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Research Article

Matlab Applications for Skew-Symmetric Matrices and Integral Curves in Lorentzian Spaces

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ABSTRACT

In [8], the authors obtained the non-zero solutions of the equation $A(x)=0$, $x \in E_1^{2n+1}$, in Lorentzian space E_1^{2n+1} , where A is a skew-symmetric matrix corresponding to the linear map A and got normal forms of the skew-symmetric matrix A , depending on the causal characters of the vector x . Taking into consideration the structure of the matrix A , we generate Matlab codes and make some Matlab applications for normal form of skew-symmetric matrix. Also, we give some Matlab codes for the linear first order system of differential equations which the solution of the system gives rise to integral curves of linear vector fields in such a space. Moreover, we give some application with respect to special case of n and causal characters of the vector x for Matlab.

Keywords: Lorentz space, Skew symmetric matrix, Vector field, Matlab.

Lorentz Uzaylarında Anti-simetrik Matrisler ve İntegral Eğrileri İçin Matlab Uygulamaları

ÖZET

[8] de, yazarlar $(2n+1)$ boyutlu Lorentz uzayında, A matrisi A lineer dönüşümüne karşılık gelen anti-simetrik matris olmak üzere, $A(x)=0$ denkleminin sıfırdan farklı çözümlerini ve x vektörünün kosul karakterlerine göre A anti-simetrik matrisinin normal formlarını elde ettiler. Bu çalışmada, A matrisinin yapısı göz önüne alınarak, anti-simetrik matrislerin normal formları için bazı Matlab uygulamaları ve Matlab kodları üretildi. Ayrıca bu uzayda çözümleri lineer vektör alanlarının integral eğrilerine karşılık gelen birinci mertebeden lineer diferansiyel denklem sistemleri için Matlab kodları verildi. Dahası özel durumlar ve x vektörünün kosul karakterlerine göre Matlab uygulamaları yapıldı.

Anahtar Kelimeler: Lorentz uzay, Anti-simetrik matris, Vektör alanı, Matlab.

I. INTRODUCTION

Lorentzian Geometry, which is simultaneously the geometry of special relativity, has an important role in differential geometry. The structural and characteristic differences of Lorentzian geometry has attracted the attention of many researches. So, there is a lot of literature dealing with the geometry of vectors, curves and surfaces with respect to their causal character in Lorentzian geometry.

We interest to normal forms of skew-symmetric matrices and vector fields on the Lorentzian space E_1^{2n+1} that are linear with respect to a chosen linear map $A: E_1^{2n+1} \rightarrow E_1^{2n+1}$. Note that when $n=3$ and the index is zero, such vector fields can be specified by integral curves. We recall that an integral curve is a parametric curve that represents a specific solution to an ordinary differential equation or system of equations. If the differential equation is represented as a vector field, then the corresponding integral curves are tangent to the field at each point. Such curves could represent the histories of small text particles, in which case they would be geodesics, or they might represent the flow lines of a fluid [2]. Integral curves are called by various other names, depending on the nature and interpretation of the differential equation or the vector field. In physics, such curves for an electric field or magnetic field are known as field lines, and for the velocity field of a fluid are known as streamlines. In dynamical systems, the integral curves for a differential equation that governs a system are referred to as trajectories or orbits [4].

The purpose of this paper is to transfer the normal forms of skew-symmetric matrices and system of differential equations obtained by Turhan and Ayyıldız in Lorentzian $(2n+1)$ -spaces into computer environment with the aid of MATLAB (short for MATrix LABoratory) program. This software is a special and important computer program optimized to perform engineering and scientific calculations [2, 5]. Also, the solutions of system of differential equations are given with the aid of such a program. So, we think that this program simplifies the works related with integral curves and skew-symmetric matrices in $(2n+1)$ -dimensional Lorentzian space.

We first recall some general notions and notations needed throughout the paper, and repeat some of the definitions mentioned in the introduction more formally. Section 3 deals with some applications on MATLAB program for the skew-symmetric matrices in such a space. Section 4 deals with some applications for the system of differential equations and their solutions. Also this section contains some examples with respect to specific values of n .

II. PRELIMINARIES

The Lorentzian space $(E^{2n+1}, \langle, \rangle) = E_1^{2n+1}$ is the $(2n+1)$ -dimensional vector space E^{2n+1} endowed with the pseudo scalar product

$$\langle v, w \rangle = -v_1 w_1 + \sum_{i=2}^{2n+1} v_i w_i \quad (1)$$

where $v = (v_1, v_2, \dots, v_{2n+1})$, $w = (w_1, w_2, \dots, w_{2n+1})$ in E_1^{2n+1} . We say that the vector $v \in E_1^{2n+1}$ is spacelike, lightlike or timelike if $\langle v, v \rangle > 0$ or $v = 0$, $\langle v, v \rangle > 0$ and $v \neq 0$, and $\langle v, v \rangle < 0$, respectively, [6]. The norm of a vector $v \in E_1^{2n+1}$ is defined by $\|v\| = \sqrt{|\langle v, v \rangle|}$.

The signature matrix S in $(2n+1)$ -dimensional Lorentzian space E_1^{2n+1} is the diagonal matrix whose diagonal entries are $s_1 = -1$ and $s_2 = s_3 = \dots = s_{2n+1} = +1$. We call that A is a skew-symmetric matrix in $(2n+1)$ -dimensional Lorentzian space if its transpose satisfies the equation $A^T = -SAS$, [6].

Let X be a vector field on E_1^{2n+1} . By an integral curve of the vector field X we understand a curve $\alpha : (a, b) \rightarrow E_1^{2n+1}$ such that its every tangent vector belongs to the vector field X . If $\frac{d\alpha}{dt} = X_{\alpha(t)}$, $\forall t \in I$, is satisfied, then the curve α is called an integral curve of the vector field X . A vector field X on E_1^{2n+1} is called linear if $X_v = -(\mathbf{SAS})(v)$ for all $v \in E_1^{2n+1}$, where \mathbf{A} is a linear mapping from E_1^{2n+1} into E_1^{2n+1} and \mathbf{S} is a linear mapping corresponding to matrix S , [7, 8].

A frame field $\emptyset = \{u_1, \dots, u_n, \dots, u_{2n}, u_{2n+1}\}$ on E_1^{2n+1} is called a pseudo orthonormal frame field, [6], if

$$\begin{aligned} \langle u_{2n}, u_{2n} \rangle &= -\langle u_{2n+1}, u_{2n+1} \rangle = -1, \langle u_{2n}, u_{2n+1} \rangle = 0, \\ \langle u_{2n}, u_i \rangle &= \langle u_{2n+1}, u_i \rangle = 0, \langle u_i, u_j \rangle = \delta_{ij}, \quad i, j = 1, \dots, n. \end{aligned} \quad (2)$$

Definition 2.1. Let $\alpha(s)$, s being the arclength parameter, be a non-null regular curve in semi-Euclidean space E_v^{2n+1} . The changing of a pseudo orthonormal frame field $\{u_1, \dots, u_n, \dots, u_{2n}, u_{2n+1}\}$ of E_v^{2n+1} along α is given by

$$\begin{aligned} u_1'(s) &= \kappa_1(s)u_2(s) \\ u_i'(s) &= -\varepsilon_{i-1}\varepsilon_i\kappa_{i-1}(s)u_{i-1}(s) + \kappa_i(s)u_{i+1}(s), \quad 2 \leq i \leq 2n, \\ u_{2n+1}'(s) &= -\varepsilon_{2n}\varepsilon_{2n+1}\kappa_{2n}(s)u_{2n}(s). \end{aligned} \quad (3)$$

These equations are called the Frenet-Serret type formulae for $\alpha(s)$, where $\kappa_i(s)$, $1 \leq i \leq 2n$, is the curvature function of α , $\kappa_i(s) = \varepsilon_{i+1} \langle u_i'(s), u_{i+1}(s) \rangle$, and ε_i is the signature of the vector u_i , $1 \leq i \leq 2n$, [10].

III. MATLAB APPLICATIONS FOR SKEW-SYMMETRIC MATRICES

In this section, we give some applications on MATLAB program for the skew-symmetric matrices provide the citation to the paper by Turhan and Ayyıldız in Lorentzian $(2n+1)$ -spaces. For the non-zero solutions of the equation $A(x)=0$, $x \in E_1^{2n+1}$, in the Lorentzian space E_1^{2n+1} , where A is the skew-symmetric matrix corresponding to the linear map \mathbf{A} , the normal forms of the matrix A with respect to causal character of x can be written as:

Case 1: If the vector x is timelike, then we get;

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -L_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & L_1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & L_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & L_n \\ 0 & 0 & 0 & 0 & 0 & \cdots & -L_n & 0 \end{bmatrix}$$

where $L_i \in \mathbb{R} - \{0\}, 1 \leq i \leq n$, [8].

Case 2: If the vector x is spacelike, then we get;

$$\begin{bmatrix} 0 & L_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ L_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & L_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -L_2 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & L_n & 0 \\ 0 & 0 & 0 & 0 & 0 & -L_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $L_i \in \mathbb{R} - \{0\}, 1 \leq i \leq n$, [9].

Case 3: If the vector x is lightlike, then we get;

$$\begin{bmatrix} 0 & 0 & L_1 & L_2 & L_3 & \cdots & L_{2n-2} & L_{2n-1} \\ 0 & 0 & L_1 & L_2 & L_3 & \cdots & L_{2n-2} & L_{2n-1} \\ L_1 & -L_1 & 0 & -L_{2n} & -L_{2n+1} & \cdots & -L_{4n-4} & -L_{4n-3} \\ L_2 & -L_2 & L_{2n} & 0 & -L_{4n-2} & \cdots & -L_{6n-7} & -L_{6n-6} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ L_3 & -L_3 & L_{2n+1} & \lambda_{4n-2} & 0 & \cdots & -L_{8n-11} & -L_{8n-10} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{2n-2} & -L_{2n-2} & L_{4n-4} & L_{6n-7} & L_{8n-11} & \cdots & 0 & -L_{n(2n-1)} \\ L_{2n-1} & -L_{2n-1} & L_{4n-3} & L_{6n-6} & L_{8n-10} & \cdots & L_{n(2n-1)} & 0 \end{bmatrix}$$

where $L_i \in \mathbb{R} - \{0\}, 1 \leq i \leq n$, [8].

MATLAB loop structures and comparison blocks were used in this application which was formed according to characteristic features of these matrices. A screenshot of this application is given below:

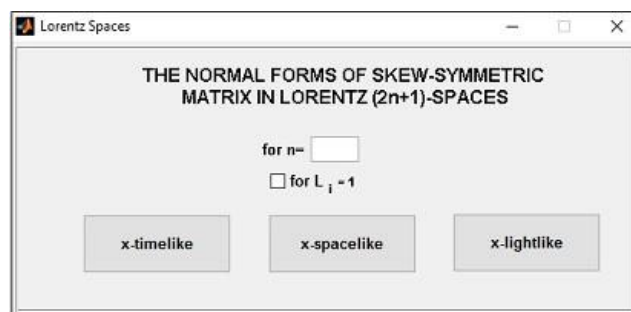


Figure 1. The screenshot for characteristic features of the matrices

For example, if we get $n = 3$, produced matrices with respect to causal character of x are given, respectively;

$$\begin{bmatrix} '0' & '0' & '0' & '0' & '0' & '0' & '0' \\ '0' & '0' & '-L1' & '0' & '0' & '0' & '0' \\ '0' & 'L1' & '0' & '0' & '0' & '0' & '0' \\ '0' & '0' & '0' & '0' & 'L2' & '0' & '0' \\ '0' & '0' & '0' & '-L2' & '0' & '0' & '0' \\ '0' & '0' & '0' & '0' & '0' & '0' & 'L3' \\ '0' & '0' & '0' & '0' & '0' & '-L3' & '0' \end{bmatrix}$$

Figure 2. The matrix for the timelike vector x

$$\begin{bmatrix} '0' & 'L1' & '0' & '0' & '0' & '0' & '0' \\ 'L1' & '0' & '0' & '0' & '0' & '0' & '0' \\ '0' & '0' & '0' & 'L2' & '0' & '0' & '0' \\ '0' & '0' & '-L2' & '0' & '0' & '0' & '0' \\ '0' & '0' & '0' & '0' & '0' & 'L3' & '0' \\ '0' & '0' & '0' & '0' & '-L3' & '0' & '0' \\ '0' & '0' & '0' & '0' & '0' & '0' & '0' \end{bmatrix}$$

Figure 3. The matrix for the spacelike vector x

$$\begin{bmatrix} '0' & '0' & 'L1' & 'L2' & 'L3' & 'L4' & 'L5' \\ '0' & '0' & 'L1' & 'L2' & 'L3' & 'L4' & 'L5' \\ 'L1' & '-L1' & '0' & '-L6' & '-L7' & '-L8' & '-L9' \\ 'L2' & '-L2' & 'L6' & '0' & '-L10' & '-L11' & '-L12' \\ 'L3' & '-L3' & 'L7' & 'L10' & '0' & '-L13' & '-L14' \\ 'L4' & '-L4' & 'L8' & 'L11' & 'L13' & '0' & '-L15' \\ 'L5' & '-L5' & 'L9' & 'L12' & 'L14' & 'L15' & '0' \end{bmatrix}$$

Figure 4. The matrix for the lightlike vector x

Meanwhile, if the checkbox is set, the value of $L_i \in \mathbb{R} - \{0\}$ is assumed as 1 and the matrices are produced for this value. A part of the code block which produce matrix for $L_i \in \mathbb{R} - \{0\}$, n parameters and the timelike vector x is as below:

```
n=str2num(get(handles.edit1,'String'));
n=(n*2)+1;
lvalue=get(handles.checkbox1,'value');
for i=1:n
    for j=1:n
        matrix(i,j)={'0'};
    end
end
if(lvalue==1)
    for i=1:fix(n/2)
        if i==1
            matrix(2*i,2*i+1)={'-1'};
            matrix(2*i+1,2*i)={'1'};
        else
            matrix(2*i,2*i+1)={'1'};
            matrix(2*i+1,2*i)={'-1'};
        end
        i=i+1;
    end
end
```

Figure 5. A part of the code block for the timelike vector x

IV. MATLAB APPLICATIONS FOR INTEGRAL CURVES

Let X be a linear vector field in E_1^{2n+1} determined by the matrix $\begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix}$ with respect to a pseudo-orthonormal frame $\{0; u_1, u_2, \dots, u_{2n+1}\}$, where A is a normal formed skew-symmetric matrix and C is a $(2n+1) \times 1$ column matrix such that

$$C = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2n-1} \\ a_{2n} \\ a_{2n+1} \end{bmatrix}.$$

The screenshot of the MATLAB program written for the differential equation system which gives integral curves of this linear vector field is given below:

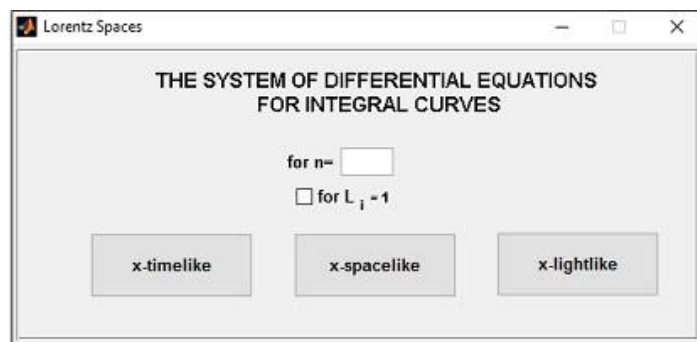


Figure 6. A screenshot for the differential equation system

Produced differential equation systems for $(n = 3)$ and the timelike vector x is given below,

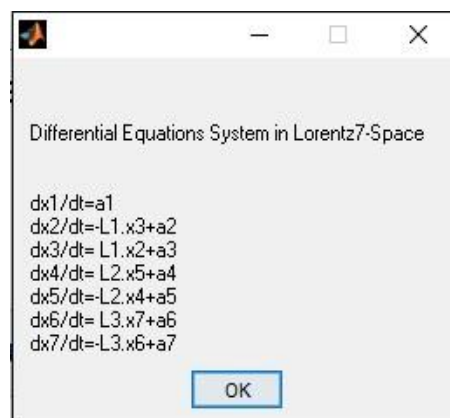


Figure 7. A screenshot for the differential equation system for the timelike vector x

A part of the code block which produces the differential equation system with respect to the timelike vector x is as below:

```
[c,d]=size(matrix);
for i=1:d-1
st=(strcat('dx',num2str(i),'dt=',matrix(1,i)));
for j=2:c
if not(isempty(matrix(j,i)))
if strcmp(matrix(j,i){1},'')==0
st=strcat(st,'+',matrix(j,i));
elseif strcmp(matrix(j,i){1},'')==1
st=strcat(st,matrix(j,i));
end
end
end
if length(st{1})<=8
st=strcat(st,'0');
end
str(i,1)=st;
end
[e,f]=size(str);
for temp=e:-1:1
str(temp+3,1)=str(temp,1);
end
str(1,1)={msg}; str(2,1)={' '}; str(3,1)={' '};
msgbox(str)
```

Figure 8. A part of the code block for differential equation for the timelike vector x .

If the checkbox is set in order to facilitate solving differential equation systems, the value of $L_i \in \mathbb{R} - \{0\}$ is assumed as 1. For instance, if we select $n = 3$, the following differential equation systems and its solution is produced by clicking x -spacelike button.

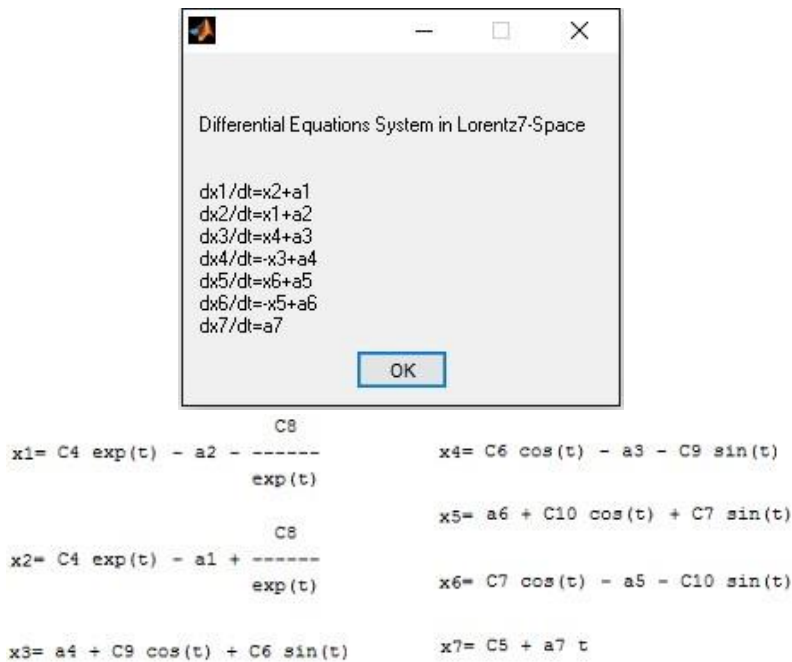


Figure 9. A screenshot for the differential equation system and its solution for the spacelike vector x

The program not only produces solutions of differential equation systems but also their graphs for $n = 1$. For example, the solution and graph for the x -spacelike is given as

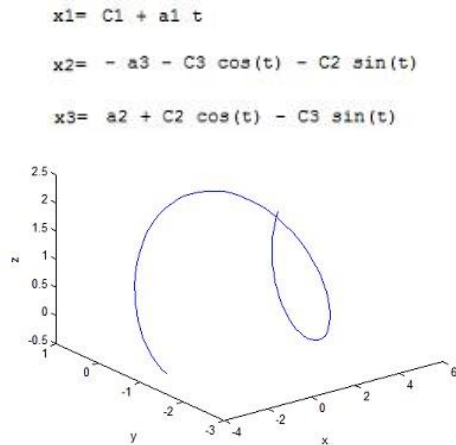


Figure 10. The solution for the differential equation system and its graph for the spacelike vector x , respectively,

Similarly, the solution and graph for the x -lightlike is given as

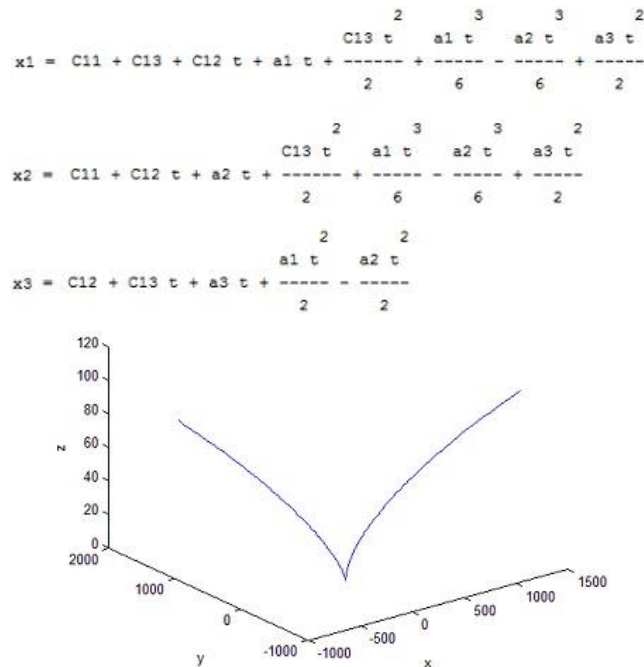


Figure 11. The solution for the differential equation system and its graph for the lightlike vector x , respectively,

Moreover, it is possible to classify integral curves of linear vector fields according to the rank of $[AC]$ matrix, [3]. The screenshot of the MATLAB programme which gives differential equation systems according to the rank of $[AC]$ matrix is as below:

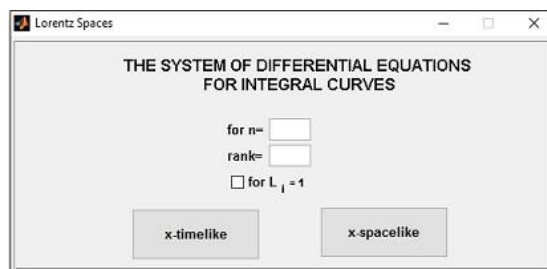


Figure 12. A screenshot for the matrix $[AC]$

If we use x -spacelike button with parameter $n = 3$ and $rank = 5$, the programme output for matrix

$\begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix}$ and differential equation system are as below:

$$\begin{bmatrix} '0' & 'L1' & '0' & '0' & '0' & '0' & '0' & 'a1' \\ 'L1' & '0' & '0' & '0' & '0' & '0' & '0' & 'a2' \\ '0' & '0' & '0' & 'L2' & '0' & '0' & '0' & 'a3' \\ '0' & '0' & '-L2' & '0' & '0' & '0' & '0' & 'a4' \\ '0' & '0' & '0' & '0' & '0' & '0' & '0' & 'a5' \\ '0' & '0' & '0' & '0' & '0' & '0' & '0' & '0' \\ '0' & '0' & '0' & '0' & '0' & '0' & '0' & '0' \\ '0' & '0' & '0' & '0' & '0' & '0' & '0' & '1' \end{bmatrix}$$

Figure 13. The matrix $[AC]$ for the spacelike vector x

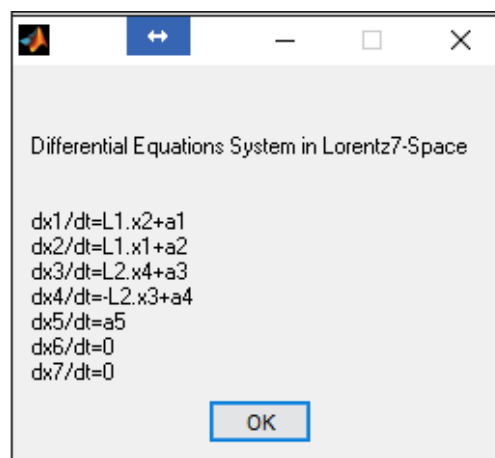


Figure 14. The differential equation system for the spacelike vector x and the rank $[AC]=5$

Also, if the checkbox is set with parameters $n = 3$ and $rank = 5$, the programme outputs for the matrix of the linear vector field and differential equation system are as below:

$$\begin{bmatrix} '0' & '1' & '0' & '0' & '0' & '0' & '0' & 'a1' \\ '1' & '0' & '0' & '0' & '0' & '0' & '0' & 'a2' \\ '0' & '0' & '0' & '1' & '0' & '0' & '0' & 'a3' \\ '0' & '0' & '-1' & '0' & '0' & '0' & '0' & 'a4' \\ '0' & '0' & '0' & '0' & '0' & '0' & '0' & 'a5' \\ '0' & '0' & '0' & '0' & '0' & '0' & '0' & '0' \\ '0' & '0' & '0' & '0' & '0' & '0' & '0' & '0' \\ '0' & '0' & '0' & '0' & '0' & '0' & '0' & '1' \end{bmatrix}$$

Figure 15. The matrix $[AC]$ for the spacelike vector x with aid of checkbox

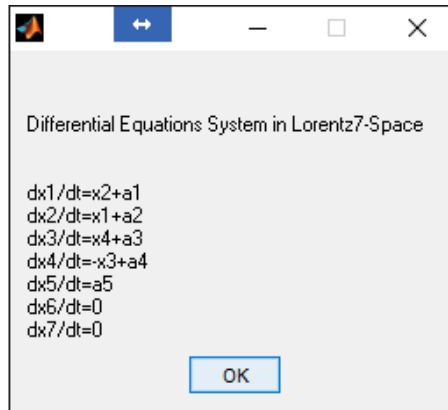


Figure 16. The differential equation system for the spacelike vector x with aid of checkbox

A part of the code block which produces the matrix $\begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix}$ with respect to the rank of the matrix $[AC]$ and the spacelike vector x is as below:

```
[a,b]=size(matrix);
if rank==1
    for i=1:a
        for j=1:b-1
            matrix(i,j)={'0'};
        end
    end
elseif rank>1
    if mod(rank,2)==0
        for i=(rank+1):(a-1)
            for j=1:b
                matrix(i,j)={'0'};
            end
        end
    end
end
end
```

Figure 17. A part of the code block for the rank of the matrix $[AC]$ for the spacelike vector x .

V. CONCLUSION

This work gives and develops the Matlab codes for the normal form of skew-symmetric matrices and system of differential equations in Lorentzian $(2n+1)$ -spaces. So, this study may shed light on future work between differential geometry and computer sciences.

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