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The Qualitative Analysis of Some Difference Equations Using Homogeneous Functions

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Abstract

This article deals with the qualitative analysis of a general class of difference equations. That is, we examine the periodicity nature and the stability character of some non-linear second-order difference equations. Homogeneous functions are used while examining the character of the solutions of introduced difference equations. Moreover, a new technique available in the literature is used to examine the periodic solutions of these equations.

1. Introduction

Although it is known that the theory of difference equations emerged with the rabbit problem introduced by the famous Italian mathematician Fibonacci in 1202 has been a field of study that has been of interest to many scientists, especially in the last 30 years (see [1, 2, 3, 4, 5, 6, 7, 8, 9]). Difference equations are an important field of study in many applied sciences, including mathematics, physics, chemistry, statistics, sociology, psychology, and engineering. Different mathematical models are needed to examine situations related to different living conditions, such as the climate crisis, the arms race, plant populations, animal populations, human populations, birth and death rates, migration rates, the spread of diseases. Here, difference equations come into play, and ecological, biological, economic, statistical, sociological and psychological mathematical models that can be used in different fields of science are created (see [10, 11, 12, 13, 14, 15, 16, 17]). In this context, the examination of difference equations (because it models various systems) is of great importance in that it is applicable not only in mathematics but also in different branches.

In recent years, many studies have been done on difference equations in mathematics, sub-branches of mathematics and other sciences (see [18, 19, 20]). Any quantitative and qualitative research, especially in the field of difference equations, is very important. Detailed qualitative studies in this field are invaluable when considering any result obtained by examining the global behavior, asymptotic behavior, boundedness nature and the stability character of solutions of difference equations. However, considering difference equation theory, it should be noted that there are not many general theorems and techniques that study difference equation classes. The structure of higher-order non-linear difference equation classes is quite complex and challenging. For this reason, although there are many articles and books on linear difference equations, there are not many sources on higher-order non-linear difference equations. On account of this, it is very important to examine various difference equations that will both contribute to the literature and expand and improve the difference equation theory.

In [21], Elsayed introduced a new method for the prime period two solutions and the prime period three solutions of the rational difference equation

$$\omega_{n+1}=\mu+\phi\frac{\omega_n}{\omega_{n-1}}+\gamma\frac{\omega_{n-1}}{\omega_n},\quad n=0,1,\ldots$$

where the parameters μ,ϕ,γ and initial values ω_{-1},ω_0 are positive real numbers. Besides, the global convergence and the boundedness nature were investigated.

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In [22], Moaaz *et al.* examined the asymptotic behavior, that is, the stability, the oscillation and the periodicity character of solutions of a general class of difference equations

$$z_{n+1} = g(z_n, z_{n-1}), \quad n = 0, 1, \ldots$$

where the initial conditions z_{-1}, z_0 are real numbers and g is a continuous homogeneous function with degree zero.
In [23], Moaaz investigated the asymptotic behavior of solutions of the following general class of difference equations

$$\omega_{n+1} = g(\omega_{n-l}, \omega_{n-k})$$

where l, k are positive integers, the initial conditions $\omega_{-\mu}, \omega_{-\mu+1}, \ldots, \omega_0$ are real numbers for $\mu = \max\{l, k\}$ and g is a continuous homogeneous real function of degree γ. Namely, the periodic solutions, the global attractiveness and the stability have been examined.
In [24], Stevic has shown that the claim given in Theorem 3.3 in [23] is not true. Essentially, he has improved and expanded global attractiveness results.
In [25], Abdelrahman *et al.* investigated the local stability, the periodicity and the boundedness character of solutions of a new class of the difference equations

$$\omega_{n+1} = \zeta \omega_{n-l} + \varphi \omega_{n-k} + g(\omega_{n-l}, \omega_{n-k}), \quad n = 0, 1, \ldots \tag{1.1}$$

where l, k are non-negative integers, the parameters ζ, φ are non-negative real numbers and the initial values $\omega_{-s}, \omega_{-s+1}, ..., \omega_0$ are positive real numbers for $s = \max\{l, k\}$ and $g : (0, \infty)^2 \to (0, \infty)$ is a continuous homogeneous function with degree zero.
In [26], Abdelrahman investigated the dynamical behavior of solutions of a general class of difference equations

$$x_{m+1} = g(x_m, x_{m-1}, \ldots, x_{m-k}), \quad m = 0, 1, \ldots$$

where $g : (0, \infty)^{k+1} \to (0, \infty)$ is a continuously homogeneous function of degree zero and k is a positive integer. That is, the stability, the periodicity and the oscillatory have been examined.
In [27], Moaaz *et al.* examined the existence and non-existence of periodic solutions of some non-linear difference equations. Especially, they studied the existence of periodic solutions of the difference equation

$$\omega_{n+1} = \gamma \omega_{n-1} F(\omega_n, \omega_{n-1})$$

where the parameter γ is positive real number, the initial values ω_{-1}, ω_0 are positive real numbers and F is a homothetic function, namely there exists a strictly increasing function $F_1 : \mathbb{R} \to \mathbb{R}$ and $F_2 : \mathbb{R}^2 \to \mathbb{R}$ are homogenous function with degree ρ, such that $F = F_1(F_2)$ and also studied the following second-order difference equation

$$\omega_{n+1} = \mu + \eta \frac{\omega_{n-1}^{\rho}}{h(\omega_n, \omega_{n-1})}$$

where ρ is a positive real number, the parameters μ, η are arbitrary real numbers, the initial values ω_{-1}, ω_0 arbitrary real numbers and h is a continuous homogeneous function with degree ρ. Finally, they obtained the periodicity results of the closed-form difference equations

$$\omega_{n+1} = \zeta(\omega_n, \omega_{n-1})$$

and

$$\omega_{n+1} = \zeta(\omega_n, \omega_{n-2})$$

where $\zeta \in C\left((0, \infty)^2, (0, \infty)\right)$ and the initial values $\omega_{-2}, \omega_{-1}, \omega_0$ are positive arbitrary real numbers.
In [28], Gümüş and Eğilmez investigated the global behavior of solutions, that is, the prime period two solutions, the prime period three solutions and the stability character of a new general class of the second-order difference equation

$$\delta_{m+1} = \omega + \zeta \frac{f(\delta_m, \delta_{m-1})}{\delta_{m-1}^{\beta}}, \quad m = 0, 1, ...$$

where the parameters $\omega, \zeta \in \mathbb{R}$, the initial conditions $\delta_{-1}, \delta_0 \in \mathbb{R}$ and $f : (0, \infty)^2 \to (0, \infty)$ is a continuous homogeneous function with degree β.
This paper aims to investigate the global dynamics of solutions for a new general class of the second-order difference equations

$$\omega_{m+1} = \sigma + \zeta \frac{g(\omega_m, \omega_{m-1})}{\omega_m^{\gamma}}, \quad m = 0, 1, \ldots \tag{1.2}$$

$$\omega_{m+1}=\sigma+\zeta\frac{\omega_m^{\gamma}}{g(\omega_m,\omega_{m-1})},\quad m=0,1,\ldots \tag{1.3}$$

where the parameters σ,ζ are arbitrary real numbers, the initial conditions ω_{-1},ω_0 are arbitrary real numbers and $g:(0,\infty)^2\to(0,\infty)$ is a continuous homogeneous function with degree γ. In other words, the prime period two solutions, the prime period three solutions and the stability character are discussed in detail. Also, periodic solutions are studied using a new technique. In addition, stability analysis of the equilibrium point is performed and new sufficient conditions for stability character are specified.
In the following, we will give a very useful theorem to examine the stability character of the solutions of difference equations, which we will benefit from in this paper.

Theorem 1.1. *[19] (Clark Theorem) Assume that $a_0,a_1\in\mathbb{R}$ and $k\in\{0,1,\ldots\}$. Then, the difference equation*

$$\gamma_{m+1}+a_0\gamma_m+a_1\gamma_{m-k}=0,\quad m=0,1,\ldots.$$

is the asymptotic stability if

$$|a_0|+|a_1|<1.$$

2. The behavior of solutions of the difference equation $\omega_{m+1}=\sigma+\zeta\frac{g(\omega_m,\omega_{m-1})}{\omega_m^{\gamma}}$

This section is devoted to investigating the dynamical behavior of solutions, that is, the two periodic solutions, the three periodic solutions and the local stability of second-order rational difference equation (1.2).
Here, we can easily find the positive equilibrium point of Eq.(1.2) as

$$\bar{\omega}=\sigma+\zeta g(1,1).$$

Now, let's define the function $f:(0,\infty)^2\to(0,\infty)$ by

$$f(u,v)=\sigma+\zeta\frac{g(u,v)}{u^{\gamma}}.$$

Hence, we get the partial derivatives of the function f

$$\frac{\partial f}{\partial u}(u,v)=\zeta\frac{ug_u(u,v)-\gamma g(u,v)}{u^{\gamma+1}}$$

and

$$\frac{\partial f}{\partial v}(u,v)=\zeta\frac{g_v(u,v)}{u^{\gamma}}.$$

In the next theorem, the locally asymptotic stability of Eq.(1.2) will be examined.

Theorem 2.1. *The equilibrium point of Eq.(1.2) $\bar{\omega}=\sigma+\zeta g(1,1)$ is locally asymptotically stable if*

$$|g_u(1,1)-\gamma g(1,1)|+|g_v(1,1)|<\left|\frac{\sigma+\zeta g(1,1)}{\zeta}\right|. \tag{2.1}$$

Proof. By using the Euler's Homogeneous Function Theorem, we obtain that

$$\begin{aligned}f_u(\bar{\omega},\bar{\omega}) &= \zeta\frac{\bar{\omega}g_u(\bar{\omega},\bar{\omega})-\gamma g(\bar{\omega},\bar{\omega})}{\bar{\omega}^{\gamma+1}}\\ &= \zeta\frac{\bar{\omega}^{\gamma}g_u(1,1)-\gamma\bar{\omega}^{\gamma}g(1,1)}{\bar{\omega}^{\gamma+1}}\\ &= \zeta\frac{g_u(1,1)-\gamma g(1,1)}{\bar{\omega}},\end{aligned}$$

and

$$\begin{aligned}f_v(\bar{\omega},\bar{\omega}) &= \zeta\frac{g_v(\bar{\omega},\bar{\omega})}{\bar{\omega}^{\gamma}}\\ &= \zeta\frac{\bar{\omega}^{\gamma-1}g_v(1,1)}{\bar{\omega}^{\gamma}}\\ &= \zeta\frac{g_v(1,1)}{\bar{\omega}}.\end{aligned}$$

Now, by applying Clark Theorem, we find

$$\left|\zeta\frac{g_u(1,1)-\gamma g(1,1)}{\bar{\omega}}\right|+\left|\zeta\frac{g_v(1,1)}{\bar{\omega}}\right|<1.$$

Since $\bar{\omega}=\sigma+\zeta g(1,1)$, we find

$$\left|\zeta\frac{g_u(1,1)-\gamma g(1,1)}{(\sigma+\zeta g(1,1))}\right|+\left|\zeta\frac{g_v(1,1)}{(\sigma+\zeta g(1,1))}\right|<1,$$

and so

$$|g_u(1,1)-\gamma g(1,1)|+|g_v(1,1)|<\left|\frac{\sigma+\zeta g(1,1)}{\zeta}\right|.$$

The proof is completed. □

In the next theorem, the two periodic solutions of Eq.(1.2) will be examined.

Theorem 2.2. *Eq.(1.2) has the prime period two solution*

$$\ldots,\phi,\vartheta,\phi,\vartheta,\ldots.$$

if and only if

$$\sigma=\zeta\frac{\Omega g\left(1,\frac{1}{\Omega}\right)-\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)} \tag{2.2}$$

where $\Omega=\frac{\phi}{\vartheta}$, $\Omega\in\mathbb{R}-\{0,\pm1\}$.

Proof. Suppose that Eq.(1.2) has a prime period two solution in the following form

$$\ldots,\phi,\vartheta,\phi,\vartheta,\ldots.$$

Let's define $\omega_{n-(2s+1)}=\phi$ and $\omega_{n-2s}=\vartheta$ for $s=0,1,2,\ldots$. From Eq.(1.2), we obtain

$$\phi=\sigma+\zeta\frac{g(\vartheta,\phi)}{\vartheta^{\gamma}},$$

and

$$\vartheta=\sigma+\zeta\frac{g(\phi,\vartheta)}{\phi^{\gamma}}.$$

Since g is a continuous homogeneous function of degree γ, we obtain

$$\phi=\sigma+\zeta\frac{\phi^{\gamma}g\left(\frac{\vartheta}{\phi},1\right)}{\vartheta^{\gamma}}\Rightarrow\phi=\sigma+\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right), \tag{2.3}$$

and

$$\vartheta=\sigma+\zeta\frac{\phi^{\gamma}g\left(1,\frac{\vartheta}{\phi}\right)}{\phi^{\gamma}}\Rightarrow\vartheta=\sigma+\zeta g\left(1,\frac{1}{\Omega}\right). \tag{2.4}$$

By using the fact $\phi-\Omega\vartheta=0$, we find

$$0=\phi-\Omega\vartheta=\sigma+\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)-\Omega\left(\sigma+\zeta g\left(1,\frac{1}{\Omega}\right)\right),$$

and so

$$\sigma(1-\Omega)=\Omega\zeta g\left(1,\frac{1}{\Omega}\right)-\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right).$$

Therefore, we get

$$\sigma=\zeta\frac{\Omega g\left(1,\frac{1}{\Omega}\right)-\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}.$$

Thus, from (2.3) and (2.4) respectively, we find

$$\begin{aligned}\phi &= \frac{\Omega\zeta g\left(1,\frac{1}{\Omega}\right)-\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}+\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right) \\ &= \zeta\frac{\Omega g\left(1,\frac{1}{\Omega}\right)-\Omega^{\gamma+1}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)},\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\vartheta &= \frac{\Omega\zeta g\left(1,\frac{1}{\Omega}\right)-\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}+\zeta g\left(1,\frac{1}{\Omega}\right) \\ &= \zeta\frac{g\left(1,\frac{1}{\Omega}\right)-\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}.\end{aligned} \tag{2.6}$$

Secondly, assume (2.2) holds. Let's choose the initial conditions

$$\omega_{-1}=\phi \text{ and } \omega_0=\vartheta,$$

where ϕ,ϑ are defined as (2.3) and (2.4), respectively. Hence, we obtain that

$$\begin{aligned}\omega_1 &= \sigma+\zeta\frac{g(\omega_0,\omega_{-1})}{\omega_0^{\gamma}} \\ &= \sigma+\zeta\frac{g(\vartheta,\phi)}{\vartheta^{\gamma}} \\ &= \frac{\Omega\zeta g\left(1,\frac{1}{\Omega}\right)-\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}+\zeta\frac{\phi^{\gamma}g\left(\frac{\vartheta}{\phi},1\right)}{\vartheta^{\gamma}} \\ &= \frac{\Omega\zeta g\left(1,\frac{1}{\Omega}\right)-\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}+\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right) \\ &= \zeta\frac{\Omega g\left(1,\frac{1}{\Omega}\right)-\Omega^{\gamma+1}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}=\phi,\end{aligned}$$

and

$$\begin{aligned}\omega_2 &= \sigma+\zeta\frac{g(\omega_1,\omega_0)}{\omega_1^{\gamma}} \\ &= \sigma+\zeta\frac{g(\phi,\vartheta)}{\phi^{\gamma}} \\ &= \frac{\Omega\zeta g\left(1,\frac{1}{\Omega}\right)-\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}+\zeta\frac{\phi^{\gamma}g\left(1,\frac{\vartheta}{\phi}\right)}{\phi^{\gamma}} \\ &= \frac{\Omega\zeta g\left(1,\frac{1}{\Omega}\right)-\zeta\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}+\zeta g\left(1,\frac{1}{\Omega}\right) \\ &= \zeta\frac{g\left(1,\frac{1}{\Omega}\right)-\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}{(1-\Omega)}=\vartheta.\end{aligned}$$

Then, by induction, we can obtain that for all $n\geq 0$

$$\omega_{2n-1}=\phi \text{ and } \omega_{2n}=\vartheta.$$

Hence, Eq.(1.2) has a prime period two solution. The proof is completed. □

In the following theorem, the prime period three solution of Eq.(1.2) will be investigated.

Theorem 2.3. *Eq.(1.2) has the prime period three solution $\{\omega_n\}_{n=-1}^{\infty}$ where*

$$\omega_n=\begin{cases}\phi, & \text{for } n=3z-1 \\ \vartheta, & \text{for } n=3z \\ \nu, & \text{for } n=3z+1\end{cases}, \quad z=0,1,\ldots$$

if and only if

$$\begin{aligned}\eta\left(\sigma+\frac{\zeta}{\psi^{\gamma}}g(\psi,\eta)\right) &= \sigma+\zeta g(1,\psi) \\ \psi\left(\sigma+\frac{\zeta}{\psi^{\gamma}}g(\psi,\eta)\right) &= \sigma+\frac{\zeta}{\eta^{\gamma}}g(1,\psi)\end{aligned} \tag{2.7}$$

where $\eta=\frac{\vartheta}{\phi}$ *and* $\psi=\frac{\nu}{\phi}$, $\eta,\psi\in\mathbb{R}-\{0,\mp1\}$.

Proof. Suppose that Eq.(1.2) has a prime period three solution in the following form

$$\ldots,\phi,\vartheta,\nu,\phi,\vartheta,\nu,\ldots.$$

From Eq.(1.2), we obtain that

$$\phi=\sigma+\zeta\frac{g(\nu,\vartheta)}{\nu^{\gamma}},$$

$$\vartheta=\sigma+\zeta\frac{g(\phi,\nu)}{\phi^{\gamma}},$$

and

$$\nu=\sigma+\zeta\frac{g(\vartheta,\phi)}{\vartheta^{\gamma}}.$$

By using the homogeneous function definition, we can find the equalities

$$\phi=\sigma+\zeta\frac{\phi^{\gamma}g(\psi,\eta)}{\nu^{\gamma}}\Rightarrow\phi=\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}$$

$$\vartheta=\sigma+\zeta\frac{\phi^{\gamma}g(1,\psi)}{\phi^{\gamma}}\Rightarrow\vartheta=\sigma+\zeta g(1,\psi)$$

and

$$\nu=\sigma+\zeta\frac{\phi^{\gamma}g(\eta,1)}{\vartheta^{\gamma}}\Rightarrow\nu=\sigma+\zeta\frac{g(\eta,1)}{\eta^{\gamma}}.$$

Therefore, we can easily see that

$$\eta=\frac{\vartheta}{\phi}=\frac{\sigma+\zeta g(1,\psi)}{\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}}$$

and

$$\psi=\frac{\nu}{\phi}=\frac{\sigma+\zeta\frac{g(\eta,1)}{\eta^{\gamma}}}{\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}}.$$

Thus, we can rewrite the equalities

$$\eta\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)=\sigma+\zeta g(1,\psi),$$

$$\psi\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)=\sigma+\zeta\frac{g(\eta,1)}{\eta^{\gamma}}.$$

Secondly, assume (2.7) holds. Let's choose the initial conditions for all $\eta,\psi\in\mathbb{R}-\{0,\mp1\}$

$$\omega_{-1}=\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}$$

and

$$\omega_0=\sigma+\zeta g(1,\psi).$$

Thus, we obtain that

$$\begin{aligned}
\omega_1 &= \sigma+\zeta\frac{g(\omega_0,\omega_{-1})}{\omega_0^{\gamma}} \\
&= \sigma+\zeta\frac{g\left(\sigma+\zeta g(1,\psi),\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)}{(\sigma+\zeta g(1,\psi))^{\gamma}} \\
&= \sigma+\zeta\frac{g\left(\eta\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right),\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)}{\left(\eta\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)\right)^{\gamma}} \\
&= \sigma+\zeta\frac{\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)^{\gamma}g(\eta,1)}{\left(\eta\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)\right)^{\gamma}} \\
&= \sigma+\zeta\frac{g(\eta,1)}{\eta^{\gamma}}=\nu,
\end{aligned}$$

$$\begin{aligned}
\omega_2 &= \sigma+\zeta\frac{g(\omega_1,\omega_0)}{\omega_1^{\gamma}} \\
&= \sigma+\zeta\frac{g\left(\sigma+\zeta\frac{g(\eta,1)}{\eta^{\gamma}},\sigma+\zeta g(1,\psi)\right)}{\left(\sigma+\zeta\frac{g(\eta,1)}{\eta^{\gamma}}\right)^{\gamma}} \\
&= \sigma+\zeta\frac{g\left(\psi\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right),\eta\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)\right)}{\left(\psi\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)\right)^{\gamma}} \\
&= \sigma+\zeta\frac{\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)^{\gamma}g(\psi,\eta)}{\left(\psi\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)\right)^{\gamma}} \\
&= \sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}=\phi,
\end{aligned}$$

and

$$\begin{aligned}
\omega_3 &= \sigma+\zeta\frac{g(\omega_2,\omega_1)}{\omega_2^{\gamma}} \\
&= \sigma+\zeta\frac{g\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}},\sigma+\zeta\frac{g(\eta,1)}{\eta^{\gamma}}\right)}{\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)^{\gamma}} \\
&= \sigma+\zeta\frac{g\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}},\psi\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)\right)}{\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)^{\gamma}} \\
&= \sigma+\zeta\frac{\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)^{\gamma}g(1,\psi)}{\left(\sigma+\zeta\frac{g(\psi,\eta)}{\psi^{\gamma}}\right)^{\gamma}} \\
&= \sigma+\zeta g(1,\psi)=\vartheta.
\end{aligned}$$

Then, by induction, we can obtain that for all $n\geq 0$.

$$\omega_{3n-1}=\phi,\ \omega_{3n}=\vartheta \text{ and } \omega_{3n+1}=\nu.$$

Hence, Eq.(1.2) has a prime period three solution. The proof is completed. □

3. The behavior of solutions of the difference equation $\omega_{m+1}=\sigma+\zeta\frac{\omega_m^{\gamma}}{g(\omega_m,\omega_{m-1})}$

This section is devoted to examining the asymptotic behavior of the solutions of non-linear rational difference equation (1.3). Here, we can easily obtain the positive equilibrium point of Eq.(1.3) as

$$\bar{\omega}=\sigma+\frac{\zeta}{g(1,1)}.$$

Now, let's define the function $z:(0,\infty)^2\to(0,\infty)$ as

$$z(u,v)=\sigma+\zeta\frac{u^{\gamma}}{g(u,v)}.$$

Therefore, we find

$$\frac{\partial z}{\partial u}(u,v)=\zeta\frac{\gamma u^{\gamma-1}g(u,v)-g_u(u,v)u^{\gamma}}{(g(u,v))^2}$$

and

$$\frac{\partial z}{\partial v}(u,v)=-\zeta\frac{g_v(u,v)u^{\gamma}}{(g(u,v))^2}.$$

In the next theorem, the locally asymptotic stability for Eq.(1.3) will be examined.

Theorem 3.1. *The equilibrium point of Eq.(1.3) $\bar{\omega}=\sigma+\frac{\zeta}{g(1,1)}$ is locally asymptotically stable if*

$$|\gamma g(1,1)-g_u(1,1)|+|g_v(1,1)|<\left|\frac{\left(\sigma+\frac{\zeta}{g(1,1)}\right)g^2(1,1)}{\zeta}\right|.$$

Proof. Since g is a homogeneous function with degree γ, the partial derivatives are of degree $\gamma-1$. Thus, we obtain that

$$\begin{aligned}
z_u(\bar{\omega},\bar{\omega}) &= \zeta\frac{\gamma\bar{\omega}^{\gamma-1}g(\bar{\omega},\bar{\omega})-g_u(\bar{\omega},\bar{\omega})\bar{\omega}^{\gamma}}{(g(\bar{\omega},\bar{\omega}))^2}\\
&= \zeta\frac{\gamma\bar{\omega}^{2\gamma-1}g(1,1)-g_u(1,1)\bar{\omega}^{2\gamma-1}}{(\bar{\omega}^{\gamma}g(1,1))^2}\\
&= \zeta\frac{\gamma g(1,1)-g_u(1,1)}{\bar{\omega}g^2(1,1)},
\end{aligned}$$

and

$$\begin{aligned}
z_v(\bar{\omega},\bar{\omega}) &= -\zeta\frac{g_v(\bar{\omega},\bar{\omega})\bar{\omega}^{\gamma}}{(g(\bar{\omega},\bar{\omega}))^2}\\
&= -\zeta\frac{g_v(1,1)\bar{\omega}^{2\gamma-1}}{\bar{\omega}^{2\gamma}g^2(1,1)}\\
&= -\zeta\frac{g_v(1,1)}{\bar{\omega}g^2(1,1)}.
\end{aligned}$$

Now, by using Clark Theorem, we obtain

$$\left|\zeta\frac{\gamma g(1,1)-g_u(1,1)}{\bar{\omega}g^2(1,1)}\right|+\left|\zeta\frac{g_v(1,1)}{\bar{\omega}g^2(1,1)}\right|<1.$$

Since the equilibrium point $\bar{\omega}=\sigma+\zeta\frac{1}{g(1,1)}$, we find

$$\left|\zeta\frac{\gamma g(1,1)-g_u(1,1)}{\left(\sigma+\frac{\zeta}{g(1,1)}\right)g^2(1,1)}\right|+\left|\zeta\frac{g_v(1,1)}{\left(\sigma+\frac{\zeta}{g(1,1)}\right)g^2(1,1)}\right|<1,$$

and so,

$$|\gamma g(1,1)-g_u(1,1)|+|g_v(1,1)|<\left|\frac{\left(\sigma+\zeta\frac{1}{g(1,1)}\right)g^2(1,1)}{\zeta}\right|.$$

This completes the proof. $\square$

In the next theorem, the prime period two solutions of Eq.(1.3) will be investigated.

Theorem 3.2. *Eq.(1.3) has the prime period two solution*

$$\ldots,\phi,\vartheta,\phi,\vartheta,\ldots$$

if and only if

$$\sigma=\frac{\zeta}{(1-\Omega)}\left(\frac{\Omega}{g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right) \tag{3.1}$$

where $\Omega=\frac{\phi}{\vartheta}$, $\Omega\in\mathbb{R}-\{0,\pm1\}$.

Proof. Suppose that Eq.(1.3) has a prime period two solution in the following form

$$\ldots,\phi,\vartheta,\phi,\vartheta,\ldots.$$

Let's define $\omega_{n-(2s+1)}=\phi$ and $\omega_{n-2s}=\vartheta$ for $s=0,1,2,\ldots$. From Eq.(1.3), we find

$$\phi=\sigma+\zeta\frac{\vartheta^{\gamma}}{g(\vartheta,\phi)},$$

and

$$\vartheta=\sigma+\zeta\frac{\phi^{\gamma}}{g(\phi,\vartheta)}.$$

From the definition of the homogeneous function, we can easily obtain that

$$\phi=\sigma+\zeta\frac{\vartheta^{\gamma}}{\phi^{\gamma}g\left(\frac{\vartheta}{\phi},1\right)}\Rightarrow\phi=\sigma+\frac{\zeta}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)} \tag{3.2}$$

and

$$\vartheta=\sigma+\zeta\frac{\phi^{\gamma}}{\phi^{\gamma}g\left(1,\frac{\vartheta}{\phi}\right)}\Rightarrow\vartheta=\sigma+\frac{\zeta}{g\left(1,\frac{1}{\Omega}\right)}. \tag{3.3}$$

Now, by using the fact $\phi-\Omega\vartheta=0$, we find

$$0=\phi-\Omega\vartheta=\sigma+\frac{\zeta}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}-\Omega\left(\sigma+\frac{\zeta}{g\left(1,\frac{1}{\Omega}\right)}\right)$$

and so,

$$\sigma(1-\Omega)=\zeta\left(\frac{\Omega}{g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right).$$

Hence, we find

$$\sigma=\frac{\zeta}{(1-\Omega)}\left(\frac{\Omega}{g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right).$$

Then, from Eq.(3.2) and (3.3), we obtain

$$\begin{aligned}\phi &= \sigma+\frac{\zeta}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)} \\ &= \frac{\zeta}{(1-\Omega)}\left(\frac{\Omega}{g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right)+\frac{\zeta}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)} \\ &= \zeta\left(\frac{\Omega}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}+\frac{(1-\Omega)}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right) \\ &= \zeta\left(\frac{\Omega}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}-\frac{\Omega}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right),\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\vartheta &= \sigma+\zeta\frac{1}{g\left(1,\frac{1}{\Omega}\right)} \\
&= \frac{\zeta}{(1-\Omega)}\left(\frac{\Omega}{g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right)+\zeta\frac{1}{g\left(1,\frac{1}{\Omega}\right)} \\
&= \zeta\left(\frac{\Omega}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}+\frac{(1-\Omega)}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}\right) \\
&= \zeta\left(\frac{1}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right).
\end{aligned} \tag{3.5}$$

On the other hand, suppose (3.1) holds. Let's choose the initial conditions

$$\omega_{-1}=\phi \text{ and } \omega_0=\vartheta,$$

where ϕ,ϑ are defined as (3.2) and (3.3), respectively. Therefore, we find

$$\begin{aligned}
\omega_1 &= \sigma+\zeta\frac{\omega_0^{\gamma}}{g(\omega_0,\omega_{-1})} \\
&= \sigma+\zeta\frac{\vartheta^{\gamma}}{g(\vartheta,\phi)} \\
&= \frac{\zeta}{(1-\Omega)}\left(\frac{\Omega}{g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right)+\zeta\frac{1}{\Omega^{\gamma}g\left(\frac{\vartheta}{\phi},1\right)} \\
&= \zeta\left(\frac{\Omega}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}+\frac{(1-\Omega)}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right) \\
&= \zeta\left(\frac{\Omega}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}-\frac{\Omega}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right)=\phi
\end{aligned}$$

and

$$\begin{aligned}
\omega_2 &= \sigma+\zeta\frac{\omega_1^{\gamma}}{g(\omega_1,\omega_0)} \\
&= \sigma+\zeta\frac{\phi^{\gamma}}{g(\phi,\vartheta)} \\
&= \frac{\zeta}{(1-\Omega)}\left(\frac{\Omega}{g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right)+\zeta\frac{1}{g\left(1,\frac{1}{\Omega}\right)} \\
&= \zeta\left(\frac{\Omega}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}+\frac{(1-\Omega)}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}\right) \\
&= \zeta\left(\frac{1}{(1-\Omega)g\left(1,\frac{1}{\Omega}\right)}-\frac{1}{(1-\Omega)\Omega^{\gamma}g\left(\frac{1}{\Omega},1\right)}\right)=\vartheta
\end{aligned}$$

Then, by induction, we can obtain that for all $n\geq 0$

$$\omega_{2n-1}=\phi \text{ and } \omega_{2n}=\vartheta.$$

Hence, Eq.(1.3) has a prime period two solution. The proof is completed. □

In the following theorem, the three periodic solutions of Eq.(1.3) will be studied.

Theorem 3.3. *Eq.(1.3) has a prime period three solution* $\{\omega_n\}_{n=-1}^{\infty}$ *where*

$$\omega_n=\begin{cases}\phi, & \text{for } n=3z-1\\ \vartheta, & \text{for } n=3z\\ \nu, & \text{for } n=3z+1\end{cases}, \quad z=0,1,\ldots$$

if and only if

$$\begin{aligned} \eta\left(\sigma+\zeta\frac{\psi^{\gamma}}{g(\psi,\eta)}\right) &= \sigma+\zeta\frac{1}{g(1,\psi)} \\ \psi\left(\sigma+\zeta\frac{\psi^{\gamma}}{g(\psi,\eta)}\right) &= \sigma+\zeta\frac{\eta^{\gamma}}{g(\eta,1)} \end{aligned} \tag{3.6}$$

where $\eta=\frac{\vartheta}{\phi}$ *and* $\psi=\frac{\nu}{\phi}, \eta, \psi \in \mathbb{R}-\{0,\pm 1\}$.

Proof. Suppose that Eq.(1.3) has a prime period three solution in the following form

$$\ldots,\phi,\vartheta,\nu,\phi,\vartheta,\nu,\ldots$$

From Eq.(1.3), we obtain that

$$\phi=\sigma+\zeta\frac{\nu^{\gamma}}{g(\nu,\vartheta)},$$

$$\vartheta=\sigma+\zeta\frac{\phi^{\gamma}}{g(\phi,\nu)}$$

and

$$\nu=\sigma+\zeta\frac{\vartheta^{\gamma}}{g(\vartheta,\phi)}.$$

Since g is a homogeneous function with degree γ, we obtain the equalities

$$\phi=\sigma+\zeta\frac{\psi^{\gamma}}{g(\psi,\eta)},$$

$$\vartheta=\sigma+\zeta\frac{1}{g(1,\psi)}$$

and

$$\nu=\sigma+\zeta\frac{\eta^{\gamma}}{g(\eta,1)}.$$

Hence, we find

$$\eta=\frac{\vartheta}{\phi}=\frac{\sigma+\zeta\frac{1}{g(1,\psi)}}{\sigma+\zeta\frac{\psi^{\gamma}}{g(\psi,\eta)}}$$

and

$$\psi=\frac{\nu}{\phi}=\frac{\sigma+\zeta\frac{\eta^{\gamma}}{g(\eta,1)}}{\sigma+\zeta\frac{\psi^{\gamma}}{g(\psi,\eta)}}.$$

Thus, we obtain that

$$\eta\left(\sigma+\zeta\frac{\psi^{\gamma}}{g(\psi,\eta)}\right)=\sigma+\zeta\frac{1}{g(1,\psi)},$$

$$\psi\left(\sigma+\zeta\frac{\psi^{\gamma}}{g(\psi,\eta)}\right)=\sigma+\zeta\frac{\eta^{\gamma}}{g(\eta,1)}.$$

Now, assume (3.6) holds. Let's choose the initial values for all $\eta,\psi\in\mathbb{R}-\{0,\pm 1\}$

$$\omega_{-1}=\sigma+\zeta\frac{\psi^{\gamma}}{g(\psi,\eta)}$$

and

$$\omega_0 = \sigma + \frac{\zeta}{g(1,\psi)}.$$

Therefore, we obtain

$$\begin{aligned}
\omega_1 &= \sigma + \zeta \frac{\omega_0^\gamma}{g(\omega_0,\omega_{-1})} \\
&= \sigma + \zeta \frac{\left(\sigma + \zeta \frac{1}{g(1,\psi)}\right)^\gamma}{g\left(\sigma + \zeta \frac{1}{g(1,\psi)}, \sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)} \\
&= \sigma + \zeta \frac{\left(\eta\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)\right)^\gamma}{g\left(\left(\eta\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)\right), \sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)} \\
&= \sigma + \zeta \frac{\left(\eta\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)\right)^\gamma}{\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)^\gamma g(\eta,1)} \\
&= \sigma + \zeta \frac{\eta^\gamma}{g(\eta,1)} = \nu,
\end{aligned}$$

$$\begin{aligned}
\omega_2 &= \sigma + \zeta \frac{\omega_1^\gamma}{g(\omega_1,\omega_0)} \\
&= \sigma + \zeta \frac{\left(\sigma + \zeta \frac{\eta^\gamma}{g(\eta,1)}\right)^\gamma}{g\left(\sigma + \zeta \frac{\eta^\gamma}{g(\eta,1)}, \sigma + \zeta \frac{1}{g(1,\psi)}\right)} \\
&= \sigma + \zeta \frac{\left(\psi\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)\right)^\gamma}{g\left(\psi\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right), \eta\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)\right)} \\
&= \sigma + \zeta \frac{\left(\psi\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)\right)^\gamma}{\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)^\gamma g(\psi,\eta)} \\
&= \sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)} = \phi
\end{aligned}$$

and

$$\begin{aligned}
\omega_3 &= \sigma + \zeta \frac{\omega_2^\gamma}{g(\omega_2,\omega_1)} \\
&= \sigma + \zeta \frac{\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)^\gamma}{g\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}, \sigma + \zeta \frac{\eta^\gamma}{g(\eta,1)}\right)} \\
&= \sigma + \zeta \frac{\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)^\gamma}{g\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}, \psi\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)\right)} \\
&= \sigma + \zeta \frac{\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)^\gamma}{\left(\sigma + \zeta \frac{\psi^\gamma}{g(\psi,\eta)}\right)^\gamma g(1,\psi)} \\
&= \sigma + \zeta \frac{1}{g(1,\psi)} = \vartheta.
\end{aligned}$$

Then, by induction, we obtain for all $n \geq 0$

$$\omega_{3n-1} = \phi,\ \omega_{3n} = \vartheta \text{ and } \omega_{3n+1} = \nu.$$

Hence, Eq.(1.3) has a prime period three solution. The proof is completed. □

4. Conclusions and suggestions

In this article, we have considered the detailed qualitative behavior of a general class of difference equations, which can be seen as an extension of [22, 23, 24, 25, 26, 27]. The qualitative behavior of the solutions of the introduced non-linear difference equations has been examined. In other words, the two periodic solutions, the three periodic solutions and the stability character of difference equations have been discussed. Qualitative research of mathematical models created using difference equations has an important place in mathematics, sub-branches of mathematics and other applied sciences. Here, the two periodic solutions of Eq.(1.2) and Eq.(1.3) in Theorem 3.2 and Theorem 2.2 and the three periodic solutions in Theorem 3.3 and Theorem 2.3 have been examined in detail. In these theorems, using the new technique, the periodicity character of Eq.(1.2) and Eq.(1.3) have been determined and necessary and sufficient conditions have been created for the existence of periodic solutions. In addition, the equilibrium points of Eq.(1.2) and Eq.(1.3) have been investigated and sufficient conditions have been obtained for the local asymptotic stability of these equilibrium points.
It can be suggested to those who do research in this field that research can be done in the equations established with the help of homogeneous functions. Difference equations created with these functions are very convenient and useful for researching general classes of difference equations.
In our future studies, we will aim to investigate some general classes of difference equations formed by homogeneous functions of different degrees.

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