http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 2, Pages 319–335 (2024)
DOI:10.31801/cfsuasmas.1336733
ISSN 1303-5991 E-ISSN 2618-6470

COMMUNICATIONS SERIES A1

Research Article; Received: August 2, 2023; Accepted: November 27, 2023

SEPARATION, COMPACTNESS, AND SOBRIETY IN THE CATEGORY OF CONSTANT LIMIT SPACES

Ayhan ERCİYES[1], Muhammad QASIM[2] and İsmail Alper GÜVEY[3]

[1,3]Department of Mathematics, Faculty of Science and Arts, Aksaray University, 68100 Aksaray, TÜRKİYE
[2]Department of Mathematics, School of Natural Sciences, National University of Sciences & Technology, H-12 Islamabad, 44000, PAKISTAN
[2]School of Mathematics, China University of Mining and Technology, Xuzhou 21189, PR CHINA

Abstract. The objective of this article is to characterize each of compact, sober, and T_i for $i = 0, 1, 2$ constant limit spaces as well as to investigate the relationships between them. Finally, we compare our results in some topological categories.

1. Introduction

The lack of natural function spaces in ***Top***, the category of topological spaces and continuous maps which is not cartesian closed has been recognized as an awkward situation for various applications in the field of functional analysis and homotopy theory. The category ***Lim*** of limit spaces and continuous maps which is cartesian closed [17] supercategory of ***Top***. Limit spaces with compatible vector space structures are used to develop a calculus for vector spaces without norm [22].

Baran, in [2], introduced the notion of (strong) closedness in terms of final lifts, initial lifts, and discrete structures which are available in a topological category. He used these notions to generalize each of compact, sober, and T_i, $i = 1, 2, 3, 4$ objects in topological categories in [2, 7, 12].

The sober spaces were introduced in [18] and used in the theory of non-T_2 spaces. In 2022, Baran and Abughalwa [12] gave various forms of sober objects in a topological category and investigated relationships among these various forms.

2020 *Mathematics Subject Classification.* 18B99, 18D15, 54A05, 54A20, 54B30, 54D10.
Keywords. Topological category, closure operator, sober object, compacteness, constant limit space.

[1] ayhanerciyes@aksaray.edu.tr-Corresponding author; 0000-0002-0942-5182
[2] muhammad.qasim@sns.nust.edu.pk; 0000-0001-9485-8072
[3] ismailalper.guvey@aksaray.edu.tr; 0009-0008-6165-643X.

The objective of this article is to characterize each of compact, sober, and T_i for $i = 0, 1, 2$ constant limit spaces as well as to investigate the relationships between them. Finally, we compare our results in some topological categories.

2. Preliminaries

Definition 1. *Let $B \neq \emptyset$, $F(B)$ be the set of filters (proper or improper) on B, and the map $K : B \longrightarrow P(F(B))$. We call (B, K) is a constant limit space if K satisfies:*

(i) $[s] \in K$, $\forall s \in B$, *where* $[s] = \{U \subset B : s \in U\}$,

(ii) *if* $\alpha \in K$ *and* $\alpha \subset \beta$, *then* $\beta \in K$,

(iii) *if* $\alpha, \beta \in K$, *then* $\alpha \cap \beta \in K$.

Let (B, K) and (C, L) be constant limit spaces. If $f(\alpha) \in L$ for every $\alpha \in K$, then a map $f : (B, K) \longrightarrow (C, L)$ is called continuous, where $f(\alpha) = \{U \subset C : \exists V \in \alpha$ such that $f(V) \subset U\}$.

We denote ***ConLim*** by the category of constant limit spaces and continuous maps.

Proposition 1. *([5])* **(1)** *Let $\{(B_i, K_i), i \in I\}$ in **ConLim**, B be a set, and $\{f_i : B \longrightarrow (B_i, K_i), i \in I\}$ be a source in **Set**. $\{f_i : (B, K) \longrightarrow (B_i, K_i), i \in I\}$ in **ConLim** is an initial lift iff $K = \{\alpha \in F(B) : f_i(\alpha) \in K_i, \forall i \in I\}$.*

(2) *An epi sink $\{f_i : (B_i, K_i) \longrightarrow (B, K)\}$ in **ConLim** is a final lift iff $\alpha \in K$ implies $\bigcap_{i=1}^{n} f(\alpha_i) \subset \alpha$ for some $\alpha_i \in K_i$, $i \in I$.*

(3) *$K = \{\alpha : \alpha = [U], U \subset B$ is finite $\}$ is discrete structure on B, where $[U] = \{V \subset B : U \subset V\}$.*

The constant limit structure on a finite set B is unique. Let $B = \{a_1, a_2, ..., a_n\}$. The discrete structure on B, $K = \{\alpha : \alpha = [U], U \subset B\} = F(B)$, the indiscrete structure on B.

3. Closed Subobjects

Let X be a set, $X^\infty = X \times X \times ...$ be the countable product of X, and $a \in X$. $\bigvee_a^\infty X$ (resp., $X \bigvee_a X$) is formed by taking countably many disjoint (resp., two distinct) copies of X identifying them at the point a.

Definition 2. *([2, 6]) Define $S_a : X \bigvee_a X \longrightarrow X^2$ by*

$$S_a(t_i) = \begin{cases} (t, t) & \text{if } i = 1 \\ (a, t) & \text{if } i = 2 \end{cases},$$

$\bigtriangledown_a : X \bigvee_a X \longrightarrow X$ *by* $\bigtriangledown_a(t_i) = t$,

$A_a^\infty : \bigvee_a^\infty X \longrightarrow X^\infty$ *by* $A_a^\infty(t_i) = (a, a, ..., a, t, a, a, ...)$,

and $\bigtriangledown_a^\infty : \bigvee_a^\infty X \longrightarrow X$ *by* $\bigtriangledown_a^\infty(t_i) = t$ *for each* $i \in I$*, where* I *is the index set* $\{i : t_i$ *is in the* i*-th component of* $\bigvee_a^\infty X\}$.

Definition 3. *([2]) Let* $\mathcal{U} : \mathcal{E} \longrightarrow$ ***Set*** *be a topological functor [1] and* X *be an object of* $\mathcal{E}$ *with* $\mathcal{U}(X) = B$.

(1) *If the initial lift of the* $\mathcal{U}$*-source* $S_a : B \bigvee_a B \longrightarrow \mathcal{U}(X^2) = B^2$ *and* $\bigtriangledown_a : B \bigvee_a B \longrightarrow \mathcal{U}\mathcal{D}((B)) = B$ *is discrete, then* X *is called* T_1 *at* a*, where* $\mathcal{D}$ *is the discrete functor.*

(2) *If the initial lift of the* $\mathcal{U}$*-source*

$$A_a^\infty : \vee_a^\infty B \longrightarrow \mathcal{U}(X^\infty) = B^\infty \quad \text{and} \quad \nabla_a^\infty : \vee_a^\infty B \longrightarrow \mathcal{U}\mathcal{D}((B)) = B$$

is discrete, then $\{a\}$ *is called closed.*

(3) *If* $\{*\}$ *is closed in* X/M*, then* $M \subset X$ *is called closed, where* X/M *is the final lift of the epi* U*-sink*

$$q : B = U(X) \to B/M = (B \backslash M) \cup \{*\},$$

identifying M *with a point *.*

(4) *If* X/M *is* T_1 *at* $*$*, then* M *is called strongly closed in* X.

(5) *If* $B = M = \emptyset$ *iff then* M *is to be (strongly) closed.*

(6) $M \subset X$ *is open (resp., strongly open) iff* M^c *is closed (resp., strongly closed) in* X.

Remark 1. **(1)** *In* ***Top****, by Corollary 2.2.6 of [2],* $M \subset B$ *is closed iff* M *is closed in the usual sense. Moreover, the notion of strong closedness implies closedness and they coincide when a topological space is* T_1 *[4].*

(2) *In an arbitrary topological category, in general, the notions of closedness and strong closedness are independent of each other [4].*

Theorem 1. *Let* $(B, K) \in$ ***ConLim****.* $\emptyset \neq M \subset B$ *is closed (open) iff* $M = B$.

Proof. Suppose $\emptyset \neq M \subset B$ and $M \neq B$. Then $\exists t \in B$ with $t \notin M$. Take $\sigma = \bigcap_{i=1}^\infty [t_i]$ with $t_i \in B/M$. We have $\nabla_* \sigma = [t]$ and $\pi_j A_*^\infty \sigma = [*] \cap [t] \in K_1$ for all i, where K_1 is the final structure on B/M. Since σ is generated by the infinite set $\{t_1, t_2, ..., t_n, ...\}$, σ does not contain a finite set which contradicts $\{*\}$ is being closed. Hence, $B = M$.

If $M = B$, then $\bigvee_*^\infty(B/M) = \{*\}$ and by Definition 3 (5), $\{*\} = \bigvee_*^\infty(B/M)$ is closed and consequently, M is closed.

The proof for openness follows from Definition 3. □

Theorem 2. *Every subset of constant limit space is both strongly closed and strongly open.*

Proof. Let $(B, K) \in \boldsymbol{ConLim}$ and $M \subset B$. If $M = \emptyset$, then by Definition 3, M is strongly open (strongly closed). Suppose $M \neq \emptyset$ and let K_1 be the quotient structure on B/M induced by $q : (B, K) \longrightarrow (B/M, K_1)$, K_q be the initial structure on $(B/M) \bigvee_*(B/M)$ induced by

$$S_* : (B/M) \vee_* (B/M) \longrightarrow ((B/M)^2, K_1^2)$$

and

$$\triangledown_* : (B/M) \vee_* (B/M) \longrightarrow (B/M, K_d),$$

where K_1^2 is structure on $(B/M)^2$ and K_d is the discrete structure on B/M.

Suppose $\sigma \in K_q$. Then by Proposition 1, $\pi_1 S_*\sigma, \pi_2 S_*\sigma \in K_1$ and $\triangledown_*\sigma \in K_d$. It follows that $\triangledown_*\sigma = [\emptyset]$ or $[U]$, $U \subset B/M$ is finite with $card(U) = m$. If $\triangledown_*\sigma = [\emptyset]$, then $\sigma = [\emptyset]$. If $\triangledown_*\sigma = [U]$, then $\exists V \in \sigma$ such that $U \supset \triangledown_* V$. Since U is finite, $card(V) \leq 2m$ and consequently, V is finite. Hence, by Definition 2, $(B/M, K_1)$ is T_1 at $*$ and M is strongly closed. The proof for strongly open follows from Definition 3. □

Theorem 3. **(1)** *Let $f : (A, L) \longrightarrow (B, K)$ be in $\boldsymbol{ConLim}$. If $M \subset B$ is (strongly) closed, then $f^{-1}(M) \subset A$ is (strongly) closed.*

(2) *Let $(B, K) \in \boldsymbol{ConLim}$. If $M \subset N$ and $N \subset B$ are (strongly) closed, then $M \subset B$ is (strongly) closed.*

(3) *Let $(B_i, K_i) \in \boldsymbol{ConLim}$ for $\forall i \in I$ and $M_i \subset B_i$ be (strongly) open (resp., closed) for each $i \in I$. Then $\prod_{i\in I} M_i$ is (strongly) open (resp., closed) in $\prod_{i\in I} B_i$.*

Proof. We get the proof from Theorems 1 and 2. □

Let X be a set and the wedge $X^2 \bigvee_\Delta X^2$ be two distinct copies of X^2 identified along the diagonal Δ [2]. Define $A : X^2 \bigvee_\Delta X^2 \longrightarrow X^3$ by

$$A((s,t)_i) = \begin{cases} (s,t,s) & \text{if } i = 1 \\ (s,s,t) & \text{if } i = 2 \end{cases},$$

$S : X^2 \bigvee_\Delta X^2 \longrightarrow X^3$ by

$$S((s,t)_i) = \begin{cases} (s,t,t) & \text{if } i=1 \\ (s,s,t) & \text{if } i=2 \end{cases},$$

and $\nabla : X^2 \bigvee_{\triangle} X^2 \longrightarrow X^2$ by

$$\bigtriangledown((s,t)_i) = (s,t)$$

for $i = 1, 2$.

Definition 4. *([2, 5])* **(1)** *If the initial lift of the $\mathcal{U}$-source*

$$A : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(X^3) = B^3 \quad \textit{and} \quad \nabla : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(\mathcal{D}(B^2)) = B^2$$

(resp.,

$$id : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(B^2 \vee_{\triangle} B^2)' = B^2 \vee_{\triangle} B^2 \quad \textit{and} \quad \nabla : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(\mathcal{D}(B^2)) = B^2)$$

is discrete, then X is called $\overline{T}_0$ (resp., $T_0^{'}$), where $(B^2 \bigvee_{\triangle} B^2)'$ is the final lift of the U-sink $\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \longrightarrow B^2 \vee_{\triangle} B^2\}$ and i_1, i_2 are the canonical injections.

(2) *If X does not contain an indiscrete subspace with (at least) two points, then X is called a T_0 object.*

(3) *If the initial lift of the $\mathcal{U}$-source*

$$S : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(X^3) = B^3 \quad \textit{and} \quad \nabla : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(\mathcal{D}(B^2)) = B^2$$

is discrete, then X is called T_1.

(4) *If the initial lift of the $\mathcal{U}$-sources $A : B^2 \bigvee_{\triangle} B^2 \longrightarrow \mathcal{U}(X^3) = B^3$ and $S : B^2 \bigvee_{\triangle} B^2 \longrightarrow \mathcal{U}(X^3) = B^3$ agree, then X is called pre$\overline{T}_2$.*

(5) *If the initial lift of the $\mathcal{U}$-source $S : B^2 \vee_{\triangle} B^2 \longrightarrow \mathcal{U}(X^3) = B^3$ and the final lift of the $\mathcal{U}$-sink $\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \longrightarrow B^2 \vee_{\triangle} B^2\}$ agree, then X is called pre$T_2^{'}$.*

(6) *X is KT_2 iff X is pre$\overline{T}_2$ and $T_0^{'}$.*

(7) *X is LT_2 iff X is pre$T_2^{'}$ and $\overline{T}_0$.*

(8) *X is NT_2 iff X is pre$\overline{T}_2$ and T_0.*

Remark 2. *In **Top**, by Theorem 2.2.11 of [2] and Remark 1.3 of [6], all of $\overline{T_0}$, T_0' and T_0 (resp., KT_2, NT_2, and LT_2) are equal to T_0 (resp., T_2). In the realm of preT_2 topological spaces, by the Theorem 2.4 of [14], all T_0, T_1, and T_2 spaces are equivalent.*

Theorem 4. *Let $(B, K) \in$ **ConLim**. Then (B, K) is LT_2 iff (B, K) is KT_2.*

Proof. Let (B,K) be KT_2. By Theorem 2.3 of [5], (B,K) is T_0'. Let K_A (resp., K_F) be the initial lift of A (resp., final lift of $\{i_1, i_2 : B^2 \longrightarrow B^2 \vee_{\triangle} B^2\}$ and $\sigma \in F(B^2 \vee_{\triangle} B^2))$ with $\sigma \in K_F$. By Proposition 1, $\exists \alpha, \beta \in K^2$ with $\sigma \supset i_1\alpha \cap i_2\beta$, where K^2 is structure on B^2. Hence,

$$\pi_1 A\sigma \supset \pi_1 A(i_1\alpha \cap i_2\beta) = \pi_1\alpha \cap \pi_1\beta,$$

$$\pi_2 A\sigma \supset \pi_2 A(i_1\alpha \cap i_2\beta) = \pi_2\alpha \cap \pi_1\beta,$$

$$\pi_3 A\sigma \supset \pi_3 A(i_1\alpha \cap i_2\beta) = \pi_1\alpha \cap \pi_2\beta.$$

Since K is a constant limit structure on B and $\pi_1\alpha, \pi_2\alpha, \pi_1\beta, \pi_2\beta \in K$, we have $\pi_1\alpha \cap \pi_1\beta$, $\pi_2\alpha \cap \pi_1\beta, \pi_1\alpha \cap \pi_2\beta \in K$, and consequently, $\pi_1 A\sigma, \pi_2 A\sigma, \pi_3 A\sigma \in K$. By Proposition 1, $\sigma \in K_A$. Hence, $K_F \subset K_A$.

Suppose $\sigma \in F(B^2 \vee_{\triangle} B^2)$ with $\sigma \in K_A$. If $\sigma = [\emptyset]$, then $\sigma \in K_F$. Suppose $\sigma \neq [\emptyset]$. Let $\alpha_{11} = \pi_1 A\sigma, \alpha_{21} = \pi_2 A\sigma$, and $\alpha_{12} = \pi_3 A\sigma$. In case of (1) of Theorem 3.8 of [3], we have $\pi_1 A\sigma = \pi_2 A\sigma$.Let $\sigma_1 = \pi_1^{-1}(\pi_1 A\sigma) \cup \pi_2^{-1}(\pi_3 A\sigma)$. Since $\pi_1 A\sigma_1 = \pi_1 A\sigma = \pi_2 A\sigma \in K$ and $\pi_2 A\sigma_1 = \pi_3 A\sigma \in K$, we get $\sigma_1 \in K^2$.

We now show $i_1\sigma_1 = (\pi_1 A)^{-1}(\pi_1 A\sigma) \cup (\pi_2 A)^{-1}(\pi_2 A\sigma) \cup (\pi_3 A)^{-1}(\pi_3 A\sigma) = \sigma_0$.

If $U \in i_1\sigma$, then $U \supset (U_1 \times U_2)_1$ for some $U_1 \in \pi_1 A\sigma = \pi_2 A\sigma$ and $U_2 \in \pi_3 A\sigma$. Since case 1 of Theorem 3.8 of [3] holds and $\pi_1 A\sigma \cup \pi_3 A\sigma$ is improper, we may assume $U_1 \cap U_2 = \emptyset$.
Note that

$$(\pi_1 A)^{-1}(U_1) \cap (\pi_2 A)^{-1}(U_1) \cap (\pi_3 A)^{-1}(U_2) = (U_1 \times U_2)_1 \in \sigma_0$$

and consequently, $U \in \sigma_0$. Hence, $i_1\sigma_1 \subset \sigma_0$.

If $U \in \sigma_0$, then $U \supset (U_1 \times U_2)_1 \bigvee ((U_1 \cap U_2) \times U_2)_2$ for some $U_1 \in \pi_1 A\sigma = \pi_2 A\sigma$ and $U_2 \in \pi_3 A\sigma$.

Since case (1) of Theorem 3.8 of [3] holds and $\pi_1 A\sigma \cup \pi_3 A\sigma$ is improper, we may assume $U_1 \cap U_2 = \emptyset$. Hence, $U \supset (U_1 \times U_2)_1$ and consequently, $U \in i_1\sigma_1$. Thus, $i_1\sigma_1 = \sigma_0$. By Corollary 3.3 of [3], $i_1\sigma_1 = \sigma_0 \subset \sigma$.

In case (2) of Theorem 3.8 of [3] holds, we have $\pi_1 A\sigma = \pi_3 A\sigma$. Let $\sigma_1 = \pi_1^{-1}(\pi_1 A\sigma) \cup \pi_2^{-1}(\pi_2 A\sigma)$.

Note that

$$\pi_1\sigma_1 = \pi_1 A\sigma \in K,$$

$$\pi_2\sigma_1 = \pi_2 A\sigma \in K.$$

Consequently, $\sigma_1 \in K^2$.

Let $\sigma_0 = (\pi_1 A)^{-1}(\pi_1 A\sigma) \cup (\pi_2 A)^{-1}(\pi_2 A\sigma) \cup (\pi_3 A)^{-1}(\pi_3 A\sigma)$. Since case (2) of Theorem 3.8 of [3] holds, then $i_2\sigma_1 = \sigma_0$ and by Corollary 3.3 of [3], $i_2\sigma_1 \subset \sigma$.

In case (3) of Theorem 3.8 of [3] holds, we have $\pi_3 A\sigma \cap \pi_2 A\sigma \subset \pi_1 A\sigma$.

Let

$$\sigma_1 = \pi_1^{-1}(\pi_3 A\sigma) \cup \pi_2^{-1}(\pi_2 A\sigma)$$

and

$$\sigma_0 = (\pi_1 A)^{-1}(\pi_3 A\sigma) \cup (\pi_2 A)^{-1}((\pi_2 A\sigma) \cap (\pi_3 A\sigma)) \cup (\pi_3 A)^{-1}(\pi_3 A\sigma).$$

By Corollary 3.3 of [3], $\sigma_0 \subset \sigma$, $\pi_1 A\sigma_0 = \pi_3 A\sigma \in K$, $\pi_2 A\sigma_0 = (\pi_2 A\sigma) \cap (\pi_3 A\sigma) \in K$, and $\pi_3 A\sigma_0 = \pi_3 A\sigma \in K$ since K is a constant limit structure on B. We show that $\sigma_0 = i_1\sigma_1 \cap i_2\sigma_1$.

If $U \in \sigma_0$, then $U \supset (U_1 \times (U_2 \cap U_3))_1 \bigvee ((U_1 \cap U_3) \times U_2)_2$ for some $U_1 \in \pi_3 A\sigma$, $U_3 \in (\pi_2 A\sigma) \cap (\pi_3 A\sigma)$, and $U_2 \in \pi_3 A\sigma$.

Note that

$$((U_1 \cap U_3) \times (U_2 \cap U_3)) \in \sigma_1,$$

$$((U_1 \cap U_3) \times (U_3 \cap U_2))_1 \in i_1\sigma_1,$$

$$((U_1 \cap U_3) \times (U_3 \cap U_2))_2 \in i_2\sigma_1,$$

and

$$((U_1 \cap U_3) \times (U_3 \cap U_2))_1 \bigvee ((U_1 \cap U_3) \times (U_3 \cap U_2))_2 \in i_1\sigma_1 \cap i_2\sigma_1.$$

Hence, $U \in i_1\sigma_1 \cap i_2\sigma_1$ and so $\sigma_0 \subset i_1\sigma_1 \cap i_2\sigma_1$.

If $U \in i_1\sigma_1 \cap i_2\sigma_1$, then $U \supset (U_1 \times U_2)_1 \bigvee (U_1 \times U_2)_2$ for some $U_3 \in \pi_2 A\sigma$ and $U_2 \in \pi_3 A\sigma$. Note that

$$U_3 \cup U_2 \in (\pi_2 A\sigma) \cap (\pi_3 A\sigma)$$

and

$$(\pi_1 A)^{-1}(U_3) \cap (\pi_2 A)^{-1}(U_3 \cup U_2) \cap (\pi_3 A)^{-1}(U_3) = (U_3 \times U_2)_1 \bigvee (U_3 \times U_2)_2 \in \sigma_0$$

and consequently, $U \in \sigma_0$. Hence, $\sigma_0 = i_1\sigma_1 \cap i_2\sigma_1 \subset \sigma$. Therefore $K_A \subset K_F$ and consequently, $K_A = K_F$. Since (B, K) is KT_2, by Definition 4, $K_S = K_A$, where K_S is the initial lift of S. Hence, by Definition 4, $K_S = K_F$ and (B, K) is LT_2.

Suppose (B, K) is LT_2. By Theorem 2.3 of [5], (B, K) is $T_0^{'}$ and by Remark 3.6 of [11], (B, K) is $pre\overline{T}_2$. Hence, by Definition 4, (B, K) is KT_2. $\square$

Let $\boldsymbol{T_0'\mathcal{E}}$ (resp., $\boldsymbol{T_0\mathcal{E}}$, $\boldsymbol{\overline{T}_0\mathcal{E}}$, $\boldsymbol{T_1\mathcal{E}}$, $\boldsymbol{KT_2\mathcal{E}}$, $\boldsymbol{LT_2\mathcal{E}}$, and $\boldsymbol{NT_2\mathcal{E}}$) be the subcategory of $\mathcal{E}$ consisting of T_0' (resp., T_0, $\overline{T}_0$, T_1, KT_2, LT_2, and NT_2) objects of $\mathcal{E}$.

Remark 3. **(1)** *By Theorem 2.3 of [5] and Theorem 4, $\overline{T_0}$, T_0' and T_1 constant limit spaces are equivalent. Furthermore, a constant limit space (B, K) is NT_2 iff B is a point or the empty set. Moreover, $NT_2 \Rightarrow KT_2 \iff LT_2$ but the converse is not true, in general. For example, let be $B = \{a, b\}$, and $K = \{[a], [b], [a] \cap [b], [\emptyset]\}$. (B, K) is LT_2 but it is not NT_2.*

(2) *By Theorem 4 and Theorem 2.3 of [5],* $\boldsymbol{T_0ConLim}$, $\boldsymbol{\overline{T}_0ConLim}$, $\boldsymbol{T_0'ConLim}$, $\boldsymbol{T_1ConLim}$, $\boldsymbol{KT_2ConLim}$, $\boldsymbol{LT_2ConLim}$, *and* $\boldsymbol{ConLim}$ *are pairwise isomorphic categories. Since* $\boldsymbol{ConLim}$ *is a cartesian closed, all of these categories are cartesian closed.*

(3) *By Theorems 1 and 4, we have Tietze Extension Theorem for constant limit spaces. If (B, K) is a KT_2 constant limit space and A is non-empty closed subspace of (B, K), then every morphism $f : (A, L) \to (\mathbb{R}, S)$ has an extension morphism $g : (B, K) \to (\mathbb{R}, S)$, where $\mathbb{R}$ is the set of real numbers and S is any constant limit structure on $\mathbb{R}$.*

(4) *By Theorem 1, we have Urysohn's Lemma for constant limit spaces. Suppose (B, K) is a KT_2 constant limit space and M and N are any nonempty disjoint subsets of B. Then there exists a morphism $f : (B, K) \to ([0, 1], L)$, where L is any constant limit structure on $[0, 1]$ with $f(w) = 0$ if $w \in M$ and $f(w) = 1$ if $w \in N$.*

Note that Tietze Extension Theorem and Urysohn's Lemma for constant filter convergence spaces (resp., extended pseudo-quasi-semi metric spaces) are presented in [21, 23, 24].

Definition 5. *Let $(B, K) \in$* $\boldsymbol{ConLim}$ *and $Z \subset B$.*

$\boldsymbol{scl}(Z) = \bigcap\{H \subset B : Z \subset H \text{ and } H \text{ is strongly closed}\}$ *is said to be the strong closure of Z.*

$\boldsymbol{cl}(Z) = \bigcap\{H \subset B : Z \subset H \text{ and } H \text{ is closed}\}$ *is said to be the closure of Z.*

$\boldsymbol{Q}(Z) = \bigcap\{H \subset B : Z \subset H, H \text{ is closed and open}\}$ *is called the quasi-component closure of Z.*

$\boldsymbol{SQ}(Z) = \bigcap\{H \subset B : Z \subset H, H \text{ is strongly closed and strongly open}\}$ *is said to be the strong quasi-component closure of Z.*

Theorem 5. *$\boldsymbol{cl} = \boldsymbol{\imath} = \boldsymbol{Q}$, the indiscrete closure operator and $\boldsymbol{scl} = \boldsymbol{\delta} = \boldsymbol{SQ}$, the discrete closure operator of* ***ConLim****.*

Proof. Combine Definition 5, Theorems 1, and 5. □

Definition 6. *([19])Let $\boldsymbol{c}$ be a closure operator of $\boldsymbol{\mathcal{E}}$.*

(1) $\boldsymbol{\mathcal{E}}_{0c} = \{W \in \boldsymbol{\mathcal{E}} : s \in \boldsymbol{c}(\{t\})$ *and* $t \in \boldsymbol{c}(\{s\})$ *implies* $s = t$ *with* $s, t \in W\}$,

(2) $\boldsymbol{\mathcal{E}}_{1c} = \{W \in \boldsymbol{\mathcal{E}} : \boldsymbol{c}(\{s\}) = \{s\}, \forall s \in W\}$,

(3) $\boldsymbol{\mathcal{E}}_{2c} = \{W \in \boldsymbol{\mathcal{E}} : \boldsymbol{c}(\triangle) = \triangle$, *the diagonal*$\}$.

Theorem 6. *A constant limit space $(B, K) \in \boldsymbol{ConLim_{icl}}$ for $i = 0, 1, 2$ iff $B = \emptyset$ or $B = \{a\}$, a one point set.*

Proof. We get the proof from Theorem 1. □

Theorem 7. *$\boldsymbol{ConLim_{iscl}}$, $i = 0, 1, 2$ are isomorphic to* ***ConLim****.*

Proof. We get the proof from Theorem 5. □

4. Sober Constant Limit Spaces

In this section, we characterize irreducible, sober, and quasi-sober constant limit spaces.

Definition 7. *([12, 16]) Let $\boldsymbol{\mathcal{E}}$ be a topological category and $X \in Ob(\boldsymbol{\mathcal{E}})$.*

(1) *X is called irreducible if Z_1, Z_2 are closed subobjects of X and $X = Z_1 \cup Z_2$, then $X = Z_1$ or $X = Z_2$.*

(2) *X is called quasi-sober if every nonempty irreducible closed subset of X is the closure of a point .*

(3) *X is called $\overline{T_0}$ sober if X is $\overline{T_0}$ and a quasi-sober.*

(4) *X is called T_0' sober if X is T_0' and a quasi-sober.*

(5) *X is called T_0 sober if X is T_0 and a quasi-sober.*

Remark 4. *In* ***Top****, by Remark 3.4 of [12], all of T_0' sober, T_0 sober, and $\overline{T_0}$ sober are equivalent and they reduce to the usual sober. Also, the notion of irreducibility reduces to notion of the usual irreducibility [16].*

Theorem 8. *Let $(B, K) \in$* ***ConLim****.*

(A) *The following are equivalent:*

(1) *A constant limit space* (B, K) *is quasi-sober.*

(2) (B, K) *is* $\overline{T_0}$ *sober.*

(3) (B, K) *is* T_0' *sober.*

(4) (B, K) *is irreducible.*

(B) *The following are equivalent:*

(1) (B, K) *is* T_0.

(2) (B, K) *is* T_0 *sober.*

(3) $card(B) \leq 1$.

Proof. **(A)** By Theorem 2.4 of [5] and Definition 7, we get (1) $\iff$ (2) $\iff$ (3).

(1) $\Longrightarrow$ (4): Suppose (B, K) is quasi-sober and $B = B_1 \cup B_2$, where B_1 and B_2 are closed subsets of B. By Theorem 1, $B_1 = B$ or $\emptyset$ and $B_2 = B$ or $\emptyset$. Hence, by Definition 7, (B, K) is irreducible.

(4) $\Longrightarrow$ (1): Suppose (B, K) is irreducible and $\emptyset \neq B_1 \subset B$ is irreducible closed. Since B_1 is closed, by Theorem 1, $B_1 = B$ and by Theorem 5, $B = B_1 = \boldsymbol{cl}(\{b\})$ for some $b \in B$. Hence, by Definition 7, (B, K) is quasi-sober. Thus, (1) $\iff$ (4).

(B) (1) $\Longrightarrow$ (2): Suppose (B, K) is T_0 and $\emptyset \neq B_1 \subset B$ is irreducible closed. Since B_1 is closed, by Theorem 1, $B_1 = B$ and hence, by Theorem 5, $B_1 = B = \boldsymbol{cl}(\{b\})$ for some $b \in B$. Hence, consequently, (B, K) is quasi-sober and by Definition 7, (B, K) is T_0 sober.

(2) $\Longrightarrow$ (3): Suppose (B, K) is T_0 sober and $B \neq \emptyset$ and $B \neq \{a\}$. Then, $\exists s, t \in B$ with $s \neq t$ and $(\{s, t\}, F(\{s, t\}))$ is the indiscrete subspace of (B, K), contradicting to (B, K) is being T_0 sober. Hence, $card(B) \leq 1$.

(3) $\Longrightarrow$ (1): If $card(B) \leq 1$, then by Definition 4, (B, K) is T_0. □

5. Compact Constant Limit Spaces

Definition 8. *([7]) Let* $\boldsymbol{\mathcal{E}}$ *be a topological category,* $A, B \in Ob(\boldsymbol{\mathcal{E}})$*, and* $f : A \longrightarrow B$ *be a morphism in* $\boldsymbol{\mathcal{E}}$*.*

(1) *If the image of every (strongly) closed subobject of A is a (strongly) closed subobject of B, then f is said to be (strongly) closed.*

(2) *If the projection $\pi_2 : A \times B \longrightarrow B$ is (strongly) closed for every object B in $\mathcal{E}$, then A is called (strongly) compact.*

Remark 5. *In **Top**, by Remark 2.2 of [7], the notion of compactness reduces to usual one, the notion of strong compactness implies compactness and they coincide when a topological space is T_1.*

Theorem 9. *A constant limit space is compact iff it is strongly compact.*

Proof. Suppose (B, K) is a compact constant limit space. We need to show that for each constant limit space (C, L), the projection $\pi_2 : (B, K) \times (C, L) \longrightarrow (C, L)$ is strongly closed. Suppose $M \subset B \times C$ is strongly closed. If $M = \emptyset$, then $\pi_2 M = \emptyset$ is strongly closed. If $M \neq \emptyset$, then by Theorem 2, $\pi_2(M)$ is strongly closed subset of C and hence, by Definition 8, $\pi_2 : (B, K) \times (C, L) \longrightarrow (C, L)$ is strongly closed and consequently, (B, K) is strongly compact.

Suppose (B, K) is a strongly compact constant limit space. We show $\pi_2 : (B, K) \times (C, L) \longrightarrow (C, L)$ is closed for each constant limit space (C, L). Suppose $M \subset B \times C$ is closed. By Theorem 1, $M = \emptyset$ or $M = B \times C$. If $M = \emptyset$, then $\pi_2 M = \emptyset$ is closed in C. If $M = B \times C$, then $C = \pi_2 M$ is closed. By Definition 8, $\pi_2 : (B, K) \times (C, L) \longrightarrow (C, L)$ is closed and hence, (B, K) is compact. □

Theorem 10. *Let $f : (B, K) \longrightarrow (C, L)$ be morphism in **ConLim**.*

(1) *If (B, K) is (strongly) compact, then the subspace $f(B)$ is (strongly) compact.*

(2) *If (B, K) is connected (resp., strongly connected, D-connected, scl-connected, cl-connected), then the subspace $f(B)$ is connected (resp., strongly connected, D-connected, scl-connected, cl-connected).*

(3) *If (B, K) is $\overline{T}_0$ (resp., T_0', T_1, KT_2 or LT_2), then the subspace $f(B)$ is $\overline{T}_0$ (resp., T_0', T_1, KT_2 or LT_2).*

Proof. It follows from Theorems 1, 2, 4, and 9.

□

6. Comparative Evaluation

We compare our findings in some topological categories and we infer:

(1) In ***Top***,

(i) By Theorem 2.2.11 of [2], Remark 1.3 of [6], and Remark 2.6 of [9],

$$\boldsymbol{Top_{2cl}} = \boldsymbol{Top_{2scl}} = \boldsymbol{LT_2Top} = \boldsymbol{NT_2Top} = \boldsymbol{KT_2Top} \subset \boldsymbol{Top_{1cl}}$$

$$= \boldsymbol{Top_{1scl}} \subset \boldsymbol{Top_{0cl}} = \boldsymbol{Top_{0scl}} = \boldsymbol{\overline{T}_0Top} = \boldsymbol{T_0'Top} = \boldsymbol{T_0Top}.$$

and

$$\boldsymbol{Top_{1Q}} = \boldsymbol{Top_{2Q}}$$

(ii) By Remark 3.4 of [12],

$$\boldsymbol{T_0'SobTop} = \boldsymbol{\overline{T}_0SobTop} = \boldsymbol{T_0SobTop}$$

(iii) By Remark 4.4 of [14], there is no implication between $preT_2$ and each of T_0, T_1 and soberity. By Theorem 4.3 of [14], in the realm of $PreT_2$ topological spaces, all T_0 T_1, T_2, and sober spaces are equivalent.

(2) In $\boldsymbol{ConLim}$,

(i) By Theorems 4 and 6,

$$\begin{aligned}\boldsymbol{ConLim_{2cl}} =& \boldsymbol{ConLim_{1cl}} = \boldsymbol{ConLim_{2Q}}\\ =& \boldsymbol{T_0ConLim} \subset \boldsymbol{ConLim_{2scl}}\\ =& \boldsymbol{ConLim_{1scl}} = \boldsymbol{ConLim_{0scl}}\\ =& \boldsymbol{\overline{T}_0ConLim} = \boldsymbol{T_0'ConLim}\\ =& \boldsymbol{T_1ConLim} = \boldsymbol{KT_2ConLim} = \boldsymbol{LT_2ConLim}\end{aligned}$$

(ii) By Theorem 8,

$$\boldsymbol{T_0ConLim} = \boldsymbol{T_0SobConLim}$$

and

$$\boldsymbol{\overline{T}_0SobConLim} = \boldsymbol{T_0'SobConLim} = \boldsymbol{QSobConLim},$$

where $\boldsymbol{QSobConLim}$ is the full subcategory of $\boldsymbol{ConLim}$ consisting of all quasi-sober constant limit spaces.

(iii) By Theorems 8, the categories $\boldsymbol{\overline{T}_0SobConLim}$, $\boldsymbol{T_0'SobConLim}$, and $\boldsymbol{QSobConLim}$ have all limits and colimits.

(iv) By Theorem 8, a T_0 sober constant limit space is T_0' sober, $\overline{T_0}$ sober, a quasi-sober, and irreducible. The constant limit space $(\mathbb{R}, F(\mathbb{R}))$ is quasi-sober, $\overline{T}_0$ sober, and T_0' sober, and irreducible but it is not T_0 sober, where $\mathbb{R}$ is the set of real numbers.

(v) By Theorem 9, a constant limit space (B, K) is compact iff it is strongly compact.

(3) In $\boldsymbol{Lim}$,

(i) By Theorem 2.10 of [9] and Theorem 2.4 of [6],

$$\boldsymbol{Lim_{2scl}} \subset \boldsymbol{LT_2Lim} = \boldsymbol{NT_2Lim} \subset \boldsymbol{KT_2Lim}$$

and

$$\boldsymbol{LT_2Lim} \subset \boldsymbol{Lim_{1cl}} = \boldsymbol{Lim_{1scl}} = \boldsymbol{T_1Lim}$$

$$\subset \boldsymbol{Lim_{0cl}} = \boldsymbol{Lim_{0scl}} = \boldsymbol{\overline{T}_0 Lim} = \boldsymbol{T_0 Lim} = \boldsymbol{T_0' Lim}$$

(4) In ***ConFCO*** (the category of constant filter convergence spaces and continuous maps), by Theorems 4.3-4.5 of [20], Theorems 2.1, 2.2, 2.9, and 2.10 of [5],

$$\boldsymbol{LT_2 ConFCO} \subset \boldsymbol{NT_2 ConFCO} \subset \boldsymbol{KT_2 ConFCO} \subset \boldsymbol{ConFCO_{2cl}} = \boldsymbol{ConFCO_{2scl}}$$

$$\subset \boldsymbol{ConFCO_{1cl}} = \boldsymbol{ConFCO_{1scl}} = \boldsymbol{T_0 ConFCO} = \boldsymbol{T_1 ConFCO}$$

$$= \boldsymbol{\overline{T}_0 ConFCO} \subset \boldsymbol{ConFCO_{0cl}} = \boldsymbol{ConFCO_{0scl}} \subset \boldsymbol{T_0' ConFCO}$$

(5) In ***FCO*** (the category of filter convergence spaces and continuous maps),
(i) By Theorems 2.9 and 2.11 of [9] and Theorem 4.10 of [11],

$$\boldsymbol{LT_2 FCO} \subset \boldsymbol{NT_2 FCO} \subset \boldsymbol{KT_2 FCO} \subset \boldsymbol{FCO_{2scl}} \subset \boldsymbol{FCO_{2cl}}$$

$$= \boldsymbol{FCO_{1cl}} = \boldsymbol{FCO_{1scl}} = \boldsymbol{T_1 FCO} \subset \boldsymbol{FCO_{0cl}}$$

$$= \boldsymbol{FCO_{0scl}} = \boldsymbol{\overline{T}_0 FCO} \subset \boldsymbol{T_0 FCO} \subset \boldsymbol{T_0' FCO}$$

(ii) By Theorem 6.3 of [10], (B, K) is strongly compact iff every ultrafilter in B converges and every filter convergence space is compact.

(6) In ***CApp*** (the category of approach spaces and contraction maps), by Theorems 4.8, 4.9, 4.12, and 4.13 of [26] and Theorems 7, 9, and 10 of [25],

$$\boldsymbol{CApp_{2scl}} \subset \boldsymbol{CApp_{1scl}} \subset \boldsymbol{CApp_{0scl}}$$

and

$$\boldsymbol{CApp_{2cl}} \subset \boldsymbol{CApp_{1cl}} \subset \boldsymbol{CApp_{0cl}} = \boldsymbol{\overline{T}_0 CApp} \subset \boldsymbol{T_0 CApp} \subset \boldsymbol{T_0' CApp}$$

(7) In ***psqMet*** (the category of extended pseudo-quasi-semi metric spaces and non-expansive maps),

(i) By Theorem 6 of [15], Theorems 3.3-3.5 and 3.15 of [23], Theorem 3.10 of [16],

$$\boldsymbol{LT_2 pqsMet} = \boldsymbol{KT_2 pqsMet} = \boldsymbol{T_1 pqsMet} = \boldsymbol{pqsMet_{1SQ}} = \boldsymbol{pqsMet_{1scl}}$$

$$= \boldsymbol{pqsMet_{2scl}} \subset \boldsymbol{pqsMet_{1cl}} = \boldsymbol{pqsMet_{2cl}} = \boldsymbol{pqsMet_{1Q}} = \boldsymbol{\overline{T}_0 pqsMet}$$

$$\subset \boldsymbol{T_0 pqsMet} \subset \boldsymbol{pqsMet_{0scl}} \subset \boldsymbol{pqsMet_{0cl}} \subset \boldsymbol{T_0' pqsMet}$$

(ii) By Theorem 3.13 of [12], $\{x\}$ is closed for all $x \in X$ and the nonempty proper irreducible closed subsets of X are exactly the one-point subsets iff an extended pseudo-quasi-semi metric space (X, d) is $\overline{T_0}$ sober,

(iii) By Theorem 3.13 of [12], (X, d) is a quasi-sober and an extended quasi-semi metric space iff (X, d) is T_0 sober.

(8) In ***RRel*** (the category of reflexive relation spaces and relation preserving functions),

(i) By Theorem 3.7 of [12] and Theorem 3.7 of [13],

$$\boldsymbol{KT_2RRel} \subset \boldsymbol{RRel_{1cl}} = \boldsymbol{T_0RRel} = \boldsymbol{\overline{T}_0RRel}$$

$$\boldsymbol{RRel_{2cl} = RRel_{2scl} = RRel_{1SQ} = RRel_{2SQ} = RRel_{2Q} = LT_2RRel = T_1RRel}$$

(ii) By Theorems 3.8 and 3.9 of [12],

$$\boldsymbol{T_0'SobRRel} = \boldsymbol{QSobRRel},$$

where ***QSobRRel*** is the subcategory of ***RRel*** consisting of quasi-sober reflexive spaces.

(iii) By Theorems 3.8 and 3.9 of [12],

$$\boldsymbol{T_0SobRRel} = \boldsymbol{\overline{T}_0SobRRel}$$

(iv) By Theorems 3.8 and 3.9 of [12], a reflexive space (B, R) is $\overline{T_0}$ sober iff the nonempty proper irreducible closed subsets of B are exactly the one-point subsets and $\{x\}$ is closed for all $x \in B$ iff (B, R) is T_0 sober.

(v) By Theorems 3.2 and 5.2 of [13], $(B, R) \in \boldsymbol{RRel_{1SQ}}$ iff it is NT_2.

(vi) By Theorem 5.2, Part (1), and Theorem of 3.8 of [12], if $(B, R) \in \boldsymbol{RRel_{1SQ}}$, then it is quasi-sober and T_0 sober.

(vii) By Theorem 5.3 of [13], $\boldsymbol{RRel_{1SQ}} \subset \boldsymbol{RRel_{1Q}}$ and also by Theorem 5.2 of [13], if $(B, R) \in KT_2$, then $(B, R) \in \boldsymbol{RRel_{1SQ}}$ iff $(B, R) \in \boldsymbol{RRel_{1Q}}$.

(viii) By Theorem 3.4 of [14], a reflexive space (A, R) is compact iff for every $x \in A$ there exist $a, b \in A$ with xRa and bRx.

(9) In ***Rel*** (the category of relation spaces and relation preserving functions),

(i) By Theorem 3.3 of [14],

$$\boldsymbol{Rel_{1cl}} = \boldsymbol{Rel_{2cl}} = \boldsymbol{Rel_{1Q}} = \boldsymbol{Rel_{2Q}} = \boldsymbol{Rel_{1SQ}} = \boldsymbol{Rel_{2SQ}}$$

(ii) By Theorem 4.5 of [14],

$$\boldsymbol{LT_2Rel} \subset \boldsymbol{NT_2Rel} \subset \boldsymbol{KT_2Rel} = \boldsymbol{\overline{T}_2Rel} = \boldsymbol{pre\overline{T}_2Rel}$$

$$\subset \boldsymbol{Rel_{1Q}} = \boldsymbol{T_1Rel} = \boldsymbol{T_0'Rel} = \boldsymbol{\overline{T}_0Rel} = \boldsymbol{Rel}$$

(iii) By Theorem 3.3 of [14],

$$\boldsymbol{\overline{T}_0SobRel} = \boldsymbol{T_0'SobRel} = \boldsymbol{QSobRel},$$

where $\boldsymbol{QSobRel}$ is the full subcategory of $\boldsymbol{Rel}$ consisting of all quasi-sober relation spaces.

(iv) By Theorem 3.3 of [14], every relation space is compact.

(10) In any topological category,

(i) By Theorem 2.7 of [6], $\overline{T_0}$ implies T_0' but the converse is not true, in general and by Theorem 3.1 of [8], $preT_2'$ implies $pre\overline{T}_2$. Furthermore, there is no relationship between $\overline{T_0}$ and T_0. Also, by Theorem 3.1 of [8], LT_2 implies KT_2 but the converse is not true, in general. Moreover, by Remark 2.8 (7) of [6], notions of KT_2 and NT_2 are independent of each other.

By Theorem 3.5 of [11], in the realm of $pre\overline{T}_2$ objects, $\overline{T_0}$, T_1, and $\overline{T}_2$ objects are equivalent.

(ii) By Theorems 3.5, 3.13 and Parts (2) and (3) of [12], every $\overline{T_0}$ sober object is T_0' sober. Also, there is no implication between T_0 sober and $\overline{T_0}$ sober.

(iii) By Remark 6.2 of [10] the notions of compactness and strongly compactness are different from each other, in general.

Author Contribution Statements The authors have made equal contributions in this work.

Declaration of Competing Interests The authors declare that they have no competing interests.

Acknowledgements The authors are thankful to referees for their valuable comments and suggestions which helped very much in improving the paper.

References

[1] Adamek, J., Herrlich. H., Strecker, G. E., Abstract and Concrete Categories, Pure and Applied Mathematics, John Wiley & Sons, New York, 1990.

[2] Baran, M., Separation properties, *Indian Journal Pure Applied Mathematics*, 23 (1991), 333–341.

[3] Baran, M., Stack and filters, *Turkish Journal of Mathematics*, 16 (1992), 95–108.

[4] Baran M., The notion of closedness in topological categories. *Commentationes Mathematicae Universitatis Carolinae*, 34 (2) (1993), 383–395.

[5] Baran, M., Separation properties in categories of constant convergence spaces, *Turkish Journal of Mathematics*, 18 (1994), 238–248.

[6] Baran, M., Altındiş, H., T_2–objects in topological categories, *Acta Mathematica Hungarica* 71 (1996), 41–48.

[7] Baran, M., A notion of compactness in topological categories, *Publicationes Mathematicae Debrecen*, 50 (1997), 221–234.

[8] Baran M., Completely regular objects and normal objects in topological categories. *Acta Mathematica Hungarica*, 80 (1998), 211–224.

[9] Baran, M., Closure operators in convergence spaces, *Acta Mathematica Hungarica*, 87 (2000), 33–45. https://doi.org/10.1023/a:1006768916033

[10] Baran, M., Kula, M., A note on separation and compactness in categories of convergence spaces, *Applied General Topology*, 4 (1) (2003), 1–13. https://doi.org/10.4995/agt.2003.2005

[11] Baran, M., $PreT_2$ objects in topological category, *Applied Category Structures*, 17 (2009), 591–602. https://doi.org/10.1007/s10485-008-9161-4

[12] Baran, M., Abughalwa, H., Sober spaces, *Turkish Journal of Mathematics*, 46 (2022), 299–310. https://doi.org/10.3906/mat-2109-95

[13] Baran, M., Separation, connectedness, and disconnectedness, *Turkish Journal of Mathematics*, 47 (1) (2023), 279–295. https://doi.org/10.55730/1300-0098.3360

[14] Baran, M., Stone spaces I, *Filomat*, 38 (2024).

[15] Baran, T. M., Local T_2 extended pseudo-quasi-semi metric spaces, *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 68 (2) (2019), 2117–2127. https://doi.org/10.31801/cfsuasmas.497701

[16] Baran, T. M., Closedness, separation and connectedness in pseudo-quasi-semi metric spaces, *Filomat*, 34 (14) (2020), 4757–4766. https://doi.org/10.2298/FIL2014757B

[17] Binz, E., Keller, H. H., Kunktionenräume in der Kategorie der Limesräume, no. 383 in Sero AI. Ann. Acad. Sci. Fennicae, 1966.

[18] Dieudonné, J. A., Grothendieck A. Eléments de Géométrie Algébrique, Springer, New York, 1971.

[19] Dikranjan, D., Tholen, W., Categorical Structure of Closure Operators, Kluwer Academic Publishers, Dordrecht, 1995.

[20] Erciyes, A., Baran, T. M., Qasim, M., Closure operators in constant filter convergence spaces, *Konuralp Journal of Mathematics*, 8 (1) (2020), 185–191.

[21] Erciyes, A., Baran, T. M., T_4, Urysohn's Lemma, and Tietze Extension Theorem for constant filter convergence Spaces, *Turkish Journal of Mathematics*, 45 (2021), 843–855. https://doi.org/10.3906/mat-2012-101

[22] Frölicher, A., Bucher, W., Calculus in Vector Spaces without Norm, Lecture Notes in Mathematics, 1966.

[23] Kula, M., Baran, T. M., Separation axioms, Urysohn's lemma and Tietze extention theorem for extended pseudo-quasi-semi metric spaces, *Filomat*, 36 (2) (2022), 703–713. https://doi.org/10.2298/FIL2202703B

[24] Ozkan, S., Kula, M., Kula, S., Baran, T. M., Closure operators, irreducibility, Urysohn's lemma, and Tietze extension theorem for proximity spaces, *Turkish Journal of Mathematics*, 47 (1) (2023), 870–882. https://doi.org/10.55730/1300-0098.3398

[25] Qasim, M., Baran, M., T_1 approach spaces, *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, 68 (1) (2019), 784–800. https://doi.org/10.31801/cfsuasmas.478632

[26] Qasim, M., Baran, M., Abughalwa, H., Closure operators in convergence approach spaces, *Turkish Journal of Mathematics*, 45 (2021), 139–152. https://doi.org/10.3906/mat-2008-6