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$\mathcal{D}$-Homothetic Deformations of Basic Classes of Almost Contact B-metric Structures

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Abstract. We study $\mathcal{D}$-homothetic deformations of almost contact manifolds with B-metric. Some basic classes are known to be invariant under these deformations. We examine the invariance of remaining basic classes. Also we investigate $\mathcal{D}$-homothetic deformations of normal and K-contact structures. We give examples of deformations of almost contact manifolds with B-metric structures in three dimensions.

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1. Introduction

Almost contact manifolds with B-metric were first classified in [3]. There are eleven basic classes of such manifolds depending on the symmetry properties of the fundamental tensor defined by using the covariant derivative of the metric. Many authors have made remarkable contributions to the study of these manifolds, see for instance [3, 4, 6, 9–11] and references therein.

Our aim in this study is to investigate $\mathcal{D}$-homothetic deformation of basic classes of almost contact manifolds with B-metric given in [1]. In [2], it was shown that the classes $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_4, \mathcal{F}_5$ are invariant under $\mathcal{D}$-homothetic deformations. We study the invariance of the remaining classes. In addition we investigate $\mathcal{D}$-homothetic deformations of normal and K-contact structures. We give examples of deformations of almost contact manifolds with B-metric structures in three dimensions. We deform the B-metric structure on a 3-dimensional Lie algebra which is of class $\mathcal{F}_8$ and obtain a structure in $\mathcal{F}_8 \oplus \mathcal{F}_{10}$. Also, we write the new covariant derivative of the deformed structure for a 3 dimensional almost contact B-metric manifold.

2. Preliminaries

An ordered triple (φ, ξ, η), where φ is an endomorphism, ξ is a vector field, η is a 1-form is called an almost contact structure on a smooth manifold M^{2n+1} if

$$\eta(\xi) = 1, \tag{2.1}$$

$$\varphi^2 = -I + \eta \otimes \xi. \tag{2.2}$$

If there also exists a compatible metric g with the property that

$$g(\varphi u, \varphi v) = -g(u, v) + \eta(u)\eta(v), \tag{2.3}$$

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where u, v are smooth vector fields on M, then $(M, \varphi, \xi, \eta, g)$ is called an almost contact manifold with B-metric. Identities (2.1), (2.2), (2.3) imply

$$\eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad \eta(u) = g(u, \xi), \quad g(\varphi u, v) = g(u, \varphi v).$$

We denote smooth vector fields and also tangent vectors by letters u, v, w.

The signature of the metric g is $(n+1, n)$.

There are 2^{11} classes of almost contact manifolds with B-metric. Let F be the tensor

$$F(u, v, w) = g((\nabla_u \varphi)(v), w),$$

for all $u, v, w \in T_pM$ where T_pM is the tangent space at p and ∇ denotes the covariant derivative of g. By (2.1), (2.2), (2.3), the tensor F satisfies the followings:

$$F(u, v, w) = F(u, w, v) \tag{2.4}$$

$$F(u, \varphi v, \varphi w) = F(u, v, w) - \eta(v)F(u, \xi, w) - \eta(w)F(u, v, \xi) \tag{2.5}$$

$$F(u, \xi, \xi) = 0. \tag{2.6}$$

The following 1-forms are Lee forms associated with F:

$$\theta(u) = g^{ij}F(e_i, e_j, u), \quad \theta^*(u) = g^{ij}F(e_i, \varphi e_j, u), \quad \omega(u) = F(\xi, \xi, u),$$

where $u \in T_pM$, $\{e_i, \xi\}$ is any basis for T_pM and (g^{ij}) is the matrix which is the inverse of g_{ij}.

Let $\mathcal{F}$ be the set of all $(0,3)$ tensors over T_pM having properties (2.4), (2.5). $\mathcal{F}$ is the direct sum of eleven subspaces $\mathcal{F}_i$, $i = 1, \ldots, 11$ with defining conditions listed below [3,6].

$$\mathcal{F}_1: \; F(u, v, w) = \frac{1}{2n}\{g(u, \varphi v)\theta(\varphi w) + g(\varphi u, \varphi v)\theta(\varphi^2 w) + g(u, \varphi w)\theta(\varphi v) + g(\varphi u, \varphi w)\theta(\varphi^2 v)\}$$

$$\begin{aligned} \mathcal{F}_2: \; & F(\xi, v, w) = F(u, \xi, w) = 0, \\ & F(u, v, \varphi w) + F(v, w, \varphi u) + F(w, u, \varphi v) = 0, \\ & \theta = 0 \end{aligned} \tag{2.7}$$

$$\mathcal{F}_3: \; F(\xi, v, w) = F(u, \xi, w) = 0, \; F(u, v, w) + F(v, w, u) + F(w, u, v) = 0 \tag{2.8}$$

$$\mathcal{F}_4: \; F(u, v, w) = -\frac{\theta(\xi)}{2n}\{g(\varphi u, \varphi v)\eta(w) + g(\varphi u, \varphi w)\eta(v)\}$$

$$\mathcal{F}_5: \; F(u, v, w) = -\frac{\theta^*(\xi)}{2n}\{g(u, \varphi v)\eta(w) + g(u, \varphi w)\eta(v)\}$$

$$\mathcal{F}_6: \; F(u, v, w) = -F(\varphi u, \varphi v, w) - F(\varphi u, v, \varphi w) = -F(v, w, u) + F(w, u, v) - 2F(\varphi u, \varphi v, w), \tag{2.9}$$

$$\theta(\xi) = 0, \theta^*(\xi) = 0 \tag{2.10}$$

or equivalently,

$$F(u, v, w) = F(u, v, \xi)\eta(w) + F(u, w, \xi)\eta(v),$$

$$F(u, v, \xi) = F(v, u, \xi) = -F(\varphi u, \varphi v, \xi), \quad \theta = \theta^* = 0$$

$$\mathcal{F}_7: \; F(u, v, w) = -F(\varphi u, \varphi v, w) - F(\varphi u, v, \varphi w) = -F(v, w, u) - F(w, u, v) \tag{2.11}$$

or equivalently,

$$\begin{aligned} & F(u, v, w) = F(u, v, \xi)\eta(w) + F(u, w, \xi)\eta(v), \\ & F(u, v, \xi) = -F(v, u, \xi) = -F(\varphi u, \varphi v, \xi) \end{aligned} \tag{2.12}$$

$$\mathcal{F}_8: \ F(u,v,w) = F(\varphi u,\varphi v,w) + F(\varphi u,v,\varphi w) = -F(v,w,u) + F(w,u,v) + 2F(\varphi u,\varphi v,w)$$

or equivalently,

$$F(u,v,w) = F(u,v,\xi)\eta(w) + F(u,w,\xi)\eta(v), \quad F(u,v,\xi) = F(v,u,\xi) = F(\varphi u,\varphi v,\xi) \tag{2.13}$$

$$\mathcal{F}_9: \ F(u,v,w) = F(\varphi u,\varphi v,w) + F(\varphi u,v,\varphi w) = -F(v,w,u) - F(w,u,v) \tag{2.14}$$

or equivalently,

$$F(u,v,w) = F(u,v,\xi)\eta(w) + F(u,w,\xi)\eta(v), \quad F(u,v,\xi) = -F(v,u,\xi) = F(\varphi u,\varphi v,\xi) \tag{2.15}$$

$$\mathcal{F}_{10}: \ F(u,v,w) = \eta(u)F(\xi,\varphi v,\varphi w) \tag{2.16}$$

$$\mathcal{F}_{11}: \ F(u,v,w) = \eta(u)\{\eta(v)\omega(w) + \eta(w)\omega(v)\}. \tag{2.17}$$

Note that, (2.6) holds for all $\mathcal{F}_i$.

Projections F^i onto each subspace $\mathcal{F}_i$ are obtained in [6]. We will write the projections in the context when needed.

An almost contact manifold with B-metric is said to be in the class $\mathcal{F}_i \oplus \mathcal{F}_j$, etc if the tensor F is in the class $\mathcal{F}_i \oplus \mathcal{F}_j$ over T_pM for all $p \in M$.

An almost contact manifold with B-metric whose characteristic vector field ξ is Killing, that is, ξ has the property that $g(\nabla_x\xi, y) + g(\nabla_y\xi, x) = 0$ for any vector fields x, y, is called a K-contact B-metric manifold. The class of K-contact B-metric manifolds is $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_7 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10}$ for manifolds of any dimension [5].

Normal structures with B-metric are those in $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6$ and for normal structures we have $d\eta = 0$ [7,8].

If (M,φ,ξ,η,g) is an almost contact manifold with B-metric and t is a positive constant, then the deformation introduced in [1]

$$\tilde{\eta} = t\eta, \quad \tilde{\xi} = \frac{1}{t}\xi, \quad \tilde{\varphi} = \varphi, \quad \tilde{g} = -tg + t(t+1)\eta\otimes\eta \tag{2.18}$$

is called a $\mathcal{D}$-homothetic deformation and also gives an almost contact B-metric structure on M. The tensor $\tilde{F}$ of the deformed structure is

$$\tilde{F}(u,v,w) = -tF(u,v,w) + \frac{t(t+1)}{2}\{d\eta(\varphi v,w)\eta(u) - d\eta(v,\varphi w)\eta(u) - d\eta(u,\varphi v)\eta(w) - d\eta(u,\varphi w)\eta(v)\}, \tag{2.19}$$

where

$$d\eta(u,v) = g(\nabla_u\xi, v) - g(\nabla_v\xi, u) = F(u,\varphi v,\xi) - F(v,\varphi u,\xi) \tag{2.20}$$

3. Deformations of Basic Classes

(M,φ,ξ,η,g) denotes an almost contact B-metric manifold and $(M,\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ is the new almost contact B-metric manifold obtained by deforming the metric by (2.18). In [2] it was shown that the classes $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_4, \mathcal{F}_5$ are invariant under $\mathcal{D}$-homothetic deformations. Note that since $\tilde{F} = -tF$ for these classes, $\tilde{\theta} = \theta$ and $\tilde{\theta}^* = \theta^*$ hold for Lee forms of the first and deformed structures, compare with Theorem 2.1 and equation (2.7) in [2] which also implies the same results. In this section, we show that all other basic classes but $\mathcal{F}_8$ remain same after $\mathcal{D}$-homothetic deformations. We use defining relation of basic classes and properties of almost contact B-metric structures.

Theorem 3.1. *Let (M,φ,ξ,η,g) be in the class $\mathcal{F}_i$ for $i \neq 0,1,4,5,8$. Then, the deformed manifold is also in the same class $\mathcal{F}_i$.*

Proof. The class $\mathcal{F}_2$: Let (M,φ,ξ,η,g) be in the class $\mathcal{F}_2$. Then, the defining relation (2.7) is satisfied. We show that $(M,\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ also has the property (2.7). Note that, $\mathcal{F}_2$ is normal and $d\eta = 0$ [7,8]. From the equation (2.19), it is clear that $\tilde{F}(u,v,w) = -tF(u,v,w)$. Thus,

$$\tilde{F}(\tilde{\xi},v,w) = -tF(\frac{1}{t}\xi,v,w) = 0 \ \text{ and } \ \tilde{F}(u,\tilde{\xi},w) = 0 \tag{3.1}$$

and

$$\tilde{F}(u,v,\tilde{\varphi}w)+\tilde{F}(v,w,\tilde{\varphi}u)+\tilde{F}(w,u,\tilde{\varphi}v)=-t\{F(u,v,\varphi w)+F(v,w,\varphi u)+F(w,u,\varphi v)\}=0$$

by (2.18) and (2.19). Now, we evaluate $\tilde{\theta}$. Let $\mathcal{B}=\{\xi,e_1,\ldots,e_n,f_1,\ldots,f_n\}$ be a g-orthonormal ordered basis such that $g(\xi,\xi)=g(e_i,e_i)=-g(f_i,f_i)=1$. Then,
$\tilde{\mathcal{B}}=\{\tilde{\xi},\tilde{e}_1,\ldots,\tilde{e}_{2n}\}=\{\tilde{\xi},\frac{1}{\sqrt{t}}f_1,\ldots,\frac{1}{\sqrt{t}}f_n,\frac{1}{\sqrt{t}}e_1,\ldots,\frac{1}{\sqrt{t}}e_n\}$ is a $\tilde{g}$-orthonormal ordered basis such that $g_{ij}=g^{ij}=\tilde{g}_{ij}=\tilde{g}^{ij}$. Since

$$\begin{aligned}\theta(u)=g^{ij}F(e_i,e_j,u)&=F(\xi,\xi,u)+F(e_1,e_1,u)+\ldots+F(e_n,e_n,u)\\&-F(f_1,f_1,u)-F(f_2,f_2,u)-\ldots-F(f_n,f_n,u)\\&=0,\end{aligned}$$

and $F(\xi,\xi,u)=0$ from (2.7), we have

$$\begin{aligned}\tilde{\theta}(u)=\tilde{g}^{ij}\tilde{F}(\tilde{e}_i,\tilde{e}_j,u)&=-\frac{1}{t}F(\xi,\xi,u)+F(e_1,e_1,u)+\ldots+F(e_n,e_n,u)\\&-F(f_1,f_1,u)-F(f_2,f_2,u)-\ldots-F(f_n,f_n,u)\\&=\theta(u)=0.\end{aligned}\tag{3.2}$$

By (3.1), (3.2) and (3.2), $(M,\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ is also in $\mathcal{F}_2$.
The class $\mathcal{F}_3$: Since ξ is Killing in $\mathcal{F}_3$ [5], (2.20) yields

$$\begin{aligned}d\eta(u,v)&=2F(u,\varphi v,\xi)\\&=2F(u,\xi,\varphi v),\end{aligned}\tag{3.3}$$

and $d\eta(u,v)=2F(u,\xi,\varphi v)=0$ by (2.8). Then, (2.19) gives $\tilde{F}=-tF$ and thus,

$$\begin{aligned}&\tilde{F}(\tilde{\xi},v,w)=\tilde{F}(u,\tilde{\xi},w)=0,\\&\tilde{F}(u,v,w)+\tilde{F}(v,w,u)+\tilde{F}(w,u,v)=0.\end{aligned}$$

The class $\mathcal{F}_6$: Note that $\mathcal{F}_6$ is normal and $d\eta=0$ [7]. From (2.19),

$$\tilde{F}(u,v,w)+\tilde{F}(\tilde{\varphi}u,\tilde{\varphi}v,w)+\tilde{F}(\tilde{\varphi}u,v,\tilde{\varphi}w)=-tF(u,v,w)-tF(\varphi u,\varphi v,w)-tF(\varphi u,v,\varphi w=0$$

and

$$\tilde{F}(u,v,w)+\tilde{F}(v,w,u)-\tilde{F}(w,u,v)+2\tilde{F}(\tilde{\varphi}u,\tilde{\varphi}v,w)=0.$$

Similar to the proof of (3.2), it can be obtained that

$$\tilde{\theta}^*=\theta^*$$

and thus,

$$\tilde{\theta}(\tilde{\xi})=\frac{1}{t}\theta(\xi)=0,\quad \tilde{\theta}^*(\tilde{\xi})=\frac{1}{t}\theta^*(\xi)=0.$$

As a result the deformed structure satisfies defining relations (2.9) and (2.10).

The class $\mathcal{F}_7$: ξ is Killing and thus, $d\eta(u,v) = 2F(u,\varphi v,\xi)$ by (2.20). Then, by using (2.11), (2.12), (2.19) and (2.20), we get

$$\begin{aligned}\tilde{F}(u,v,w)+\tilde{F}(\tilde{\varphi}u,\tilde{\varphi}v,w)+\tilde{F}(\tilde{\varphi}u,v,\tilde{\varphi}w) &= -tF(u,v,w)+\frac{t(t+1)}{2}\{d\eta(\varphi v,w)\eta(u)-d\eta(v,\varphi w)\eta(u)\\ &- d\eta(u,\varphi v)\eta(w)-d\eta(u,\varphi w)\eta(y)\}\\ &- tF(\varphi u,\varphi v,w)+\frac{t(t+1)}{2}\{-d\eta(\varphi u,\varphi^2 v)\eta(w)\}\\ &- tF(\varphi u,v,\varphi w)+\frac{t(t+1)}{2}\{-d\eta(\varphi u,\varphi^2 w)\eta(v)\}\\ &= \frac{t(t+1)}{2}\{d\eta(\varphi v,w)\eta(u)-d\eta(v,\varphi w)\eta(u)\\ &- d\eta(u,\varphi v)\eta(w)-d\eta(u,\varphi w)\eta(v)\\ &- d\eta(\varphi u,\varphi^2 v)\eta(w)-d\eta(\varphi u,\varphi^2 w)\eta(v)\}\\ &= t(t+1)\{\eta(u)F(\varphi v,\varphi w,\xi)-\eta(u)F(v,\varphi^2 w,\xi)\\ &- \eta(w)F(u,\varphi^2 v,\xi)-\eta(v)F(u,\varphi^2 w,\xi)\\ &- \eta(w)F(\varphi u,\varphi(\varphi^2 v),\xi)-\eta(v)F(\varphi u,\varphi(\varphi^2 w),\xi)\}\\ &= t(t+1)\{-\eta(u)F(v,w,\xi)+\eta(u)F(v,w,\xi)+\eta(w)F(u,v,\xi)\\ &+ \eta(v)F(u,w,\xi)-\eta(w)F(u,v,\xi)-\eta(v)F(u,w,\xi)\\ &= 0.\end{aligned}$$

Similarly,

$$\tilde{F}(u,v,w)+\tilde{F}(v,w,u)+\tilde{F}(w,u,v)=0.$$

The class $\mathcal{F}_9$: By (2.20) and (2.15), we have

$$\begin{aligned}d\eta(v,\varphi w) &= F(v,\varphi^2 w,\xi)-F(\varphi w,\varphi v,\xi)\\ &= F(v,-w+\eta(w)\xi,\xi)+F(\varphi v,\varphi w,\xi)\\ &= -F(v,w,\xi)+F(v,w,\xi)\\ &= 0.\end{aligned}$$

Then,

$$\tilde{F}(u,v,w)+\tilde{F}(v,w,u)+\tilde{F}(w,u,v) = -t\{F(u,v,w)+F(v,w,u)+F(w,u,v)\} = 0$$

and

$$\tilde{F}(u,v,w)-\tilde{F}(\tilde{\varphi}u,\tilde{\varphi}v,w)-\tilde{F}(\tilde{\varphi}u,v,\tilde{\varphi}w) = -t\{F(u,v,w)-F(\varphi u,\varphi v,w)-F(\varphi u,v,\varphi w)\} = 0,$$

that is the deformed structure satisfies (2.14).

The class $\mathcal{F}_{10}$: From (2.16)

$$\begin{aligned}d\eta(u,v) &= F(u,\varphi v,\xi)-F(v,\varphi u,\xi)\\ &= \eta(u)F(\xi,\varphi^2 v,\varphi\xi)-\eta(v)F(\xi,\varphi^2 u,\varphi\xi) = 0.\end{aligned}$$

Thus $\tilde{F}(u,v,w) = -tF(u,v,w)$ and

$$\begin{aligned}\tilde{\eta}(u)\tilde{F}(\xi,\tilde{\varphi}v,\tilde{\varphi}w) &= t\eta(u)\{-tF(\frac{1}{t}\xi,\varphi v,\varphi w)\} = -t\eta(u)F(\xi,\varphi v,\varphi w)\\ &= -tF(u,v,w) = \tilde{F}(u,v,w).\end{aligned}$$

The class $\mathcal{F}_{11}$: From the equation (2.17), we get

$$F(u,\varphi v,\xi) = \eta(u)\{\eta(\varphi))\omega(\xi)+\eta(\xi)\omega(\varphi v)\} = \eta(u)F(\xi,\xi,\varphi v)$$

and

$$d\eta(u,v) = F(u,\varphi v,\xi)-F(v,\varphi u,\xi) = \eta(u)F(\xi,\xi,\varphi v)-\eta(v)F(\xi,\xi,\varphi u) \tag{3.4}$$

and thus,

$$\begin{aligned}
d\eta(\varphi v, w) &= \eta(\varphi v)F(\xi,\xi,\varphi w) - \eta(w)F(\xi,\xi,\varphi^2 v) = \eta(w)F(\xi,\xi,v),\\
d\eta(v,\varphi w) &= -\eta(v)F(\xi,\xi,w),\\
d\eta(u,\varphi v) &= -\eta(u)F(\xi,\xi,v),\\
d\eta(u,\varphi w) &= -\eta(u)F(\xi,\xi,w).
\end{aligned}$$

Then,

$$\begin{aligned}
\tilde{F}(u,v,w) &= -tF(u,v,w) + \frac{t(t+1)}{2}\{d\eta(\varphi v,w)\eta(u) - d\eta(v,\varphi w)\eta(u)\\
&\quad - d\eta(u,\varphi v)\eta(w) - d\eta(u,\varphi w)\eta(v)\}\\
&= -tF(u,v,w) + t(t+1)\{\eta(u)\eta(w)F(\xi,\xi,v) + \eta(u)\eta(v)F(\xi,\xi,w)\}\\
&= t^2F(u,v,w).
\end{aligned}$$

Since (2.17) holds, we have

$$F(u,v,w) = \eta(u)\eta(w)F(\xi,\xi,v) + \eta(u)\eta(v)F(\xi,\xi,w).$$

On the other hand, since F is in $\mathcal{F}_{11}$, the equation (3.4) implies

$$d\eta(\xi,\varphi w) = -F(\xi,\xi,w)$$

and thus,

$$\begin{aligned}
\tilde{F}(\xi,\xi,w) &= -tF(\xi,\xi,w) + \frac{t(t+1)}{2}\{d\eta(\varphi\xi,w)\eta(\xi) - d\eta(\xi,\varphi w)\eta(\xi)\\
&\quad - d\eta(\xi,\varphi\xi)\eta(w) - d\eta(\xi,\varphi w)\eta(\xi)\}\\
&= -tF(\xi,\xi,w) + t(t+1)\{-d\eta(\xi,\varphi w)\}\\
&= -tF(\xi,\xi,w) + t(t+1)F(\xi,\xi,w)\\
&= t^2F(\xi,\xi,w).
\end{aligned}$$

As a result,

$$\begin{aligned}
\tilde{\eta}(u)\{\tilde{\eta}(v)\tilde{\omega}(w) + \tilde{\eta}(w)\tilde{\omega}(v)\} &= t\eta(u)\{t\eta(v)\tilde{F}(\tilde{\xi},\tilde{\xi},w) + t\eta(w)\tilde{F}(\tilde{\xi},\tilde{\xi},v)\}\\
&= t\eta(u)\{t\eta(v)\tilde{F}(\frac{1}{t}\xi,\frac{1}{t}\xi,w) + t\eta(w)\tilde{F}(\frac{1}{t}\xi,\frac{1}{t}\xi,v)\}\\
&= t^2\eta(u)\{\eta(v)\omega(w) + \eta(w)\omega(v)\}\\
&= t^2F(u,v,w)\\
&= \tilde{F}(u,v,w).
\end{aligned}$$

□

The only class which is not invariant under a $\mathcal{D}$-homothetic deformation is $\mathcal{F}_8$.

Theorem 3.2. *Assume that (M,φ,ξ,η,g) belongs to $\mathcal{F}_8$. Then, $(M,\tilde{\varphi},\tilde{\xi},\tilde{\eta},\tilde{g})$ is not in $\mathcal{F}_8$.*

Proof. Since ξ is Killing [5], the equation (3.3) holds. So,
$d\eta(\varphi v,\xi) = 2F(v,\varphi\xi,\xi) = 0$, $d\eta(\varphi v,w) = 2F(\varphi v,\varphi w,\xi) = 2F(v,w,\xi)$ and $d\eta(v,\varphi w) = -2F(v,w,\xi)$. Then,

$$\begin{aligned}
\tilde{F}(u,v,w) - \tilde{F}(u,v,\tilde{\xi})\tilde{\eta}(w) - \tilde{F}(u,w,\tilde{\xi})\tilde{\eta}(v) &= -tF(u,v,w) + \frac{t(t+1)}{2}\{d\eta(\varphi v,w)\eta(u) - d\eta(v,\varphi w)\eta(u)\\
&\quad - d\eta(u,\varphi v)\eta(w) - d\eta(u,\varphi w)\eta(v)\}\\
&\quad - \eta(w)\{-tF(u,v,\xi) + \frac{t(t+1)}{2}\{d\eta(\varphi v,\xi)\eta(u) - d\eta(u,\varphi v)\}\}\\
&\quad - \eta(v)\{-tF(u,w,\xi) + \frac{t(t+1)}{2}\{d\eta(\varphi w,\xi)\eta(u) - d\eta(u,\varphi w)\}\}\\
&= \frac{t(t+1)}{2}\{d\eta(\varphi v,w)\eta(u) - d\eta(v,\varphi w)\eta(u)\}\\
&= 2t(t+1)F(v,w,\xi),
\end{aligned}$$

which is nonzero, since if $F(v, w, \xi) = 0$, we have $F = 0$ by (2.13). Thus, $\tilde{F}$ does not satisfy the defining relation (2.13). □

4. Deformations of Normal Structures

Assume that $(M, \varphi, \xi, \eta, g)$ is almost contact with B-metric. Then, the structure is normal if and only if

$$F(u, v, \xi) = F(v, u, \xi), \tag{4.1}$$

$$\sum_{\text{cyc}} \{F(u, v, \varphi w) - F(u, \varphi v, \xi)\eta(w)\} = 0, \tag{4.2}$$

where $\sum_{\text{cyc}}$ denotes the cyclic sum over u, v, w, see [12].

Theorem 4.1. *Let $(M, \varphi, \xi, \eta, g)$ be normal. Then, the deformed manifold is also normal.*

Proof. Since $d\eta = 0$ for normal structures,

$$\tilde{F}(u, v, \tilde{\xi}) = -tF(u, v, \frac{1}{t}\xi) = -F(u, v, \xi) = -F(v, u, \xi) = \tilde{F}(v, u, \tilde{\xi})$$

and

$$\tilde{F}(u, v, \tilde{\varphi}w) - \tilde{F}(u, \tilde{\varphi}v, \tilde{\xi})\tilde{\eta}(w) = -tF(u, v, \varphi w) + tF(u, \varphi v, \xi)\eta(w),$$

which implies

$$\sum_{\text{cyc}} \{\tilde{F}(u, v, \tilde{\varphi}w) - \tilde{F}(u, \tilde{\varphi}v, \tilde{\xi})\tilde{\eta}(w)\} = -t \sum_{\text{cyc}} \{F(u, v, \varphi w) - F(u, \varphi v, \xi)\eta(w)\} = 0.$$

Thus, the deformed structure satisfies (4.1) and (4.2). □

In [1], the following equation is obtained for the covariant derivative of $\tilde{g}$:

$$\begin{aligned} \tilde{g}(\tilde{\nabla}_u v, w) = -tg(\nabla_u v, w) + \frac{t(t+1)}{2}\{&2u[\eta(v)]\eta(w) \\ &+ \eta(u)d\eta(v, w) + \eta(v)d\eta(u, w) - \eta(w)d\eta(u, v)\}. \end{aligned} \tag{4.3}$$

We write the new covariant derivative for normal structures explicitly. Since $d\eta = 0$, from the equation (4.3), we have

$$-tg(\tilde{\nabla}_u v, w) + t(t+1)\eta(\tilde{\nabla}_u v)\eta(w) = -tg(\nabla_u v, w) + t(t+1)g(u[\eta(v)]\xi, w).$$

Also by (4.3), we get

$$\begin{aligned} \eta(\tilde{\nabla}_u v) &= \frac{1}{t}\tilde{\eta}(\tilde{\nabla}_u v) = \frac{1}{t^2}\tilde{g}(\tilde{\nabla}_u v, \xi) \\ &= -\frac{1}{t}g(\nabla_u v, \xi) + \frac{(t+1)}{t}u[\eta(v)]. \end{aligned}$$

Therefore,

$$\tilde{\nabla}_u v = \nabla_u v + \frac{(t+1)}{t}g(\nabla_u \xi, v)\xi. \tag{4.4}$$

Note that, to obtain (4.4), we only use the property that $d\eta = 0$, so for any deformation of an almost contact B-metric structure with $d\eta = 0$, the new covariant derivative is the same as (4.4).

Although normal structures remain invariant under $\mathcal{D}$-homothetic deformations, they can be very different with regard to curvature properties.

For a normal structure (or more generally when $d\eta = 0$), we calculate the curvature tensor $\tilde{R}$:

$$\begin{aligned} \tilde{R}(u, v)w &= \tilde{\nabla}_u \tilde{\nabla}_v w - \tilde{\nabla}_v \tilde{\nabla}_u w - \tilde{\nabla}_{[u,v]} w \\ &= R(u, v)w + \frac{(t+1)}{t}g(R(u, v)\xi, w)\xi \\ &+ \frac{(t+1)}{t}g(\nabla_v \xi, w)\nabla_u \xi - \frac{(t+1)}{t}g(\nabla_u \xi, w)\nabla_v \xi. \end{aligned}$$

5. Deformations of K-contact Structures

For an almost contact B-metric structure (φ, ξ, η, g), ξ is said to be torse-forming if one has $\nabla_u \xi = -f\varphi^2 u$ and the manifold is in $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_{10}$, see [4]. Then,

$$\begin{aligned}\tilde{\nabla}_u v &= \nabla_u v + \frac{(t+1)}{t} g(-f\varphi^2 u, v)\xi \\ &= \nabla_u v - f\frac{(t+1)}{t} g(\varphi u, \varphi v)\xi,\end{aligned}$$

which implies

$$\tilde{\nabla}_u \tilde{\xi} = -\frac{f}{t}\varphi^2 u = -\tilde{f}\tilde{\varphi}^2 u,$$

where $\tilde{f} = \frac{f}{t}$. Thus, for a torse-forming vector field ξ, $\tilde{\xi}$ also is. In addition, we have

$$\begin{aligned}\tilde{R}(u, v)w = R(u, v)w &+ \frac{(t+1)}{t} g(R(u, v)\xi, w)\xi \\ &+ \frac{(t+1)}{t}\{g(\varphi v, \varphi w)(-f\varphi^2 u) - g(\varphi u, \varphi w)(-f\varphi^2 v)\}.\end{aligned}$$

Let the vector field ξ be Killing for an almost contact B-metric manifold $(M, \varphi, \xi, \eta, g)$. In this case, we write the covariant derivative by Kozsul's formula:

$$\begin{aligned}2\tilde{g}(\tilde{\nabla}_u v, w) &= -2tg(\nabla_u v, w) + t(t+1)\{2u[\eta(v)]\eta(w) \\ &\quad +\eta(u)d\eta(v, w) + \eta(v)d\eta(u, w) - \eta(w)d\eta(u, v)\} \\ &= -2tg(\nabla_u v, w) \\ &\quad + t(t+1)\{2\eta(w)g(\nabla_u \xi, v) + 2\eta(w)g(\xi, \nabla_u v) \\ &\quad +2\eta(u)g(\nabla_v \xi, w) + 2\eta(v)g(\nabla_u \xi, w) - 2\eta(w)g(\nabla_u \xi, v)\}\end{aligned}$$

implies that

$$\begin{aligned}\tilde{g}(\tilde{\nabla}_u v, w) = g\,(-t\nabla_u v + t(t+1)\,\{&g(\nabla_u \xi, v)\xi \\ &+ g(\nabla_u v, \xi)\xi + \eta(u)\nabla_v \xi + \eta(v)\nabla_u \xi \\ &-g(\nabla_u \xi, v)\xi\}\,, w)\,.\end{aligned} \tag{5.1}$$

On the other hand, from the definition of $\tilde{g}$, we have

$$\tilde{g}(\tilde{\nabla}_u v, w) = -tg(\tilde{\nabla}_u v, w) + t(t+1)\eta(\tilde{\nabla}_u v)\eta(w) \tag{5.2}$$

and since

$$\eta(\tilde{\nabla}_u v) = \frac{1}{t^2}\tilde{g}(\tilde{\nabla}_u v, \xi),$$

from equation (5.1), we get

$$\eta(\tilde{\nabla}_u v) = g(\nabla_u v, \xi).$$

Arranging (5.2), we have

$$\tilde{g}(\tilde{\nabla}_u v, w) = g(-t\tilde{\nabla}_u v + t(t+1)g(\nabla_u v, \xi)\xi, w). \tag{5.3}$$

Since g is non-degenerate, comparing (5.1) and (5.3) implies

$$\tilde{\nabla}_u v = \nabla_u v - \frac{(t+1)}{t}\{\eta(u)\nabla_v \xi + \eta(v)\nabla_u \xi\}. \tag{5.4}$$

Then, we state:

Theorem 5.1. *If ξ is Killing with respect to the metric g, then $\tilde{\xi}$ is also Killing with respect to the deformed metric $\tilde{g}$.*

Proof.

$$\tilde{g}(\tilde{\nabla}_u \tilde{\xi}, v) = -\tilde{g}(\tilde{\nabla}_v \tilde{\xi}, u)$$

follows from (5.1). □

Thus, the class of K-contact B-metric strucutures is invariant under $\mathcal{D}$-homothetic deformations.

Now we write the curvature tensor of the new structure after a $\mathcal{D}$-homothetic deformation when ξ is Killing by (5.4).

$$\begin{aligned}\tilde{R}(u,v)w = R(u,v)w &- \frac{(t+1)}{t}\{\eta(u)\nabla_{\nabla_v w}\xi - \eta(v)\nabla_{\nabla_u w}\xi\}\\ &+ \frac{(t+1)^2}{t^2}\left\{\eta(w)\eta(u)\nabla_{\nabla_v \xi}\xi - \eta(w)\eta(v)\nabla_{\nabla_u \xi}\xi\right\}\\ &- \frac{(t+1)}{t}\eta(w)R(u,v)\xi - \frac{(t+1)}{t}\{\eta(v)\nabla_u\nabla_w\xi - \eta(u)\nabla_v\nabla_w\xi\}\\ &+ \frac{(t+1)}{t}\{g(\nabla_v\xi,w)\nabla_u\xi - g(\nabla_u\xi,w)\nabla_v\xi - 2g(\nabla_u\xi,v)\nabla_w\xi\}.\end{aligned}$$

6. Examples

We investigate the classes of deformed structures with B-metric.

Example 6.1. Consider the Lie algebra $\mathfrak{l}$ of a 3-dimensional real connected Lie group L. Let $\{e_0, e_1, e_2\}$ be a basis for $\mathfrak{l}$. The triple (φ, ξ, η) given by

$$\varphi e_0 = 0,\ \ \varphi e_1 = e_2,\ \ \varphi e_2 = -e_1,\ \ \xi = e_0$$
$$\eta(e_0) = 1,\ \ \eta(e_1) = \eta(e_2) = 0$$

is an almost contact structure on $\mathfrak{l}$. Together with the compatible semi-Riemannian metric g satisfying

$$g(e_0,e_0) = g(e_1,e_1) = -g(e_2,e_2) = 1,$$
$$g(e_0,e_1) = g(e_0,e_2) = g(e_1,e_2) = 0,$$

$(L, \varphi, \xi, \eta, g)$ is an almost contact B-metric manifold in $\mathcal{F}_8$ if and only if brackets of the Lie algebra $\mathfrak{l}$ are

$$[e_0,e_1] = \alpha e_2,\ \ [e_0,e_2] = \alpha e_1,\ \ [e_1,e_2] = -2\alpha e_0.$$

Nonzero covariant derivatives are

$$\nabla_{e_1}e_2 = -\nabla_{e_2}e_1 = -\alpha e_0,\ \ \nabla_{e_1}e_0 = -\alpha e_2,\ \ \nabla_{e_2}e_0 = -\alpha e_1,$$

see [5].

Consider a $\mathcal{D}$-homothetic deformation $(L, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ of $(L, \varphi, \xi, \eta, g)$. We determine the class of the deformed structure by evaluating projections $\tilde{F}^i$.

Since ξ is Killing, by Theorem 5.1, $\tilde{\xi}$ is also Killing, that is, the deformed structure is also K-contact. The classes of K-contact manifolds in 3 dimensions is $\mathcal{F}_1 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10}$, [5,6], so it is enough to write $\tilde{F}^1$, $\tilde{F}^8$ and $\tilde{F}^{10}$.

Since the structure is in $\mathcal{F}_8$, $F^i = 0$ for $i \neq 8$. We have

$$F^1(u,v,w) = 0$$

and

$$\begin{aligned}F^2(u,v,w) &= -\frac{1}{4}\left\{F(\varphi^2u,\varphi^2v,\varphi^2w) + F(\varphi^2v,\varphi^2w,\varphi^2u) - F(\varphi v,\varphi^2w,\varphi u)\right.\\ &\left.+F(\varphi^2u,\varphi^2w,\varphi^2v) + F(\varphi^2w,\varphi^2v,\varphi^2u) - F(\varphi w,\varphi^2v,\varphi u)\right\}\\ &- F^1(u,v,w)\\ &= 0.\end{aligned}$$

Since the deformed structure is in $\mathcal{F}_1 \oplus \mathcal{F}_8 \oplus \mathcal{F}_{10}$, $\tilde{F}^2 = 0$ and this gives

$$\begin{aligned}\tilde{F}^1(u,v,w) &= -\frac{1}{4}\left\{\tilde{F}(\tilde{\varphi}^2u,\tilde{\varphi}^2v,\tilde{\varphi}^2w) + \tilde{F}(\tilde{\varphi}^2v,\tilde{\varphi}^2w,\tilde{\varphi}^2u) - \tilde{F}(\tilde{\varphi}v,\tilde{\varphi}^2w,\tilde{\varphi}u)\right.\\ &\left.+\tilde{F}(\tilde{\varphi}^2u,\tilde{\varphi}^2w,\tilde{\varphi}^2v) + \tilde{F}(\tilde{\varphi}^2w,\tilde{\varphi}^2v,\tilde{\varphi}^2u) - \tilde{F}(\tilde{\varphi}w,\tilde{\varphi}^2v,\tilde{\varphi}u)\right\}\\ &- \tilde{F}^2(u,v,w)\\ &= -tF^2(u,v,w)\\ &= 0.\end{aligned}$$

$\tilde{F}^8$ can be evaluated in terms of F^8 for any dimension as follows:

$$\begin{aligned}\tilde{F}^8(u,v,w) &= \frac{1}{4}\left\{\tilde{F}(\tilde{\varphi}^2u,\tilde{\varphi}^2v,\tilde{\xi}) + \tilde{F}(\tilde{\varphi}^2v,\tilde{\varphi}^2u,\tilde{\xi}) + \tilde{F}(\tilde{\varphi}u,\tilde{\varphi}v,\tilde{\xi}) + \tilde{F}(\tilde{\varphi}v,\tilde{\varphi}u,\tilde{\xi})\right\}\tilde{\eta}(w)\\ &+ \frac{1}{4}\left\{\tilde{F}(\tilde{\varphi}^2u,\tilde{\varphi}^2w,\tilde{\xi}) + \tilde{F}(\tilde{\varphi}^2w,\tilde{\varphi}^2u,\tilde{\xi}) + \tilde{F}(\tilde{\varphi}u,\tilde{\varphi}w,\tilde{\xi}) + \tilde{F}(\tilde{\varphi}w,\tilde{\varphi}u,\tilde{\xi})\right\}\tilde{\eta}(v)\\ &= -tF^8(u,v,w) + \frac{1}{4}t(t+1)\left\{d\eta(\varphi^2u,\varphi v)\eta(w) + d\eta(\varphi^2v,\varphi u)\eta(w) + d\eta(\varphi^2u,\varphi w)\eta(v) + d\eta(\varphi^2w,\varphi u)\eta(v)\right\}\end{aligned} \tag{6.1}$$

Since $d\eta(u,v) = 2F(u,\varphi v,\xi)$ and $\mathfrak{l} \in \mathcal{F}_8$, we get $d\eta(\varphi^2u,\varphi v) = 2F(\varphi^2u,\varphi^2v,\xi) = 2F(u,v,\xi)$.
Thus, equation (6.1) together with the defining relation of $\mathcal{F}_8$ implies

$$\begin{aligned}\tilde{F}^8(u,v,w) &= -tF^8(u,v,w) + t(t+1)\{F(u,v,\xi)\eta(w) + F(u,w,\xi)\eta(v)\}\\ &= -tF^8(u,v,w) + t(t+1)F(u,v,w)\end{aligned}$$

Since $\mathfrak{l} \in \mathcal{F}_8$, $F = F^8$ and we have $\tilde{F}^8(u,v,w) = -tF^8(u,v,w) + t(t+1)F^8(u,v,w) = t^2F^8(u,v,w) \neq 0$. Next, we calculate $\tilde{F}^{10}$. Since $\mathfrak{l} \in \mathcal{F}_8$,

$$\begin{aligned}\tilde{F}^{10}(u,v,w) &= \tilde{\eta}(u)\tilde{F}(\tilde{\xi},\tilde{\varphi}^2v,\tilde{\varphi}^2w)\\ &= \eta(u)\tilde{F}(\xi,\varphi^2v,\varphi^2w)\\ &= \eta(u)\left\{-tF(\xi,\varphi^2v,\varphi^2w) + \frac{t(t+1)}{2}\left\{d\eta(\varphi v,w)\eta(u) - d\eta(v,\varphi w)\eta(u) - d\eta(u,\varphi v)\eta(w) - d\eta(u,\varphi w)\eta(v)\right\}\right\}\\ &= \frac{t(t+1)}{2}\eta(u)\left\{4\eta(u)F(v,w,\xi) + 2\eta(w)F(u,v,\xi) + 2\eta(v)F(u,w,\xi)\right\}.\end{aligned}$$

For $u = e_0$, $v = e_1$, $w = e_1$, we see that $\tilde{F}^{10}(u,v,w) = \tilde{F}^{10}(e_0,e_1,e_1) = 2t(t+1)\eta(e_0)\eta(e_0)F(e_1,e_1,e_0) = -2\alpha t(t+1) \neq 0$.
To sum up, $l \in F_8 \oplus F_{10}$.

If a manifold belongs to a class which is a direct sum of basic classes, we can check whether it is invariant under a $\mathcal{D}$-homothetic deformation by calculating projections $\tilde{F}^i$. For example by direct calculation,

$$\begin{aligned}\tilde{F}^9(u,v,w) &= \frac{1}{4}\left\{\tilde{F}(\tilde{\varphi}^2u,\tilde{\varphi}^2v,\tilde{\xi}) - \tilde{F}(\tilde{\varphi}^2v,\tilde{\varphi}^2u,\tilde{\xi}) + \tilde{F}(\tilde{\varphi}u,\tilde{\varphi}v,\tilde{\xi}) - \tilde{F}(\tilde{\varphi}v,\tilde{\varphi}u,\tilde{\xi})\right\}\tilde{\eta}(w)\\ &+ \frac{1}{4}\left\{\tilde{F}(\tilde{\varphi}^2u,\tilde{\varphi}^2w,\tilde{\xi}) - \tilde{F}(\tilde{\varphi}^2w,\tilde{\varphi}^2u,\tilde{\xi}) + \tilde{F}(\tilde{\varphi}u,\tilde{\varphi}w,\tilde{\xi}) - \tilde{F}(\tilde{\varphi}w,\tilde{\varphi}u,\tilde{\xi})\right\}\tilde{\eta}(v)\\ &= -tF^9(u,v,w).\end{aligned}$$

Thus, $\tilde{F}^9 = 0 \iff F^9 = 0$.
That is, the deformed structure contains a summand from the subclass F_9 if and only if the first structure also contains a summand from this class. This is not true for each subclass.

Consider for example the projection F^7.

$$\begin{aligned}\tilde{F}^7(u,v,w) &= \frac{1}{4}\left\{\tilde{F}(\tilde{\varphi}^2u,\tilde{\varphi}^2v,\tilde{\xi}) - \tilde{F}(\tilde{\varphi}^2v,\tilde{\varphi}^2u,\tilde{\xi}) - \tilde{F}(\tilde{\varphi}u,\tilde{\varphi}v,\tilde{\xi}) + \tilde{F}(\tilde{\varphi}v,\tilde{\varphi}u,\tilde{\xi})\right\}\tilde{\eta}(w)\\ &+ \frac{1}{4}\left\{\tilde{F}(\tilde{\varphi}^2u,\tilde{\varphi}^2w,\tilde{\xi}) - \tilde{F}(\tilde{\varphi}^2w,\tilde{\varphi}^2u,\tilde{\xi}) - \tilde{F}(\tilde{\varphi}u,\tilde{\varphi}w,\tilde{\xi}) + \tilde{F}(\tilde{\varphi}w,\tilde{\varphi}u,\tilde{\xi})\right\}\tilde{\eta}(v)\\ &= -tF^7(u,v,w) + \frac{t(t+1)}{4}\left\{\eta(w)\{d\eta(\varphi^2u,\varphi v) + d\eta(\varphi u,\varphi^2v)\}\right.\\ &+ \left.\eta(v)\{d\eta(\varphi^2u,\varphi w) + d\eta(\varphi u,\varphi^2w)\}\right\}\end{aligned}$$

and the terms

$$\eta(w)\{d\eta(\varphi^2u,\varphi v) + d\eta(\varphi u,\varphi^2v)\} + \eta(v)\{d\eta(\varphi^2u,\varphi w) + d\eta(\varphi u,\varphi^2w)\}$$

may give summands from other subclasses.

Example 6.2. Assume that $(M, \varphi, \xi, \eta, g)$ is a three-dimensional almost contact B-metric manifold. Consider a basis $\{e_0 = \xi, e_1 = e, e_2 = \varphi(e)\}$ of T_pM satisfying

$$g(e_0, e_0) = g(e_1, e_1) = -g(e_2, e_2) = 1, \quad g(e_i, e_j) = 0, i \neq j.$$

Denote the components of F with respect to the given basis by $F_{ijk} = F(e_i, e_j, e_k)$. This example and the projections F^i of the tensor F are given in [6]. We deform this structure and by direct calculations, we get

$$\begin{aligned}
\tilde{F}(u,v,w) &= -tF(u,v,w) + \frac{t(t+1)}{2}\Big\{d\eta(\varphi v, w)\eta(u) - d\eta(v, \varphi w)\eta(u) - d\eta(u, \varphi v)\eta(w) - d\eta(u, \varphi w)\eta(v)\Big\} \\
&= -tF(u,v,w) + \frac{t(t+1)}{2}\{u_0v_1w_0F_{010} + u_0v_2w_0F_{020} + u_0v_0w_1F_{010} \\
&+ u_0v_0w_2F_{020} + u_1v_1w_0F_{220} + u_2v_2w_0F_{110} \\
&+ u_0v_1w_0F_{010} + u_0v_2w_0F_{020} + u_1v_1w_0F_{110} \\
&+ u_2v_2w_0F_{220} + u_1v_0w_1F_{220} + u_2v_0w_2F_{110} \\
&+u_0v_0w_1F_{010} + u_0v_0w_2F_{020} + u_1v_0w_1F_{110} + u_2v_0w_2F_{220}\} \\
&= -tF(u,v,w) + \frac{t(t+1)}{2}\Big\{F^4(u,v,w) + F^8(u,v,w) + 2F^{11}(u,v,w)\Big\},
\end{aligned}$$

where $u = \sum u_ie_i$, $v = \sum v_ie_i$ and $w = \sum w_ie_i$ are vectors in T_pM. This result is in accordance with our results. For example, if $M \in \mathcal{F}_5$, then $F^i = 0$ for $i \neq 2$. Since $F^4 = F^8 = F^{11} = 0$, $\tilde{F} = -tF = F^2$ and M remains in the same class.

Conflicts of Interest

The author declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the manuscript.

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