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ON DARBOUX FRAMES OF INDICATRICES OF SPACELIKE SALKOWSKI CURVE WITH SPACELIKE BINORMAL IN E_1^3

Birkan AKSAN[1], Sümeyye GÜR MAZLUM[2]*

[1]Gümüşhane University, Institute of Graduate Education, Department of Mathematical Engineering, 29100, Gümüşhane, Türkiye
[2]Gümüşhane University, Kelkit Aydın Doğan Vocational School, Department of Computer Technology, 29600, Gümüşhane, Türkiye

Abstract: The aim of this study is to examine Darboux frames and some other geometric properties (geodesic curvatures, geodesic torsions, normal curvatures, Darboux derivative formulas, Darboux vectors, angles, etc.) of the spherical indicatrices on Lorentzian unit sphere S_1^2 and hyperbolic unit sphere H_0^2 of the spacelike Salkowski curve with spacelike binormal in Lorentzian 3-space E_1^3. In this context, new and interesting results have been obtained for this curve. Thus, relationships between the newly obtained curvatures and torsions and the curvature and torsion of the original curve are given. Moreover, the matrix relationship between the Darboux and Frenet frames of these indicatrices is shown. Finally, the Darboux vectors belong to the Darboux frames and the Darboux vectors belong to the Frenet frames of these curves are compared.

Keywords: Lorentzian space, Salkowski curve, Darboux frame, Spherical indicatrices, Curvatures

***Corresponding author:** Gümüşhane University, Kelkit Aydın Doğan Vocational School, Department of Computer Technology, 29600, Gümüşhane, Türkiye
E mail: sumeyyegur@gumushane.edu.tr (S. GÜR MAZLUM)
Birkan AKSAN https://orcid.org/0000-0002-1533-6557
Sümeyye GÜR MAZLUM https://orcid.org/0000-0003-2471-1627

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1. Introduction

Darboux frame, one of the tools used to study the differential geometry of a curve on a surface, consists of the tangent of the curve, the normal of the surface, and a third vector consisting of vector product of these two vectors. Therefore, the Darboux frame is an element of curves passing through the tangent plane or surface's normal at any point on the surface. Using Darboux frame's vectors, the normal and geodesic curvatures and geodesic torsion of the curve on a surface are obtained. Studies on the Darboux frame of a curve are available in (Uğurlu and Kocayiğit, 1996; Uğurlu and Çalışkan, 2012; Şentürk and Yüce, 2015; Yakıcı et al., 2016; Özdemir, 2020; Li et al., 2023). There are relationships between these elements and the main curvatures of the curve. Another relationship is found between the Frenet and Darboux frames of the curve. In addition, just as the Darboux instantaneous rotation axis belong to the Frenet frame can be calculated, a rotation axis of the Darboux frame can also be calculated (Fenchel, 1951). While drawing a curve in E^3, at every moment t, the Frenet vectors of the curve also form curves on unit sphere S^2. These curves are called spherical indicatrices of the main curve. Geometric features such as curvatures and radii of curvature of the surface on it are examined with the spherical indicatrices. Moreover, Darboux frames of the curves and other geometric elements mentioned above can be calculated, since the spherical indicatrices of any curve are on the surface. Studies on the spherical indicatrices of the curve can be found in sources (Hacısalihoğlu, 1983; Aksan and Gür Mazlum, 2023; Bilici and Çalışkan, 2019; Kula and Yaylı, 2005; Gür and Şenyurt, 2010). The aim of this study is to examine the Darboux frames and some geometric properties (curvatures, torsions, Darboux vectors, angles etc.) of the spherical indicatrices on unit spheres S_1^2 and H_0^2 of the spacelike Salkowski curve with spacelike binormal, which is one of the forms in Lorentzian 3-space of the well-known Salkowski curve (Salkowski, 1909; Monterde 2009) in differential geometry. For basic information and various studies on Lorentzian space, which is one of the current fields in which physicists and geometers frequently work, the sources (O'Neill, 1983; Birman and Nomizu, 1984; Ratcliffe, 1994; Uğurlu and Kocayiğit, 1996; Uğurlu, 1997; Kahveci and Yaylı, 2002; Bükcü and Karacan, 2007; Uğurlu and Çalışkan, 2012; Lopez, 2014; Yüksel et al., 2014; Yakıcı et al., 2016; Babaarslan and Yaylı, 2017; Li et al., 2023) can be examined. In addition, other studies on the Salkowski curves are available in (Gür and Şenyurt, 2010; Ali, 2011; Gür Mazlum et al., 2022; Aksan and Gür Mazlum, 2023).

2. Preliminaries

The inner and vector product functions are defined as respectively:

$$\langle , \rangle : E_1^3 \times E_1^3 \to R, \langle \overrightarrow{Z_1}, \overrightarrow{Z_2} \rangle = Z_{11}Z_{21} + Z_{12}Z_{22} - Z_{13}Z_{23}, \tag{1}$$

$$\wedge : E_1^3 \times E_1^3 \to E_1^3 ,$$
$$\overrightarrow{Z_1} \wedge \overrightarrow{Z_2} = (Z_{13}Z_{22} - Z_{12}Z_{23},\ Z_{11}Z_{23} - Z_{13}Z_{21},\ Z_{11}Z_{22} - Z_{12}Z_{21}), \tag{2}$$

for the vectors $\overrightarrow{Z_1}=(Z_{11}, Z_{12}, Z_{13})\in E_1^3$ and $\overrightarrow{Z_2}=(Z_{21}, Z_{22}, Z_{23})\in E_1^3$. Here, $\langle , \rangle$ is Lorentzian metric. E^3 with the metric is called Lorentzian 3-space and is denoted by E_1^3. $\overrightarrow{Z_1}\in E_1^3$ is spacelike (sl) vector, if $\langle \overrightarrow{Z_1}, \overrightarrow{Z_1}\rangle>0$ or $\overrightarrow{Z_1}=0$, $\overrightarrow{Z_1}\in E_1^3$ is timelike (tl) vector, if $\langle \overrightarrow{Z_1}, \overrightarrow{Z_1}\rangle<0$, $\overrightarrow{Z_1}\in E_1^3$ is lightlike or null vector, if $\langle \overrightarrow{Z_1}, \overrightarrow{Z_1}\rangle=0$ and $\overrightarrow{Z_1}\neq 0$. Besides, $\overrightarrow{Z_1}\in E_1^3$ is future pointing (fp) timelike vector, if $\langle \overrightarrow{Z_1}, \overrightarrow{E}\rangle<0$ or $\overrightarrow{Z_1}$ is past pointing (pp) timelike vector, if $\langle \overrightarrow{Z_1}, \overrightarrow{E}\rangle>0$, where $\overrightarrow{E}=(0, 0, 1)$. The vectors $\overrightarrow{Z_1}, \overrightarrow{Z_2}\in E_1^3$ are Lorentz orthogonal vectors, if $\langle \overrightarrow{Z_1}, \overrightarrow{Z_2}\rangle=0$. Let $\overrightarrow{Z_1}, \overrightarrow{Z_2}\in E_1^3$ be nonzero Lorentz orthogonal vectors in E_1^3, if $\overrightarrow{Z_1}$ is timelike, then $\overrightarrow{Z_2}$ is spacelike, (Ratcliffe, 1994). The norm of $\overrightarrow{Z_1}\in E_1^3$ is $\|\overrightarrow{Z_1}\|=\sqrt{|\langle \overrightarrow{Z_1}, \overrightarrow{Z_1}\rangle|}$. If $\|\overrightarrow{Z_1}\|=1$, $\overrightarrow{Z_1}\in E_1^3$ is a unit vector. The sets

$S_1^2=\{\overrightarrow{Z_1}\in E_1^3 \mid \overrightarrow{Z_1} \text{ is unit spacelike vector}\}$,

$H_0^2=\{\overrightarrow{Z_1}\in E_1^3 \mid \overrightarrow{Z_1} \text{ is unit timelike vector}\}$,

$\Lambda=\{\overrightarrow{Z_1}\in E_1^3 \mid \overrightarrow{Z_1} \text{ is unit lightlike vector}\}$

are Lorentzian and hyperbolic unit spheres, and light cone, respectively. In E_1^3, timelike vectors are located inside the light cone, lightlike vectors are located on the light cone, and spacelike vectors are located outside the light cone, Figure 1.

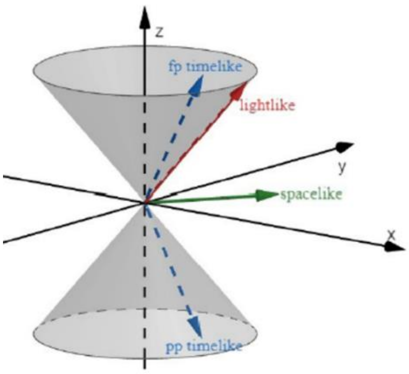

Figure 1. The vectors in E_1^3.

In E_1^3, the angle between two vectors is defined as follows: (O'Neill, 1983; Ratcliffe, 1994).

- If $\overrightarrow{Z_1}, \overrightarrow{Z_2}\in E_1^3$ are future pointing (or past pointing) timelike vectors at the same time, then the hyperbolic angle between the vectors is
$\langle \overrightarrow{Z_1}, \overrightarrow{Z_2}\rangle=-\|\overrightarrow{Z_1}\|\|\overrightarrow{Z_2}\|\cosh\varphi.$

- If $\overrightarrow{Z_1}\in E_1^3$ is future pointing and $\overrightarrow{Z_2}\in E_1^3$ is past pointing timelike vector, then the hyperbolic angle between the vectors is
$\langle \overrightarrow{Z_1}, \overrightarrow{Z_2}\rangle=\|\overrightarrow{Z_1}\|\|\overrightarrow{Z_2}\|\cosh\varphi.$

- If $\overrightarrow{Z_1}, \overrightarrow{Z_2}\in E_1^3$ are spacelike vectors lying in a spacelike plane, then the real angle between the vectors is
$\langle \overrightarrow{Z_1}, \overrightarrow{Z_2}\rangle=\|\overrightarrow{Z_1}\|\|\overrightarrow{Z_2}\|\cos\varphi.$

- If $\overrightarrow{Z_1}, \overrightarrow{Z_2}\in E_1^3$ are spacelike vectors lying in a timelike plane, then the hyperbolic angle between the vectors is
$|\langle \overrightarrow{Z_1}, \overrightarrow{Z_2}\rangle|=\|\overrightarrow{Z_1}\|\|\overrightarrow{Z_2}\|\cosh\varphi.$

- If $\overrightarrow{Z_1}\in E_1^3$ is a spacelike and $\overrightarrow{Z_2}\in E_1^3$ is a timelike vector, then the hyperbolic angle between the vectors is
$|\langle \overrightarrow{Z_1}, \overrightarrow{Z_2}\rangle|=\|\overrightarrow{Z_1}\|\|\overrightarrow{Z_2}\|\sinh\varphi.$

An curve (α) in E_1^3 is timelike, spacelike or lightlike curve, if all of the velocity vector of the curve are the timelike, spacelike or lightlike, respectively. When plotting a spacelike curve with the spacelike binormal $(\vec{\alpha})$ in E_1^3, the endpoints of the spacelike tangent, timelike principal normal and spacelike binormal vectors $\vec{T}$, $\vec{N}$ and $\vec{B}$ of $(\vec{\alpha})$ draw the spacelike indicatrices $(\vec{T})$ and $(\vec{B})$ on the Lorentz unit sphere S_1^2 and the timelike indicatrix $(\vec{N})$ on the hyperbolic unit sphere H_0^2 (Uğurlu and Çalışkan, 2012). A surface in E_1^3 is a spacelike (timelike), if the normal vector field of the surface at every points is timelike (spacelike). Let's assume that the regular spacelike curve $(\vec{\alpha})=\vec{\alpha}(t)$ is on a timelike surface. The Darboux frame of this curve is $\{\vec{T}(t), \vec{g}(t), \vec{n}(t)\}$, where $\vec{n}(t)$ is the spacelike normal vector of the timelike surface and $g(t)=\vec{n}(t)\wedge\vec{T}(t)$ is timelike (Uğurlu and Çalışkan, 2012).

3. On Darboux Frames of Indicatrices of Spacelike Salkowski Curve with Spacelike Binormal in E_1^3

Definition 2.1. For $m\in R$ and $n=\dfrac{m}{\sqrt{m^2-1}}$, the parametric equation of spacelike Salkowski curve with the spacelike binormal in E_1^3 is given as (Ali, 2011):

$$\overrightarrow{\gamma_m}(t)=\Big(2\sin t-\frac{1+n}{1-2n}\sin[(1-2n)t]-\frac{1-n}{1+2n}\sin[(1+2n)t],\; 2\cos t-\frac{1+n}{1-2n}\cos[(1-2n)t]-\frac{1-n}{1+2n}\cos[(1+2n)t],\; \frac{1}{m}\cos(2nt)\Big),$$

where, *m > 1* or *m < -1*, Figure 2. Besides

$$\|\overrightarrow{\gamma_m'}(t)\|=v(t)=\frac{\sin(nt)}{\sqrt{m^2-1}}.$$

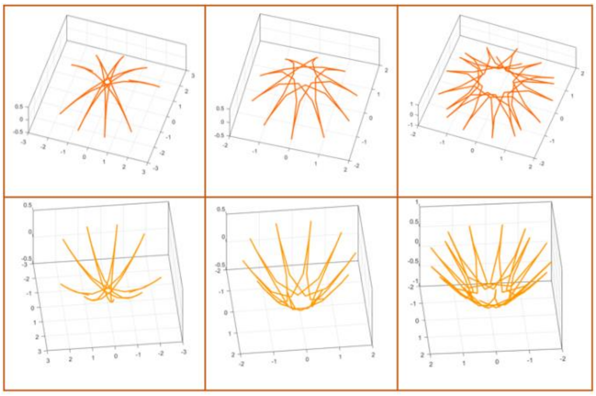

Figure 2. Salkowski curves for $m=-\frac{5}{2},-\frac{5}{4},-\frac{9}{8},\frac{5}{2},\frac{5}{4},\frac{9}{8}$.

The Frenet vectors of $\overrightarrow{\gamma_m}(t)$ are (Ali, 2011)

$$\begin{cases} \vec{T}(t)=\big(\sin(nt)\cos t-n\cos(nt)\sin t, \\ \qquad -\sin(nt)\sin t-n\cos(nt)\cos t,-\frac{n}{m}\cos(nt)\big) \quad (\text{sl}), \\ \vec{N}(t)=\frac{n}{m}(\sin t,\cos t,m) \quad (\text{tl}), \\ \vec{B}(t)=\big(-\cos(nt)\cos t-n\sin(nt)\sin t, \\ \qquad \cos(nt)\sin t-n\sin(nt)\cos t,-\frac{n}{m}\sin(nt)\big) \quad (\text{sl}). \end{cases} \tag{3}$$

The timelike tangent indicatrix $(\vec{T})$ of $\overrightarrow{\gamma_m}(t)$ is a helix on S_1^2, the spacelike principal normal indicatrix $(\vec{N})$ of $\overrightarrow{\gamma_m}(t)$ is a planar circle of radius $\frac{n}{m}$ on H_0^{2+} and the timelike binormal indicatrix $(\vec{B})$ of $\overrightarrow{\gamma_m}(t)$ is a helix on S_1^2 (Aksan and Gür Mazlum, 2023), Figure 3. Besides, for the curves

$$\begin{cases} v_T(t)=\left\|\vec{T}'(t)\right\|=\frac{n}{m}\sin(nt), \\ v_N(t)=\left\|\vec{N}'(t)\right\|=\frac{n}{m}, \\ v_B(t)=\left\|\vec{B}'(t)\right\|=\frac{n}{m}\cos(nt), \end{cases} \tag{4}$$

(Aksan and Gür Mazlum, 2023).

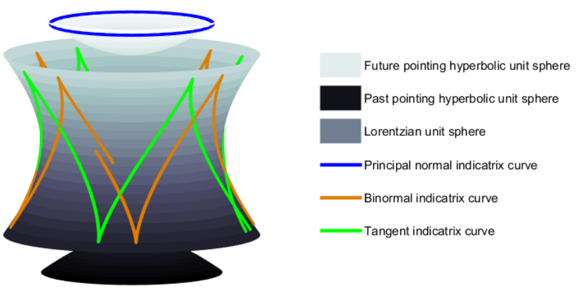

Figure 3. Spherical indicatrices on S_1^2 and H_0^2 of $\overrightarrow{\gamma_m}(t)$.

3.1. The Darboux Frame of the Tangent Indicatrix $(\vec{T})$ of $\overrightarrow{\gamma_m}(t)$

The Frenet frame $\{\overrightarrow{T_T}(t), \overrightarrow{N_T}(t), \overrightarrow{B_T}(t)\}$, curvature $\kappa_T(t)$ and torsion $\tau_T(t)$ of the timelike tangent indicatrix $(\vec{T})$ on S_1^2 of the curve $\overrightarrow{\gamma_m}(t)$in E_1^3 are as follows, respectively (Aksan and Gür Mazlum, 2023):

$$\begin{cases} \overrightarrow{T_T}(t)=\left(\frac{n}{m}\sin t,\ \frac{n}{m}\cos t,\ n\right) \quad (\text{tl}), \\ \overrightarrow{N_T}(t)=(\cos t,\ -\sin t,\ 0) \quad (\text{sl}), \\ \overrightarrow{B_T}(t)=\left(n\sin t,\ n\cos t,\ \frac{n}{m}\right) \quad (\text{sl}), \end{cases} \tag{5}$$
$$\overrightarrow{F_T}(t)=(0,\ 0,\ 1) \quad (\text{tl}),$$

$$\kappa_T(t)=\frac{1}{\sin(nt)} \quad \text{and} \quad \tau_T(t)=\frac{m}{\sin(nt)}. \tag{6}$$

Theorem 3.1. The Darboux frame $\{\overrightarrow{T_T}(t), \overrightarrow{g_T}(t), \overrightarrow{n_T}(t)\}$ of the timelike tangent indicatrix $(\vec{T})$ on S_1^2 is as follows:

$$\begin{cases} \overrightarrow{T_T}(t)=\left(\frac{n}{m}\sin t,\ \frac{n}{m}\cos t,\ n\right) \quad (\text{tl}), \\ \overrightarrow{g_T}(t)=\big(-\cos(nt)\cos t-n\sin(nt)\sin t, \\ \qquad \cos(nt)\sin t-n\sin(nt)\cos t,-\frac{n}{m}\sin(nt)\big) \quad (\text{sl}), \\ \overrightarrow{n_T}(t)=\big(\sin(nt)\cos t-n\cos(nt)\sin t, \\ \qquad -\sin(nt)\sin t-n\cos(nt)\cos t,-\frac{n}{m}\cos(nt)\big) \quad (\text{sl}). \end{cases} \tag{7}$$

Proof: Since $\overrightarrow{T_T}(t)$ is timelike and $\vec{T}(t)$ is spacelike, $\overrightarrow{g_T}(t)=-\vec{T}(t)\wedge\overrightarrow{T_T}(t)$. From (3) and (5), we get the vector $\overrightarrow{g_T}(t)$ in (7). Besides, $\vec{T}(t)$ can be taken as the surface's normal for $(\vec{T})$ on S_1^2, that is $\overrightarrow{n_T}(t)=\vec{T}(t)$ as in (3). Thus, the Darboux frame is obtained as in (7).

Theorem 3.2. The normal curvature $(\kappa_n)_T(t)$ of the tangent indicatrix $(\vec{T})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$(\kappa_n)_T(t)=1 \tag{8}$$

Proof: The normal curvature of $(\vec{T})$ is calculated by

$$(\kappa_n)_T(t)=\frac{\left\langle \overrightarrow{n_T}(t),\ \vec{T}''(t)\right\rangle}{v_T^2(t)}, \tag{9}$$

(Uğurlu and Çalışkan, 2012). From (3), we get

$$\overrightarrow{T''}(t)=\left(\frac{n^2}{m^2}\sin(nt)\cos t+\frac{n^3}{m^2}\cos(nt)\sin t,\right.$$
$$\left.-\frac{n^2}{m^2}\sin(nt)\sin t+\frac{n^3}{m^2}\cos(nt)\cos t,\ \frac{n^3}{m}\cos(nt)\right). \tag{10}$$

From (7) and (10), we have

$$\left\langle \overrightarrow{n_T}(t),\ \overrightarrow{T''}(t)\right\rangle=\frac{n^2}{m^2}\sin^2(nt). \tag{11}$$

If we substitute (4) and (11) in (9), we obtain the normal curvature of $(\vec{T})$ as in (8).

Theorem 3.3. The geodesic curvature $(\kappa_g)_T(t)$ of the tangent indicatrix $(\vec{T})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$is as follows:

$$(\kappa_g)_T(t)=-\frac{\cos(nt)}{\sin(nt)}. \tag{12}$$

Proof: The geodesic curvature $(\kappa_g)_T(t)$ of the tangent indicatrix $(\vec{T})$ is calculated by

$$(\kappa_g)_T(t)=\frac{\left\langle \overrightarrow{g_T}(t),\ \overrightarrow{T''}(t)\right\rangle}{v_T^2(t)}. \tag{13}$$

From (7) and (10), we have

$$\left\langle \overrightarrow{g_T}(t),\ \overrightarrow{T''}(t)\right\rangle=-\frac{n^2}{m^2}\cos(nt)\sin(nt). \tag{14}$$

If we substitute (4) and (14) in (13), we obtain the geodesic curvature of $(\vec{T})$ as in (12).

Theorem 3.4. The geodesic torsion $(\tau_g)_T(t)$ of the tangent indicatrix $(\vec{T})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$(\tau_g)_T(t)=0. \tag{15}$$

Proof: The geodesic torsion $(\tau_g)_T(t)$ of $(\vec{T})$ is calculated by

$$(\tau_g)_T(t)=-\frac{\left\langle \overrightarrow{g_T}(t),\ \overrightarrow{n_T'}(t)\right\rangle}{v_T^2(t)}. \tag{16}$$

From (7), we have

$$\overrightarrow{n_T'}(t)=\left(\frac{n^2}{m^2}\sin(nt)\sin t,\ \frac{n^2}{m^2}\sin(nt)\cos t,\ \frac{n^2}{m}\sin(nt)\right). \tag{17}$$

So, from (7) and (17), we get

$$\left\langle \overrightarrow{g_T}(t),\ \overrightarrow{n_T'}(t)\right\rangle=0. \tag{18}$$

If we substitute (4) and (18) in (16), we obtain the geodesic torsion of $(\vec{T})$ as in (15).

Corollary 3.1. The timelike tangent indicatrix $(\vec{T})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$ is a curvature line.

Theorem 3.5. Let $\{\overrightarrow{T_T}(t),\ \overrightarrow{g_T}(t),\ \overrightarrow{n_T}(t)\}$ be Darboux frame of the timelike tangent indicatrix $(\vec{T})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$. Darboux frame equations of $(\vec{T})$ are as follows:

$$\begin{bmatrix}\overrightarrow{T_T'}(t)\\ \overrightarrow{g_T'}(t)\\ \overrightarrow{n_T'}(t)\end{bmatrix}=\begin{bmatrix}0 & -\frac{n}{m}\cos(nt) & \frac{n}{m}\sin(nt)\\ \frac{n}{m}\cos(nt) & 0 & 0\\ \frac{n}{m}\sin(nt) & 0 & 0\end{bmatrix}\begin{bmatrix}\overrightarrow{T_T}(t)\\ \overrightarrow{g_T}(t)\\ \overrightarrow{n_T}(t)\end{bmatrix}. \tag{19}$$

Proof: We can construct the following matrix equation between Darboux vectors and their derivatives:

$$\begin{bmatrix}\overrightarrow{T_T'}(t)\\ \overrightarrow{g_T'}(t)\\ \overrightarrow{n_T'}(t)\end{bmatrix}=\begin{bmatrix}a_1 & b_1 & c_1\\ d_1 & e_1 & f_1\\ k_1 & l_1 & m_1\end{bmatrix}\begin{bmatrix}\overrightarrow{T_T}(t)\\ \overrightarrow{g_T}(t)\\ \overrightarrow{n_T}(t)\end{bmatrix}, \tag{20}$$

where $a_1,\ b_1,\ c_1,\ d_1,\ e_1,\ f_1,\ k_1,\ l_1,\ m_1$ are arbitrary functions of t. Now let's find these coefficients in order. From (20), we write the equation below:

$$\overrightarrow{T_T'}(t)=a_1\overrightarrow{T_T}(t)+b_1\overrightarrow{g_T}(t)+c_1\overrightarrow{n_T}(t). \tag{21}$$

Let's apply the inner product of the vectors $\overrightarrow{T_T}(t)$, $\overrightarrow{g_T}(t)$ and $\overrightarrow{n_T}(t)$ to both sides of (21), respectively. So, we have

$$\begin{aligned}a_1&=\left\langle \overrightarrow{T_T}(t),\ \overrightarrow{T_T'}(t)\right\rangle,\\ b_1&=-\left\langle \overrightarrow{g_T}(t),\ \overrightarrow{T_T'}(t)\right\rangle,\\ c_1&=\left\langle \overrightarrow{n_T}(t),\ \overrightarrow{T_T'}(t)\right\rangle.\end{aligned} \tag{22}$$

From (7), we get

$$\overrightarrow{T_T'}(t)=\left(\frac{n}{m}\cos t,\ -\frac{n}{m}\sin t,\ 0\right). \tag{23}$$

From (7), (22) and (23), we have

$$a_1=0,\qquad b_1=-\frac{n}{m}\cos(nt),\qquad c_1=\frac{n}{m}\sin(nt). \tag{24}$$

From (20), we write the equation below:

$$\overrightarrow{g_T'}(t)=d_1\overrightarrow{T_T}(t)+e_1\overrightarrow{g_T}(t)+f_1\overrightarrow{n_T}(t) \tag{25}$$

Let's apply the inner product of the vectors $\overrightarrow{T_T}(t)$, $\overrightarrow{g_T}(t)$ and $\overrightarrow{n_T}(t)$ to both sides of (25), respectively. So, we have

$$d_1=\left\langle \overrightarrow{T_T}(t),\ \overrightarrow{g_T}'(t)\right\rangle,$$
$$e_1=\left\langle \overrightarrow{g_T}(t),\ \overrightarrow{g_T}'(t)\right\rangle, \quad (26)$$
$$f_1=\left\langle \overrightarrow{n_T}(t),\ \overrightarrow{g_T}'(t)\right\rangle.$$

From (7), we get

$$\overrightarrow{g_T}'(t)=-\frac{n^2}{m^2}\cos(nt)(\sin t,\cos t,1). \quad (27)$$

From (7), (26) and (27), we have

$$d_1=\frac{n}{m}\cos(nt),\qquad e_1=0,\qquad f_1=0. \quad (28)$$

From (20), we write the equation below:

$$\overrightarrow{n_T}'(t)=k_1\overrightarrow{T_T}(t)+l_1\overrightarrow{g_T}(t)+m_1\overrightarrow{n_T}(t). \quad (29)$$

Let's apply the inner product of the vectors $\overrightarrow{T_T}(t)$, $\overrightarrow{g_T}(t)$ and $\overrightarrow{n_T}(t)$ to both sides of (29), respectively. So, we have

$$k_1=\left\langle \overrightarrow{T_T}(t),\ \overrightarrow{n_T}'(t)\right\rangle,$$
$$l_1=\left\langle \overrightarrow{g_T}(t),\ \overrightarrow{n_T}'(t)\right\rangle, \quad (30)$$
$$m_1=\left\langle \overrightarrow{n_T}(t),\ \overrightarrow{n_T}'(t)\right\rangle.$$

From (7), (17) and (30), we have

$$k_1=\frac{n}{m}\sin(nt),\qquad l_1=0,\qquad m_1=0. \quad (31)$$

If we substitute (24), (28) and (31) in (20), we get the expression (19).

Theorem 3.6. Let $\{\overrightarrow{T_T}(t),\ \overrightarrow{N_T}(t),\ \overrightarrow{B_T}(t)\}$ and $\{\overrightarrow{T_T}(t),\ \overrightarrow{g_T}(t),\ \overrightarrow{n_T}(t)\}$ be the Frenet and Darboux frames of the timelike tangent indicatrix $(\vec{T})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$, respectively. The real angle $\theta_T(t)$ between $\overrightarrow{n_T}(t)$ and $\overrightarrow{N_T}(t)$ is as follows:

$$\theta_T(t)=\frac{\pi}{2}+nt. \quad (32)$$

Proof: For the real angle $\theta_T(t)$ between the spacelike vectors $\overrightarrow{n_T}(t)$ and $\overrightarrow{N_T}(t)$, Figure 4, we write

$$\left\langle \overrightarrow{n_T}(t),\ \overrightarrow{N_T}(t)\right\rangle=\left\|\overrightarrow{n_T}(t)\right\|\left\|\overrightarrow{N_T}(t)\right\|\cos\theta_T(t). \quad (33)$$

Since $\overrightarrow{n_T}(t)$ and $\overrightarrow{N_T}(t)$ are unit vectors and using (5) and (7), from the inner product of these vectors in (33), we have

$$\left\langle \overrightarrow{n_T}(t),\ \overrightarrow{N_T}(t)\right\rangle=\sin(nt)=\cos\theta_T. \quad (34)$$

Similarly, using (5) and (7), from the inner product of $\overrightarrow{n_T}(t)$ and $\overrightarrow{B_T}(t)$, we get

$$\left\langle \overrightarrow{n_T}(t),\ \overrightarrow{B_T}(t)\right\rangle=\sin\theta_T(t)=-\cos(nt). \quad (35)$$

So, from (34) and (35), we obtain the expression (32).

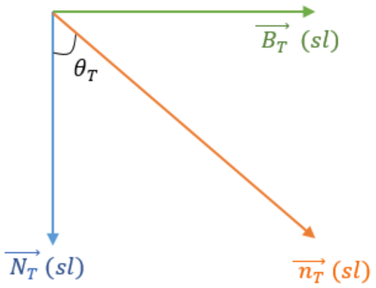

Figure 4. The real angle $\theta_T(t)$ between $\overrightarrow{n_T}(t)$ and $\overrightarrow{N_T}(t)$.

Theorem 3.7. The spacelike Darboux vector $\overrightarrow{W_T}(t)$ of Darboux frame of the timelike tangent indicatrix $(\vec{T})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$\overrightarrow{W_T}(t)=\left(-\frac{n^2}{m}\sin t,\ -\frac{n^2}{m}\cos t,\ -\frac{n^2}{m^2}\right)\quad (\mathrm{sl}). \quad (36)$$

Proof: For the Darboux vector $\overrightarrow{W_T}(t)$ belong to the Darboux frame, we know

$$\overrightarrow{T_T}'(t)=\overrightarrow{W_T}(t)\wedge\overrightarrow{T_T}(t),$$
$$\overrightarrow{g_T}'(t)=\overrightarrow{W_T}(t)\wedge\overrightarrow{g_T}(t), \quad (37)$$
$$\overrightarrow{n_T}'(t)=\overrightarrow{W_T}(t)\wedge\overrightarrow{n_T}(t),$$

(Uğurlu and Çalışkan, 2012). From (37), using (7) and (17), we have (36).

Corollary 3.2. The spacelike Darboux vector $\overrightarrow{W_T}(t)$ of Darboux frame of timelike tangent indicatrix $(\vec{T})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$\overrightarrow{W_T}(t)=\frac{n}{m}\sin(nt)\overrightarrow{g_T}(t)+\frac{n}{m}\cos(nt)\overrightarrow{n_T}(t). \quad (38)$$

Proof: We write the vector $\overrightarrow{W_T}(t)$ as follows:

$$\overrightarrow{W_T}(t)=\rho\overrightarrow{T_T}(t)+\sigma\overrightarrow{g_T}(t)+\varsigma\overrightarrow{n_T}(t), \quad (39)$$

where ρ, σ, ζ are arbitrary functions of t. For (32) and (37) to be satisfied simultaneously, we have

$$\rho = \nu_T(t)\left(\tau_g\right)_T(t),$$
$$\sigma = \nu_T(t)\left(\kappa_n\right)_T(t),$$
$$\varsigma = -\nu_T(t)\left(\kappa_g\right)_T(t)$$

in (39). If we substitute these values in (39), we obtain

$$\overrightarrow{W_T}(t) = \nu_T(t)\begin{bmatrix}\left(\tau_g\right)_T(t)\overrightarrow{T_T}(t) + \left(\kappa_n\right)_T(t)\overrightarrow{g_T}(t)\\ -\left(\kappa_g\right)_T(t)\overrightarrow{n_T}(t)\end{bmatrix}, \tag{40}$$

(Uğurlu and Çalışkan, 2012). If we substitute (4), (8), (12) and (15) in (40), we get (38). Moreover, if we substitute (8) in (39) also, it is clear that we get the expression (37).

Theorem 3.8. There are the following relations between the normal curvature$(K_n)_T(t)$, geodesic curvature $\left(K_g\right)_T(t)$, geodesic torsion $\left(\tau_g\right)_T(t)$ and torsion $\tau_T(t)$, curvature $K_T(t)$ of the timelike tangent indicatrix $(\vec{T})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$:

$$\begin{cases}\left(\kappa_n\right)_T(t) = \kappa_T(t)\sin(nt),\\ \left(\kappa_g\right)_T(t) = -\kappa_T(t)\cos(nt),\\ \left(\tau_g\right)_T(t) = \tau_T(t) - \dfrac{m}{\sin(nt)}.\end{cases} \tag{41}$$

Proof: For Darboux vector $\overrightarrow{F_T}(t)$ of Frenet frame of $(\vec{T})$, we write

$$\overrightarrow{T_T'}(t) = \overrightarrow{F_T}(t) \wedge \overrightarrow{T_T}(t), \tag{42}$$

(Uğurlu and Çalışkan, 2012). From the equality of (37) and (42), we have

$$\overrightarrow{T_T}(t) \wedge \left(\overrightarrow{F_T}(t) - \overrightarrow{W_T}(t)\right) = \vec{0}, \tag{43}$$

where

$$\overrightarrow{F_T}(t) \;=(0,\ 0,\ 1) = \nu_T(t)\left[\tau_T(t)\overrightarrow{T_T}(t) - \kappa_T(t)\overrightarrow{B_T}(t)\right] \tag{44}$$

(Aksan and Gür Mazlum, 2023). If we substitute (40) and (44) in (43), we get

$$\kappa_T(t)\overrightarrow{N_T}(t) - \left(\kappa_n\right)_T(t)\overrightarrow{n_T}(t) - \left(\kappa_g\right)_T(t)\overrightarrow{g_T}(t) = \vec{0}. \tag{45}$$

If we apply the inner product with $\overrightarrow{n_T}(t)$ and $\overrightarrow{g_T}(t)$ on both sides of (45), respectively, we get

$$\begin{cases}\left(\kappa_n\right)_T(t) = \kappa_T(t)\left\langle\overrightarrow{N_T}(t), \overrightarrow{n_T}(t)\right\rangle\\ \qquad -\left(\kappa_g\right)_T(t)\left\langle\overrightarrow{g_T}(t), \overrightarrow{n_T}(t)\right\rangle,\\ \left(\kappa_g\right)_T(t) = \kappa_T(t)\left\langle\overrightarrow{N_T}(t), \overrightarrow{g_T}(t)\right\rangle\\ \qquad -\left(\kappa_n\right)_T(t)\left\langle\overrightarrow{g_T}(t), \overrightarrow{n_T}(t)\right\rangle.\end{cases} \tag{46}$$

From (5) and (7), we get

$$\left\langle\overrightarrow{N_T}(t), \overrightarrow{g_T}(t)\right\rangle = -\cos(nt). \tag{47}$$

If we substitute (34) and (47) in (46), we have

$$\left(\kappa_n\right)_T(t) = \kappa_T(t)\sin(nt),$$
$$\left(\kappa_g\right)_T(t) = -\kappa_T(t)\cos(nt). \tag{48}$$

Besides, if we take the derivatıve of both sides of $\left\langle\overrightarrow{n_T}(t),\ \overrightarrow{N_T}(t)\right\rangle = \sinh\theta_T(t)$ in (34), we get

$$\left\langle\overrightarrow{N_T}(t),\ \overrightarrow{n_T'}(t)\right\rangle + \left\langle\overrightarrow{N_T'}(t),\ \overrightarrow{n_T}(t)\right\rangle = -\sin\theta_T(t)\frac{d\theta_T(t)}{dt}. \tag{49}$$

From the derivative formulas (Uğurlu and Çalışkan, 2012), we obtain

$$\nu_T(t)\left(\left(\tau_g\right)_T(t)\left\langle\overrightarrow{N_T}(t),\ \overrightarrow{g_T}(t)\right\rangle - \tau_T(t)\left\langle\overrightarrow{B_T}(t),\ \overrightarrow{n_T}(t)\right\rangle\right)$$
$$= -\sin\theta_T(t)\frac{d\theta_T(t)}{dt}. \tag{50}$$

If we substitute (32), (35) and (47) in (50), we have

$$\left(\tau_g\right)_T(t) = \tau_T(t) - \frac{1}{\nu_T(t)}\frac{d\theta_T}{dt}. \tag{51}$$

Also, from (4) and (32), we get

$$\left(\tau_g\right)_T(t) = \tau_T(t) - \frac{m}{\sin(nt)}. \tag{52}$$

From (48) and (52), we obtain (41).

Theorem 3.9. Let $\left\{\overrightarrow{T_T}(t),\ \overrightarrow{N_T}(t),\ \overrightarrow{B_T}(t)\right\}$ and $\left\{\overrightarrow{T_T}(t),\ \overrightarrow{g_T}(t),\ \overrightarrow{n_T}(t)\right\}$ be the Frenet and Darboux frames of the timelike tangent indicatrix $(\vec{T})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$, respectively. There is the following relationship between these frames:

$$\begin{bmatrix} \overrightarrow{T_T}(t) \\ \overrightarrow{N_T}(t) \\ \overrightarrow{B_T}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\cos(nt) & \sin(nt) \\ 0 & -\sin(nt) & -\cos(nt) \end{bmatrix} \begin{bmatrix} \overrightarrow{T_T}(t) \\ \overrightarrow{g_T}(t) \\ \overrightarrow{n_T}(t) \end{bmatrix}. \tag{53}$$

Proof: We can construct the following matrix equation between the vectors of the Frenet and Darboux frames:

$$\begin{bmatrix} \overrightarrow{T_T}(t) \\ \overrightarrow{N_T}(t) \\ \overrightarrow{B_T}(t) \end{bmatrix} = \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ k_2 & l_2 & m_2 \end{bmatrix} \begin{bmatrix} \overrightarrow{T_T}(t) \\ \overrightarrow{g_T}(t) \\ \overrightarrow{n_T}(t) \end{bmatrix}, \tag{54}$$

where $a_2, b_2, c_2, d_2, e_2, f_2, k_2, l_2, m_2$ are arbitrary functions of t. Now let's find these coefficients in order. From (54), we write the equation below:

$$\overrightarrow{T_T}(t) = a_2\overrightarrow{T_T}(t) + b_2\overrightarrow{g_T}(t) + c_2\overrightarrow{n_T}(t). \tag{55}$$

Let's apply the inner product of the vectors $\overrightarrow{T_T}(t)$, $\overrightarrow{g_T}(t)$ and $\overrightarrow{n_T}(t)$ to both sides of (55), respectively. So, it is clear that

$$a_2 = 1, \quad b_2 = 0, \quad c_2 = 0. \tag{56}$$

Similarly, from (54), we write

$$\overrightarrow{N_T}(t) = d_2\overrightarrow{T_T}(t) + e_2\overrightarrow{g_T}(t) + f_2\overrightarrow{n_T}(t). \tag{57}$$

Let's apply the inner product of the vectors $\overrightarrow{T_T}(t)$, $\overrightarrow{g_T}(t)$ and $\overrightarrow{n_T}(t)$ to both sides of (57), respectively. From (34) and (47), we get

$$d_2 = 0, \quad e_2 = -\cos(nt), \quad f_2 = \sin(nt). \tag{58}$$

Similarly, from (54), we write

$$\overrightarrow{B_T}(t) = k_2\overrightarrow{T_T}(t) + l_2\overrightarrow{g_T}(t) + m_2\overrightarrow{n_T}(t). \tag{59}$$

Let's apply the inner product of the vectors $\overrightarrow{T_T}(t)$, $\overrightarrow{g_T}(t)$ and $\overrightarrow{n_T}(t)$ to both sides of (59), respectively. From (35) and (59), we have

$$k_2 = 0, \quad l_2 = -\sin(nt), \quad m_2 = \cos(nt). \tag{60}$$

If we substitute (56), (58) and (60) in (54), we obtain the expression (53).

Theorem 3.10. Let $\{\overrightarrow{T_T}(t), \overrightarrow{g_T}(t), \overrightarrow{n_T}(t)\}$ be the Darboux frame of the timelike tangent indicatrix $(\vec{T})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$. There is the following relationship between the Darboux vectors $\overrightarrow{F_T}(t)$ and $\overrightarrow{W_T}(t)$ belong to the Frenet and Darboux frames, respectively:

$$\overrightarrow{W_T}(t) = \overrightarrow{F_T}(t) - n\overrightarrow{T_T}(t). \tag{61}$$

Proof: It is clear that (5), (36) and (44).

Theorem 3.11. The real angle $\varphi_T(t)$ between $\overrightarrow{W_T}(t)$ and $\overrightarrow{g_T}(t)$ of the timelike tangent indicatrix $(\vec{T})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$\varphi_T(t) = \arcsin\left(\frac{n}{m}\right) + nt. \tag{62}$$

Proof: For the real angle $\varphi_T(t)$ between the unit spacelike vectors $\overrightarrow{W_T}(t)$ and $\overrightarrow{g_T}(t)$, Figure 5, we write

$$\left\langle \overrightarrow{g_T}(t), \overrightarrow{W_T}(t) \right\rangle = \cos\varphi_T(t). \tag{63}$$

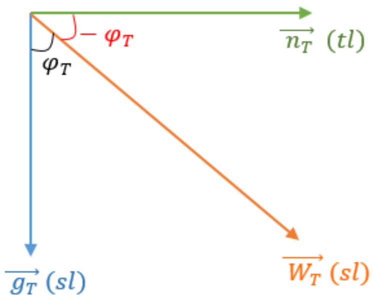

Figure 5. The real angle $\varphi_T(t)$ between $\overrightarrow{W_T}(t)$ and $\overrightarrow{g_T}(t)$.

Besides, since timelike $\overrightarrow{n_T}(t)$ and spacelike $\overrightarrow{g_T}(t)$ are perpendicular, the angle between them is zero in the Lorentzian sense, we write

$$\left\langle \overrightarrow{n_T}(t), \overrightarrow{W_T}(t) \right\rangle = \sinh(-\varphi_T(t)) = -\sinh\varphi_T(t). \tag{64}$$

On the other hand, from (7) and (36), we get

$$\begin{aligned} \left\langle \overrightarrow{g_T}(t), \overrightarrow{W_T}(t) \right\rangle &= \frac{n}{m}\sin(nt), \\ \left\langle \overrightarrow{n_T}(t), \overrightarrow{W_T}(t) \right\rangle &= \frac{n}{m}\cos(nt). \end{aligned} \tag{65}$$

From (66), (67) and (68), we obtain

$$\begin{aligned} \cosh\varphi_T(t) &= \frac{n}{m}\sin(nt), \\ \sinh\varphi_T(t) &= -\frac{n}{m}\cos(nt). \end{aligned} \tag{66}$$

From (66), we get

$$\begin{aligned} &\cosh\varphi_T(t)\sin(nt) - \sinh\varphi_T(t)\cos(nt) \\ &= \sin(\varphi_T(t) - nt) = \frac{n}{m}. \end{aligned} \tag{67}$$

3.2. The Darboux Frame of the Principal Normal indicatrix of $\overrightarrow{\gamma_m}(t)$

The Frenet frame $\{\overrightarrow{T_N}(t), \overrightarrow{N_N}(t), \overrightarrow{B_N}(t)\}$, Darboux vector belong to Frenet frame $F_N(t)$, curvature $K_N(t)$ and torsion $\tau_N(t)$ of spacelike principal normal indicatrix $(\vec{N})$ on H_0^{2+} of $\overrightarrow{\gamma_m}(t)$ are

$$\begin{cases} \overrightarrow{T_N}(t)=(\cos t,\ -\sin t,\ 0) & (\text{sl}), \\ \overrightarrow{N_N}(t)=(-\sin t,\ -\cos t,\ 0) & (\text{sl}), \\ \overrightarrow{B_N}(t)=(0,\ 0,\ -1) & (\text{tl}), \end{cases} \qquad (68)$$
$$\overrightarrow{F_N}(t)=(0,\ 0,\ 1) \quad (\text{tl}),$$

$$\kappa_N(t)=\frac{m}{n} \quad \text{and} \quad \tau_N(t)=0, \qquad (69)$$

respectively, (Uğurlu and Çalışkan, 2012).

Theorem 4.1. The Darboux frame $\{\overrightarrow{T_N}(t), \overrightarrow{g_N}(t), \overrightarrow{n_N}(t)\}$ of spacelike principal normal indicatrix $(\vec{N})$ on H_0^{2+} of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$\begin{cases} \overrightarrow{T_N}(t)=(\cos t,\ -\sin t,\ 0) & (\text{sl}), \\ \overrightarrow{g_N}(t)=\left(n\sin t,\ n\cos t,\ \frac{n}{m}\right) & (\text{sl}), \\ \overrightarrow{n_N}(t)=\left(\frac{n}{m}\sin t,\ \frac{n}{m}\cos t,\ n\right) & (\text{tl}). \end{cases} \qquad (70)$$

Proof: Since $\overrightarrow{T_N}(t)$ is spacelike and $\vec{N}(t)$ is timelike, $\overrightarrow{g_N}(t) = -N(t) \wedge \overrightarrow{T_N}(t)$. From (3) and (68), we get the vector $\overrightarrow{g_N}(t)$ in (70). Besides, $\vec{N}(t)$can be taken as the normal vector of the surface for $(\vec{N})$ on H_0^{2+}, that is $\vec{n}(t) = \vec{N}(t)$ as in (3). Thus, the Darboux frame is as in (70).

Theorem 4.2. The normal curvature $(K_n)_N(t)$of principal normal indicatrix $(\vec{N})$ on H_0^{2+} of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$(\kappa_n)_N(t)=1. \qquad (71)$$

Proof: The normal curvature of $(\vec{N})$ is calculated by

$$(\kappa_n)_N(t)=-\frac{\left\langle \overrightarrow{n_N}(t),\ \overrightarrow{N''}(t)\right\rangle}{v_N^2(t)}, \qquad (72)$$

(Uğurlu and Çalışkan, 2012). From (3), we obtain

$$\overrightarrow{N''}(t)=\left(-\frac{n}{m}\sin t,\ -\frac{n}{m}\cos t,\ 0\right). \qquad (73)$$

From (70) and (73), we have

$$\left\langle \overrightarrow{n_N}(t),\ \overrightarrow{N''}(t)\right\rangle=-\frac{n^2}{m^2}. \qquad (74)$$

If we substitute (4) and (74) in (72), we obtain the normal curvature of $(\vec{N})$ as in (71).

Theorem 4.3. The geodesic curvature $(K_g)_N(t)$ of principal normal indicatrix $(\vec{N})$ on H_0^{2+} of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$(\kappa_g)_N(t)=-m. \qquad (75)$$

Proof: $(K_g)_N(t)$ for $(\vec{N})$ is calculated by

$$(\kappa_g)_N(t)=\frac{\left\langle \overrightarrow{g_N}(t),\ \overrightarrow{N''}(t)\right\rangle}{v_N^2(t)}, \qquad (76)$$

(Uğurlu and Çalışkan, 2012). From (70) and (73), we get

$$\left\langle \overrightarrow{g_N}(t),\ \overrightarrow{N''}(t)\right\rangle=-\frac{n^2}{m}. \qquad (77)$$

If we substitute (4) and (77) in (76), we obtain the geodesic curvature of $(\vec{N})$ as in (75).

Theorem 4.4. The geodesic torsion $(\tau_g)_N(t)$ of principal normal indicatrix $(\vec{N})$ on H_0^{2+} of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$(\tau_g)_N(t)=0. \qquad (78)$$

Proof: $(\tau_g)_N(t)$ for $(\vec{N})$ is calculated by

$$(\tau_g)_N(t)=-\frac{\left\langle \overrightarrow{g_N}(t),\ \overrightarrow{n_N'}(t)\right\rangle}{v_N^2(t)}, \qquad (79)$$

(Uğurlu and Çalışkan, 2012). From (7), we have

$$\overrightarrow{n_N'}(t)=\left(\frac{n}{m}\cos t,\ -\frac{n}{m}\sin t,\ 0\right). \qquad (80)$$

So, from (7) and (80), we get

$$\left\langle \overrightarrow{g_N}(t),\ \overrightarrow{n_N'}(t)\right\rangle=0. \qquad (81)$$

If we substitute (4) and (81) in (79), we obtain the geodesic torsion of $(\vec{N})$ as in (78).

Corollary 4.1. The spacelike principal normal indicatrix $(\vec{N})$ on H_0^{2+} of $\overrightarrow{\gamma_m}(t)$ is a curvature line.

Theorem 4.5. Let $\{\overrightarrow{T_N}(t), \overrightarrow{g_N}(t), \overrightarrow{n_N}(t)\}$ be Darboux frame of spacelike principal normal indicatrix $(\vec{N})$ on H_0^{2+} of $\overrightarrow{\gamma_m}(t)$. Darboux frame equations of $(\vec{N})$ are as follows:

$$\begin{bmatrix} \overrightarrow{T_N}'(t) \\ \overrightarrow{g_N}'(t) \\ \overrightarrow{n_N}'(t) \end{bmatrix} = \begin{bmatrix} 0 & -n & \frac{n}{m} \\ n & 0 & 0 \\ \frac{n}{m} & 0 & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{T_N}(t) \\ \overrightarrow{g_N}(t) \\ \overrightarrow{n_N}(t) \end{bmatrix}.$$

Proof: The proof can be done similarly to the proof of Theorem 3.5.

Theorem 4.6. Let $\{\overrightarrow{T_N}(t), \overrightarrow{N_N}(t), \overrightarrow{B_N}(t)\}$ and $\{\overrightarrow{T_N}(t), \overrightarrow{g_N}(t), \overrightarrow{n_N}(t)\}$ be Frenet and Darboux frames of spacelike principal normal indicatrix $(\vec{N})$ on H_0^{2+} of $\overrightarrow{\gamma_m}(t)$, respectively. The hyperbolic angle $\theta_N(t)$ between $\overrightarrow{B_N}(t)$ and $\overrightarrow{n_N}(t)$ is as follows:

$$\theta_N(t) = \operatorname{arctan h}\left(\frac{1}{m}\right) = \frac{1}{2}\ln\left(\frac{m+1}{m-1}\right). \tag{82}$$

Proof: From the inner product of $\overrightarrow{B_N}(t)$ and $\overrightarrow{n_N}(t)$ in (68) and (70), for the angle $\theta_N(t)$ between unit timelike vectors $\overrightarrow{B_N}(t)$ (future pointing) and $\overrightarrow{n_N}(t)$ (past pointing), Figure 6, we have

$$\left\langle \overrightarrow{B_N}(t), \ \overrightarrow{n_N}(t) \right\rangle = \cosh\theta_N = n. \tag{83}$$

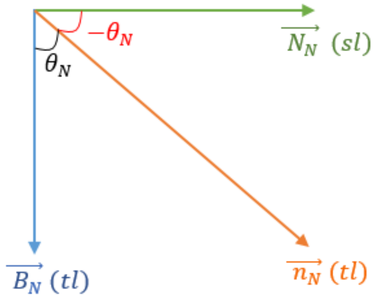

Figure 6. The real angle $\theta_N(t)$ between $\overrightarrow{B_N}(t)$ and $\overrightarrow{n_N}(t)$.

Similarly, from inner product of unit spacelike $\overrightarrow{N_N}(t)$ and unit timelike $\overrightarrow{n_N}(t)$ vectors, (here since timelike$\overrightarrow{B_N}(t)$ and spacelike $\overrightarrow{N_N}(t)$ are perpendicular, the angle between them is zero in the Lorentzian sense), we have

$$\left\langle \overrightarrow{N_N}(t), \ \overrightarrow{n_N}(t) \right\rangle = \sinh(-\theta_N) = -\sinh(\theta_N) = -\frac{n}{m}. \tag{84}$$

So, from (83) and (84), we obtain the expression (82).

Theorem 4.7. The timelike Darboux vector $\overrightarrow{W_N}(t)$ of Darboux frame of spacelike principal normal indicatrix $(\vec{N})$ on H_0^{2+} of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$\overrightarrow{W_N}(t) = (0, \ 0, \ 1) \qquad \text{(tl)}. \tag{85}$$

Proof: For the Darboux vector $\overrightarrow{W_N}(t)$ belong to the Darboux frame, we know

$$\begin{aligned} \overrightarrow{T_N}'(t) &= \overrightarrow{W_N}(t) \wedge \overrightarrow{T_N}(t), \\ \overrightarrow{g_N}'(t) &= \overrightarrow{W_N}(t) \wedge \overrightarrow{g_N}(t), \\ \overrightarrow{n_N}'(t) &= \overrightarrow{W_N}(t) \wedge \overrightarrow{n_N}(t), \end{aligned} \tag{86}$$

(Uğurlu and Çalışkan, 2012). From (86), using (70) and (80), we get (85).

Corollary 4.2. The timelike Darboux vector $\overrightarrow{W_N}(t)$ of Darboux frame of spacelike principal normal indicatrix $(\vec{N})$ on H_0^{2+} of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$\overrightarrow{W_N}(t) = -\frac{n}{m}\overrightarrow{g_N}(t) + n\overrightarrow{n_N}(t).$$

Proof: The proof can be done similarly to Corollary 3.2.

Theorem 4.8. There are the following relations between the normal curvature $(K_n)_N(t)$, geodesic curvatures $(K_g)_N(t)$, geodesic torsion $(\tau_g)_N(t)$ and curvature $K_N(t)$, torsion $\tau_N(t)$ of spacelike principal normal indicatrix $(\vec{N})$ on H_0^{2+} of $\overrightarrow{\gamma_m}(t)$:

$$\begin{cases} (\kappa_n)_N(t) = \frac{n}{m}\kappa_N(t), \\ (\kappa_g)_N(t) = -n\kappa_N(t), \\ (\tau_g)_N(t) = \tau_N(t). \end{cases}$$

Proof: The proof is similar to Theorem 3.8.

Theorem 4.9. Let $\{\overrightarrow{T_N}(t), \overrightarrow{N_N}(t), \overrightarrow{B_N}(t)\}$ and $\{\overrightarrow{T_N}(t), \overrightarrow{g_N}(t), \overrightarrow{n_N}(t)\}$ be the Frenet and Darboux frames of spacelike principal normal indicatrix $(\vec{N})$ on H_0^{2+} of $\overrightarrow{\gamma_m}(t)$, respectively. There is the following relation between the frames:

$$\begin{bmatrix} \overrightarrow{T_N}(t) \\ \overrightarrow{N_N}(t) \\ \overrightarrow{B_N}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -n & -\frac{n}{m} \\ 0 & -\frac{n}{m} & -n \end{bmatrix} \begin{bmatrix} \overrightarrow{T_N}(t) \\ \overrightarrow{g_N}(t) \\ \overrightarrow{n_N}(t) \end{bmatrix}.$$

Proof: The proof can be done similarly to Theorem 3.9.

Theorem 4.10. Let $\{\overrightarrow{T_N}(t), \overrightarrow{g_N}(t), \overrightarrow{n_N}(t)\}$ be Darboux frame of spacelike principal normal indicatrix $(\vec{N})$ on H_0^{2+} of $\overrightarrow{\gamma_m}(t)$. There is the following relation between Darboux vectors $\overrightarrow{F_N}(t)$ and $\overrightarrow{W_N}(t)$ belong to the Frenet and Darboux frames, respectively:

$$\overrightarrow{W_N}(t) = \overrightarrow{F_N}(t).$$

Proof: It is clear that (68) and (85).

Theorem 4.11. The hyperbolic angle $\varphi_N(t)$ between $\overrightarrow{W_N}(t)$ and $\overrightarrow{n_N}(t)$ of spacelike principal normal indicatrix $(\vec{N})$ on H_0^{2+} of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$\varphi_N(t) = \operatorname{arctanh}\left(\frac{1}{m}\right). \tag{87}$$

Proof: For hyperbolic angle $\varphi_N(t)$ between unit timelike (past pointing) vectors $\overrightarrow{W_N}(t)$ and $\overrightarrow{n_N}(t)$, Figure 7, we know

$$\left\langle \overrightarrow{n_N}(t), \overrightarrow{W_N}(t) \right\rangle = -\cosh\varphi_N(t). \tag{88}$$

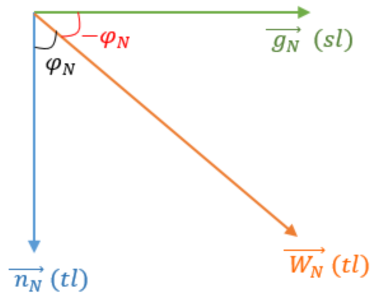

Figure 7. The real angle $\varphi_N(t)$ between $\overrightarrow{W_N}(t)$ and $\overrightarrow{n_N}(t)$.

Besides, since timelike $\overrightarrow{n_N}(t)$ and spacelike $\overrightarrow{g_N}(t)$ are perpendicular, the angle between them is zero in the Lorentzian sense, we have

$$\left\langle \overrightarrow{g_N}(t), \overrightarrow{W_N}(t) \right\rangle = \sinh(-\varphi_N(t)) = -\sinh\varphi_N(t). \tag{89}$$

On the other hand, from (70) and (85), we get

$$\left\langle \overrightarrow{n_N}(t), \overrightarrow{W_N}(t) \right\rangle = -n, \qquad \left\langle \overrightarrow{g_N}(t), \overrightarrow{W_N}(t) \right\rangle = -\frac{n}{m}. \tag{90}$$

From (88), (89) and (90), we obtain

$$\cosh\varphi_N(t) = n \quad \text{and} \quad \sinh\varphi_N(t) = \frac{n}{m}. \tag{91}$$

From (91), we have (87).

3.3. The Darboux Frame of Binormal Indicatrix $(\vec{B})$ of $\overrightarrow{\gamma_m}(t)$

The Frenet frame $\{\overrightarrow{T_B}(t), \overrightarrow{N_B}(t), \overrightarrow{B_B}(t)\}$, Darboux vector belong to Frenet frame $\overrightarrow{F_B}(t)$, curvature $K_B(t)$ and torsion $\tau_B(t)$ of the timelike binormal indicatrix $(\vec{B})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$ are

$$\begin{cases} \overrightarrow{T_B}(t) = \left(-\frac{n}{m}\sin t,\ -\frac{n}{m}\cos t,\ -n\right) & (\text{tl}), \\ \overrightarrow{N_B}(t) = (-\cos t,\ \sin t,\ 0) & (\text{sl}), \\ \overrightarrow{B_B}(t) = \left(n\sin t,\ n\cos t,\ \frac{n}{m}\right) & (\text{sl}), \end{cases} \tag{92}$$

$$\overrightarrow{F_B}(t) = (0,\ 0,\ 1),$$

$$\kappa_B(t) = \frac{1}{\cos(nt)} \quad \text{and} \quad \tau_B(t) = -\frac{m}{\cos(nt)} \tag{93}$$

respectively, (Uğurlu and Çalışkan, 2012).

Theorem 5.1. The Darboux frame $\{\overrightarrow{T_B}(t), \overrightarrow{g_B}(t), \overrightarrow{n_B}(t)\}$, of the timelike binormal indicatrix $(\vec{B})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$\begin{cases} \overrightarrow{T_B}(t) = \left(-\frac{n}{m}\sin t, -\frac{n}{m}\cos t, -n\right) \quad (\text{tl}), \\ \overrightarrow{g_B}(t) = \Big(\sin(nt)\cos t - n\cos(nt)\sin t, \\ \qquad \sin(nt)\sin t - n\cos(nt)\cos t, -\frac{n}{m}\cos(nt)\Big) \quad (\text{sl}), \\ \overrightarrow{n_B}(t) = \Big(-\cos(nt)\cos t - n\sin(nt)\sin t, \\ \qquad \cos(nt)\sin t - n\sin(nt)\cos t, -\frac{n}{m}\sin(nt)\Big) \quad (\text{sl}). \end{cases} \tag{94}$$

Proof: Since $\overrightarrow{T_B}(t)$ is timelike and $\vec{B}(t)$ is spacelike, $\overrightarrow{g_B}(t) = -B(t) \wedge \overrightarrow{T_B}(t)$. From (3) and (92), we get the vector $\overrightarrow{g_B}(t)$ in (94). Besides $\vec{B}(t)$ of $\overrightarrow{\gamma_m}(t)$ can be taken as the normal vector of the surface for $(\vec{B})$ on S_1^2, that is $\overrightarrow{n_B}(t) = \vec{B}(t)$ as in (3). Thus, Darboux frame is as in (94).

Theorem 5.2. The normal curvature $(K_n)_B(t)$ of binormal indicatrix $(\vec{B})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$(\kappa_n)_B(t) = 1. \tag{95}$$

Proof: The normal curvature of $(\vec{B})$ is calculated by

$$(\kappa_n)_B(t) = \frac{\left\langle \overrightarrow{n_B}(t), \overrightarrow{B''}(t) \right\rangle}{v_B^2(t)}, \tag{96}$$

(Uğurlu and Çalışkan, 2012). From (3), we get

$$\overrightarrow{B''}(t) = \frac{n^2}{m^2}\Big(-\cos(nt)\cos t + n\sin(nt)\sin t, \\ \cos(nt)\sin t + n\sin(nt)\cos t, nm\sin(nt)\Big). \tag{97}$$

From (94) and (97), we have

$$\left\langle \overrightarrow{n_B}(t), \overrightarrow{B''}(t) \right\rangle = \frac{n^2}{m^2}\cos^2(nt). \tag{98}$$

If we substitute (4) and (98) in (96), we obtain the normal curvature of $(\vec{B})$ as in (95).

Theorem 5.3. The geodesic curvature $(K_g)_B(t)$ of binormal indicatrix $(\vec{B})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$(\kappa_g)_B(t) = -\frac{\sin(nt)}{\cos(nt)}. \tag{99}$$

Proof: $(K_g)_B(t)$ for $(\vec{B})$ is calculated by

$$\left(\kappa_g\right)_B(t)=\frac{\left\langle \overrightarrow{g_B}(t),\ \vec{B}''(t)\right\rangle}{v_B^2(t)}. \tag{100}$$

(Uğurlu and Çalışkan, 2012). From (94) and (97), we have

$$\left\langle \overrightarrow{g_B}(t),\ \vec{B}''(t)\right\rangle=-\frac{n^2}{m^2}\cos(nt)\sin(nt). \tag{101}$$

If we substitute (4) and (101) in (100), we obtain the geodesic curvature of $(\vec{B})$ as in (99).

Theorem 5.4. The geodesic torsion $(\tau_g)_B(t)$ of binormal indicatrix $(\vec{B})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$\left(\tau_g\right)_B(t)=0. \tag{102}$$

Proof: $(\tau_g)_B(t)$ for $(\vec{B})$ is calculated by

$$\left(\tau_g\right)_B(t)=-\frac{\left\langle \overrightarrow{g_B}(t),\ \overrightarrow{n_B}'(t)\right\rangle}{v_B^2(t)}, \tag{103}$$

(Uğurlu and Çalışkan, 2012). From (7), we have

$$\overrightarrow{n_B}'(t)=-\left(\frac{n^2}{m^2}\cos(nt)\sin t,\frac{n^2}{m^2}\cos(nt)\cos t,\frac{n^2}{m}\cos(nt)\right). \tag{104}$$

So, from (7) and (104), we get

$$\left\langle \overrightarrow{n_B}'(t),\ \overrightarrow{g_B}(t)\right\rangle=0. \tag{105}$$

If we substitute (4) and (105) in (103), we obtain the geodesic torsion of $(\vec{B})$ as in (102).

Corollary 5.1. The timelike binormal indicatrix $(\vec{B})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$ is a curvature line.

Theorem 5.5. Let $\{\overrightarrow{T_B}(t),\overrightarrow{g_B}(t),\overrightarrow{n_B}(t)\}$ be Darboux frame of timelike binormal indicatrix $(\vec{B})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$. Darboux frame equations of $(\vec{B})$ are as follows:

$$\begin{bmatrix}\overrightarrow{T_B}'(t)\\ \overrightarrow{g_B}'(t)\\ \overrightarrow{n_B}'(t)\end{bmatrix}=\begin{bmatrix}0 & -\frac{n}{m}\sin(nt) & \frac{n}{m}\cos(nt)\\ -\frac{n}{m}\sin(nt) & 0 & 0\\ \frac{n}{m}\cos(nt) & 0 & 0\end{bmatrix}\begin{bmatrix}\overrightarrow{T_B}(t)\\ \overrightarrow{g_B}(t)\\ \overrightarrow{n_B}(t)\end{bmatrix}$$

Proof: The proof can be done similarly to Theorem 3.5.

Theorem 5.6. Let $\{\overrightarrow{T_B}(t),\overrightarrow{N_B}(t),\overrightarrow{B_B}(t)\}$ and $\{\overrightarrow{T_B}(t),\overrightarrow{g_B}(t),\overrightarrow{n_B}(t)\}$ be the Frenet and Darboux frames of the timelike binormal indicatrix $(\vec{B})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$, respectively. The angle $\theta_B(t)$ between $\overrightarrow{N_B}(t)$ and $\overrightarrow{n_B}(t)$ is as follows:

$$\theta_B(t)=-nt. \tag{106}$$

Proof: From the inner product of $\overrightarrow{N_B}(t)$ and $\overrightarrow{n_B}(t)$ in (92) and (94), for the real angle $\theta_B(t)$ between unit spacelike vectors $\overrightarrow{N_B}(t)$ and $\overrightarrow{n_B}(t)$, Figure 8, we get

$$\left\langle \overrightarrow{N_B}(t),\ \overrightarrow{n_B}(t)\right\rangle=\cos\theta_B(t)=\cos(nt). \tag{107}$$

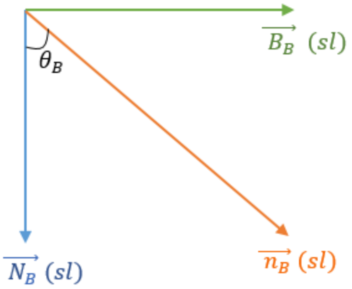

Figure 8. The real angle $\theta_B(t)$ between $\overrightarrow{N_B}(t)$ and $\overrightarrow{n_B}(t)$.

Similarly, we have

$$\left\langle \overrightarrow{B_B}(t),\ \overrightarrow{n_B}(t)\right\rangle=\sin\theta_B(t)=-\sin(nt). \tag{108}$$

So, from (107) and (108), we obtain the expression (106).

Theorem 5.7. The spacelike Darboux vector $\overrightarrow{W_B}(t)$ of Darboux frame of the timelike binormal indicatrix $(\vec{B})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$\overrightarrow{W_B}(t)=\left(-\frac{n^2}{m}\sin t,\ -\frac{n^2}{m}\cos t,\ -\frac{n^2}{m}\right)\quad \text{(sl)}. \tag{109}$$

Proof: For the Darboux vector $\overrightarrow{W_B}(t)$ belong to the Darboux frame, we know

$$\begin{aligned}\overrightarrow{T_B}'(t)&=\overrightarrow{W_B}(t)\wedge\overrightarrow{T_B}(t),\\ \overrightarrow{g_B}'(t)&=\overrightarrow{W_B}(t)\wedge\overrightarrow{g_B}(t),\\ \overrightarrow{n_B}'(t)&=\overrightarrow{W_B}(t)\wedge\overrightarrow{n_B}(t),\end{aligned} \tag{110}$$

(Uğurlu and Çalışkan, 2012). From (110), using (94) and (104), we get the expression (109).

Corollary 5.2. The spacelike Darboux vector $\overrightarrow{W_B}(t)$ of Darboux frame of timelike binormal indicatrix $(\vec{B})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$is as follows:

$$\overrightarrow{W_B}(t)=\frac{n}{m}\cos(nt)\overrightarrow{g_B}(t)+\frac{n}{m}\sin(nt)\overrightarrow{n_B}(t).$$

Proof: The proof can be done similarly to Corollary 3.2.

Theorem 5.8. There are the following relations between the normal curvature $(K_n)_B(t)$, geodesic curvatures $(K_g)_B(t)$, geodesic torsion $(\tau_g)_B(t)$ and curvature $K_B(t)$, torsion $\tau_B(t)$ of the timelike binormal indicatrix $(\vec{B})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$:

$$\begin{cases} (\kappa_n)_B(t) = \kappa_B(t)\cos(nt), \\ (\kappa_g)_B(t) = -\kappa_B(t)\sin(nt), \\ (\tau_g)_B(t) = \tau_B(t) + \dfrac{m}{\cos(nt)}. \end{cases}$$

Proof: The proof is similar to Theorem 3.8.

Theorem 5.9. Let $\{\overrightarrow{T_B}(t), \overrightarrow{N_B}(t), \overrightarrow{B_B}(t)\}$ and $\{\overrightarrow{T_B}(t), \overrightarrow{g_B}(t), \overrightarrow{n_B}(t)\}$ be Frenet and Darboux frames of timelike binormal indicatrix $(\vec{B})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$, respectively. There is the following relation between the frames:

$$\begin{bmatrix} \overrightarrow{T_B}(t) \\ \overrightarrow{N_B}(t) \\ \overrightarrow{B_B}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\sin(nt) & \cos(nt) \\ 0 & -\cos(nt) & -\sin(nt) \end{bmatrix} \begin{bmatrix} \overrightarrow{T_B}(t) \\ \overrightarrow{g_B}(t) \\ \overrightarrow{n_B}(t) \end{bmatrix}.$$

Proof: The proof can be done similarly to Theorem 3.9.

Theorem 5.10. Let $\{\overrightarrow{T_B}(t), \overrightarrow{g_B}(t), \overrightarrow{n_B}(t)\}$ be the Darboux frame of the timelike binormal indicatrix $(\vec{B})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$. There is the following relationship between the Darboux vectors $\overrightarrow{F_B}(t)$ and $\overrightarrow{W_B}(t)$ belong to the Frenet and Darboux frames, respectively:

$$\overrightarrow{W_B}(t) = \overrightarrow{F_B}(t) + n\overrightarrow{T_B}(t).$$

Proof: It is clear that (92) and (109).

Theorem 5.11. The angle $\varphi_B(t)$ between $\overrightarrow{W_B}(t)$ and $\overrightarrow{g_B}(t)$ of timelike binormal indicatrix $(\vec{B})$ on S_1^2 of $\overrightarrow{\gamma_m}(t)$ is as follows:

$$\varphi_B(t) = \arccos\left(\frac{n}{m}\right) + nt. \tag{111}$$

Proof: For the real angle $\varphi_B(t)$ between unit spacelike vectors $\overrightarrow{W_B}(t)$ and $\overrightarrow{g_B}(t)$, Figure 9, we know

$$\left\langle \overrightarrow{W_B}(t), \overrightarrow{g_B}(t) \right\rangle = \cos\varphi_B(t). \tag{112}$$

Besides, we have

$$\left\langle \overrightarrow{W_B}(t), \overrightarrow{n_B}(t) \right\rangle = \sin\varphi_B(t). \tag{113}$$

On the other hand, from (70) and (85), we get

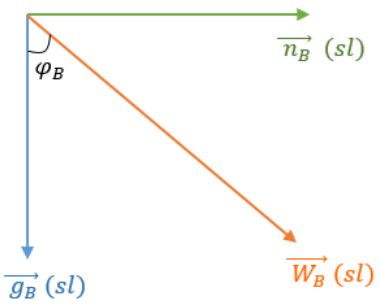

Figure 9. The real angle $\varphi_B(t)$ between $\overrightarrow{W_B}(t)$ and $\overrightarrow{g_B}(t)$.

$$\left\langle \overrightarrow{W_B}(t), \overrightarrow{g_B}(t) \right\rangle = \frac{n}{m}\cos(nt), \quad \left\langle \overrightarrow{W_B}(t), \overrightarrow{n_B}(t) \right\rangle = \frac{n}{m}\sin(nt). \tag{114}$$

From (112), (113) and (114), we obtain

$$\cos\varphi_B(t)\cos(nt) + \sin\varphi_B(t)\sin(nt) = \cos(\varphi_B(t) - nt) = \frac{n}{m} \tag{115}$$

From (115), we have (111).

4. Discussion and Conclusion

In this study, the some geometric elements (Darboux frames, curvatures, torsions, Darboux vectors, angles etc.) of timelike tangent indicatrix $(\vec{T})$ and timelike binormal indicatrix $(\vec{B})$ on the Lorentzian unit sphere S_1^2, spacelike principal normal indicatrix $(\vec{N})$ on the hyperbolic unit sphere H_0^2 of the spacelike Salkowski curve with spacelike binormal in Lorentzian 3-space E_1^3 are obtained. And the relationships between these elements are studied. Similar studies can also be done on other types of Salkowski curves in Lorentzian 3-space or other well-done curves.

Author Contributions

The percentage of the author(s) contributions is presented below. All authors reviewed and approved the final version of the manuscript.

	B.A.	S.G.M.
C	10	90
D	50	50
S		100
DCP	60	40
DAI		100
L	70	30
W	40	60
CR		100
SR		100

C=Concept, D= design, S= supervision, DCP= data collection and/or processing, DAI= data analysis and/or interpretation, L= literature search, W= writing, CR= critical review, SR= submission and revision, PM= project management, FA= funding acquisition.

Conflict of Interest

The authors declared that there is no conflict of interest.

Ethical Consideration

Ethics committee approval was not required for this study because of there was no study on animals or humans.

References

Aksan B, Gür Mazlum S. 2023. On the Spherical Indicatrix Curves of the Spacelike Salkowski Curve with Timelike Principal Normal in Lorentzian 3-Space. Honam Math J, 45(3): 513-541.

Ali AT. 2011. Spacelike Salkowski and anti-Salkowski curves with timelike principal normal in Minkowski 3-space. Math Aeterna, 1(4): 201-210.

Babaarslan M, Yaylı Y. 2017. On space-like constant slope surfaces and bertrand curves in Minkowski 3-space. Analele Stiintifice ale Universitatii Al I Cuza din Iasi-Matematica, 63(F2): 323-339.

Bilici M, Çalışkan M. 2019. Some new results on the curvatures of the spherical indicatrix curves of the involutes of a spacelike curve with a spacelike binormal in Minkowski 3-space. MathLAB J, 2(1): 110-119.

Birman GS, Nomizu K. 1984. Trigonometry in Lorentzian geometry. Ann Math Mont, 91: 534-549.

Bükcü B, Karacan MK. 2007. On the involute and evolute curves of the spacelike curve with a spacelike binormal in Minkowski 3-space. Int J Contemp Math Sci, 2(5): 221-232.

Fenchel W. 1951. On the differential geometry of closed space curves. Bull Am Math Soc, 57: 44-54.

Gür Mazlum S, Şenyurt S, Bektaş M. 2022. Salkowski curves and their modified orthogonal frames in E3. J New Theory, 40: 12-26.

Gür Mazlum S. 2023. Geometric properties of timelike surfaces in Lorentz-Minkowski 3-space. Filomat, 37(17): 5735-5749.

Gür S, Şenyurt S. 2010. Frenet vectors and geodesic curvatures of spheric indicatrix curves of Salkowski curve in E3. Hadronic J, 33(5): 485-512.

Hacısalihoğlu HH. 1983. Differential geometry. İnönü University, Publication of Faculty of Sciences and Arts, Malatya, Türkiye.

Kahveci D, Yaylı Y. 2002. Geometric kinematics of persistent rigid motions in three-dimensional Minkowski space. Mechanism Machine Theory, 167: 104535.

Kula L, Yaylı Y. 2005. On slant helix and its spherical indicatrix. Appl Math Comput, 169(1): 600-607.

Li Y, Gür Mazlum S, Şenyurt S. 2023. The Darboux trihedrons of timelike surfaces in the Lorentzian 3-space. Internationa J Geomet Methods Modern Physics, 20(2): 2350030-82.

Lopez R. 2014. Differential geometry of curves and surfaces in Lorentz-Minkowski space. Int E-J Geomet, 7: 44-107.

Monterde J. 2009. Salkowski curves revisited: A family of curves with constant curvature and non-constant torsion. Comp Aided Geomet Design, 26(3): 271-278.

O'Neill B. 1983. Semi-Riemannian geometry with applications to relativity. Academic Press, London, UK, pp: 488.

Özdemir M. 2020. Diferansiyel geometri. Altın Nokta Yayınevi, İzmir, Türkiye, pp: 132.

Ratcliffe JG. 1994. Foundations of hyperbolic manifolds. Springer-Verlag, Tokyo, Japan, pp: 779.

Salkowski E. 1909. Zur transformation von raumkurven. Math Annalen, 66(4): 517-557.

Şentürk GY, Yüce S. 2015. Characteristic properties of the ruled surface with Darboux frame in E-3. Kuwait J Sci, 42(2), 14-33.

Uğurlu HH, Çalışkan A. 2012. Darboux ani dönme vektörleri ile spacelike ve timelike yüzeyler geometrisi. Celal Bayar University Press, Manisa, Türkiye, pp: 12.

Uğurlu HH, Kocayiğit H. 1996. The Frenet and Darboux instantaneous rotation vectors of curves on time-like surface. Math Comp Appl, 1(2): 133-141.

Uğurlu HH. 1997. On the geometry of time-like surfaces. Communications, Faculty of Sciences, University of Ankara, A1 Series, No: 46, pp: 211-223.

Yakıcı Topbaş ES, Gök İ, Ekmekci FN, Yaylı Y. 2016. Darboux frame of a curve lying on a lightlike surface. Math Sci Appl E-Notes, 4(2): 121-130.

Yüksel N, Saltık B, Damar E. 2014. Parallel curves in Minkowski 3-space. Gümüşhane Univ J Sci Tech, 12(2): 480-486.