.000000053	0.000000093	0.000000067	0.000000067
x10	0.000000002	0.000000008	0.000000013	0.000000010	0.000000010
x11	0.000000000	0.000000002	0.000000003	0.000000002	0.000000002
x12	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000

Figure 2. Numerical iteration for Example 3.8

7. Conclusion

In the present manuscript, we discuss common fixed point results for a pair of self-mappings within the framework of multiplicative $\mathfrak{m}$-metric spaces, building upon previous fixed point results. We establish the multiplicative partial metric structure as a special case of a multiplicative $\mathfrak{m}$-metric space and prove common fixed point results. To validate our findings, we provide an illustrative example with discontinuous self mappings and present numerical iterations to approximate the common fixed point, supported by graphs. Our results have consequences for common fixed point results in m-metric spaces and partial metric spaces, and the results generalize several fixed point results in the literature. Additionally, we utilize these results to verify the existence and uniqueness of solutions to a system of multiplicative integral equations.

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References

[1] M. Abbas, B. Ali and Y.I. Suleman, Common fixed point of locally contractive mappings in multiplicative metric spaces with application, Int. J. Math. Math. Sci. **2015**, 218683, 2015.

[2] M. Abbas, M. Da La Sen and T. Nazir, Common fixed point of generalized rational type cocyclic mappings in multiplicative metric space, Discrete Dyn. Nat. Soc. **2015**, 532725, 2015.

[3] A.A.N. Abdou, Fixed point theorems for generalized contractive mappings in multiplicative metric spaces, J. Nonlinear Sci. Appl. **9** (5), 2347–2363, 2016.

[4] M. Asadi, E. Karpinar, and P. Salimi, New extension of p-metric space with some fixed-point results on M-metric spaces, J. Inequal. Appl. **2014**, Art. No. 18, 2014.

[5] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. **3** (1), 133–181, 1922.

[6] A.E. Bashirov, E.M. Kurpinar and A. Ozyapici, Multiplicative calculus and its applications, J. Math. Anal. Appl. **337** (1), 36–48, 2008.

[7] S.K. Chatterjea, Fixed point theorems, C.R. Acad. Bulgare Sci. **25**, 727–730, 1972.

[8] M. Grossman and R. Katz, Non-Newtonian Calculus, Lee Press, Pigeon Cove, MA, 1972.

[9] X. He, M. Song and D. Chen, Common fixed points for weak commutative mappings on multiplicative metric space, Fixed Point Theory Appl. **2014**, Art. No. 48, 2014 .

[10] H. Huang, G. Deng, T. Doenovi and N. Hussain, Note on recent common coupled fixed point results in multiplicative metric spaces, Appl. Math. Nonlinear Sci. **3** (2), 659–668, 2018.

[11] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. **60**, 71–76, 1968.

[12] S.G. Matthews, Partial metric topology, Ann. N. Y. Acad. Sci. **728** (1), 183–197, 1994.

[13] C. Mongkolkeha and W. Sintunavarat, Best proximity points for multipliative proximal contraction mapping on multiplicative metric spaces, J. Nonlinear Sci. Appl. **8** (6), 1134–1140, 2015.

[14] M. Ozavsar and A.C. Cevikel, Fixed points of multiplicative contraction mappings on multiplicative metric spaces, Journal of Engineering Technology and Applied Sciences **2** (2), 65–79, 2017.

[15] S. Reich, Some remarks concerning contraction mappings, Can. Math. Bull. **14**, 121–124, 1971.

[16] A. Robinson, Non-Standard Analysis, Kon. Nederl. Akad. Wetensch. Amsterdam Proc. **23**, 432440, 1966.

[17] M. Sarwar and Badhah-e-Rome, Some unique fixed point theorems in multiplicative metric space, arXiv:1410.3384v2[math.GM], 2014.

[18] D. Stanley, A Multiplicative Calculus, Primus, IX (4), 310–326, 1999.

[19] M. Tariq, M. Abbas, A. Hussain, M. Arshad, A. Ali and H. Al-Sulami, Fixed points of non-linear set-valued (α_*, φ_M)-contraction mappings and related applications, AIMS Math. **7** (5), 8861–8878, 2022.

[20] K. Yadav and D. Kumar, Multiplicative $\mathfrak{m}$-metric space, fixed point theorems with applications in multiplicative integrals equation and numerical results, J. Appl. Anal. **30** (1), 173–186, 2024.

[21] B. Zada and U. Riaz, Some Fixed Point Results on Multiplicative (b)-metric-like Spaces, Turk. J. Math. **4** (5), 118–131, 2016.