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B-LIFT CURVES AND INVOLUTE CURVES IN LORENTZIAN 3-SPACE

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Abstract. The involute of a curve is often called the perpendicular trajectories of the tangent vectors of a unit speed curve. Furthermore, the B-Lift curve is the curve acquired by combining the endpoints of the binormal vectors of a unit speed curve. In this study, we investigate the correspondences between the Frenet vectors of a curve's B-lift curve and its involute. We also give an illustration of a helix that resembles space in Lorentzian 3-space and show how to visualize these curves by deriving the B-Lift curve and its involute.

1. Introduction

The Lorentz-Minkowski space was expressed in a special metric by the German mathematician Hermann Minkowski in 1907. Unlike the Euclidean space, this space has a temporal dimension. Studies in the Lorentzian space have many physical applications. For example, Lorentzian space is used to formalize Einstein's relativity theory. The character of a vector in Lorentzian space is also defined as spacelike, timelike or lightlike (null).

C. Huygens carried out the curvature of the plane curves at any point in Euclidean space. Sir Isaac Newton defined the curve depending on a parameter and expressed the curvature of the curve. The differential geometry of curves in Euclidean or Lorentzian spaces has been the subject of numerous investigations. [1–9]. Especially at the mutual point of the two curves, new ideas were put forward by establishing connections between Frenet operators. Involute curves and natural lift curves are some of them.

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The involute of a curve is generally referred to as the orthogonal trajectories of the tangent vectors of a unit speed curve. In 1668, the idea of involute curves was first discovered by C. Huygens in optical studies. Afterward, Millman and Parker (1977) [10] and Hacısalihoğlu (1983) [11] clarified the known theorems and results. A basic study on the involute-evolute curves was examined by Çalışkan and Bilici in 2002 [12].They looked into the relationship between the main curve's Frenet operators and its involute curve. They also introduced some important results in 2009 [13], such as curvature and torsion for involute curves, Frenet vectors of non-null curves in Lorentzian space.

By definition, a natural lift curve is created by joining the ends of a unit speed curve's tangent vectors. [14]. The natural lift curve has been investigated by many mathematicians [15–19]. In [18], the authors identified the correlations between the Frenet vectors of the natural lift curve and the main curve. They also gave the characterizations between the natural lift and involute of a curve [19].

In this article, we present the relationships between the B-Lift curve and the involute curve's Frenet vectors in Minkowski 3-space. In this context, the results show that the Frenet vectors of the B-Lift curve and the involute curves are the same; only their signs are different. Additionally, we illustrate our curves and provide an example based on these findings.

2. Preliminaries

The real vector space $\mathbb{R}^3$ that is supplied with a Lorentzian inner product is known as the Lorentzian 3-space $\mathbb{R}^3_1$ and is defined as

$$\langle x, y\rangle_{\mathbb{L}} = -x_1y_1 + x_2y_2 + x_3y_3$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are in $\mathbb{R}^3$ [20].

Let $x = (x_1, x_2, x_3)$ be a vector in $\mathbb{R}^3_1$. Then, x is considered timelike if $\langle$x, x$\rangle < 0$, lightlike if $\langle$x, x$\rangle = 0$ and x $\neq 0$, spacelike if $\langle$x, x$\rangle > 0$ or x=0 [20].

If $\gamma^{'}(s)$ is timelike, lightlike, or spacelike at any $s \in I$, then a curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3_1$ is either timelike, lightlike, or spacelike, respectively. Using the Lorentzian inner product, the norm of the vector $x = (x_1, x_2, x_3)$ is defined as [20]

$$\|x\|_{\mathbb{L}}= \sqrt{|\langle x, x\rangle|}.$$

If $\|x\|_{\mathbb{L}}$=1, the vector x is called a unit vector. The definition of the Lorentzian vector product of the vectors x and y for the vectors x and y in $\mathbb{R}^3_1$ is [21]

$$x \times y = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Assume that γ is a unit speed curve. Given by tangent, primary normal, and binormal vectors, respectively, the set $\{T(s), N(s), B(s)\}$ is known as the Frenet frame. For any unit speed curve γ, the Darboux vector represented by W, and we call $W(s) = \tau(s)T(s)+\kappa(s)B(s)$. Let θ be the angle formed by the binormal vector B and the Darboux vector W, then we have

$$\kappa = \|W\|\cos\theta, \tau = \|W\|\sin\theta.$$

We now look at Frenet-Serret formulas based on the curve's Lorentzian characteristics [22]:

i) Suppose that γ is a unit speed spacelike curve and B is a spacelike vector. As a result, N is a timelike vector, while T and B are spacelike vectors. In that condition, we have:

$$N \times B = -T, \quad T \times N = -B, \quad B \times T = -N.$$

The Frenet-Serret formulas follow as

$$\begin{aligned} T^{'} &= \kappa N, \\ N^{'} &= \kappa T + \tau B, \\ B^{'} &= \tau N. \end{aligned}$$

ii) Assume that γ is unit speed spacelike curve and B is a timelike vector. Then, T and N are spacelike vectors, B is a timelike vector. In that case, we can write

$$N \times B = -T, \quad T \times N = B, \quad B \times T = -N.$$

Here are the Frenet-Serret formulas

$$\begin{aligned} T^{'} &= \kappa N, \\ N^{'} &= -\kappa T + \tau B, \\ B^{'} &= \tau N. \end{aligned}$$

iii) Assume that γ is a unit speed timelike curve. Then, N and B are spacelike vectors and T is a timelike vector. In that case, we have

$$N \times B = T, \quad T \times N = -B \quad B \times T = -N.$$

Here are the Frenet-Serret formulas

$$\begin{aligned} T^{'} &= \kappa N, \\ N^{'} &= \kappa T + \tau B, \\ B^{'} &= -\tau N. \end{aligned}$$

Lemma 1 ([23])**.** *Assume that x and y are linearly independent spacelike vectors that span a spacelike vector subspace in $\mathbb{R}^3_1$. In that case, we get the following inequality:*

$$|\langle x, y\rangle| \leq \|x\|_{\mathbb{L}} \cdot \|y\|_{\mathbb{L}}.$$

Hence we can write

$$\langle x, y\rangle = \|x\|_{\mathbb{L}} \cdot \|y\|_{\mathbb{L}} \cos\varphi,$$

where the angle amongst x and y is φ.

Lemma 2 ([23]). *Assume that x and y are linearly independent spacelike vectors that span a timelike vector subspace in $\mathbb{R}_1^3$. Thus we get*

$$|\langle x, y\rangle| > \|x\|_{\mathbb{L}} \cdot \|y\|_{\mathbb{L}}.$$

Therefore we can write

$$|\langle x, y\rangle| = \|x\|_{\mathbb{L}} \cdot \|y\|_{\mathbb{L}} \cosh \varphi,$$

where the angle amongst x and y is φ

Lemma 3 ([23]). *Assume that x is a spacelike vector and y is a timelike vector in $\mathbb{R}_1^3$. In that condition, we can write*

$$|\langle x, y\rangle| = \|x\|_{\mathbb{L}} \cdot \|y\|_{\mathbb{L}} \sinh \varphi,$$

where the angle amongst x and y is φ

Lemma 4 ([23]). *Suppose that x and y are timelike vectors in $\mathbb{R}_1^3$. In that case, we can write*

$$\langle x, y\rangle = -\|x\|_{\mathbb{L}} \cdot \|y\|_{\mathbb{L}} \cosh \varphi,$$

where the angle amongst x and y is φ

Definition 1 ([19]). *Let $\gamma = (\gamma(s); T(s), N(s), B(s))$ and $\gamma^* = (\gamma^*(s^*); T^*(s^*), N^*(s^*), B^*(s^*))$ are regular curves in $\mathbb{R}_1^3$. $\gamma^*(s^*)$ is called the involute of $\gamma(s)$ ($\gamma(s)$ is called the evolute of $\gamma^*(s^*)$) if $\langle T(s), T^*(s^*)\rangle = 0$. In that case, (γ, γ^*) is called involute-evolute curve couple.*

Proposition 1 ([19]). *Assume that γ is a timelike curve. Then, γ^* is a spacelike curve and B^* is a timelike or spacelike vector. We are aware of the following equations connecting the Frenet frames $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ of curves γ and γ^*:*

i) *Assume that γ is a spacelike curve and B is a spacelike vector.*

a) *If W Darboux vector is timelike, then we can write*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sinh \varphi & 0 & \cosh \varphi \\ -\cosh \varphi & 0 & -\sinh \varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is spacelike, then we can write*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \cosh \varphi & 0 & \sinh \varphi \\ -\sinh \varphi & 0 & -\cosh \varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

ii) *Let γ be a spacelike curve and B be a timelike vector.*

a) *If W Darboux vector is timelike, then we can write*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\sinh \varphi & 0 & -\cosh \varphi \\ -\cosh \varphi & 0 & -\sinh \varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is spacelike, then we can write*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cosh\varphi & 0 & -\sinh\varphi \\ -\sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Proposition 2 ([19])**.** *Let γ be a spacelike curve and B be spacelike or timelike vector. Then γ^* is a spacelike curve. We know the following equations:*
i) *Let γ be a spacelike curve and B be spacelike vector.*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \cos\varphi & 0 & \sin\varphi \\ \sin\varphi & 0 & -\cos\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

ii) *Let γ be a spacelike curve and B be timelike vector.*
a) *If W Darboux vector is timelike, then we have*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \cosh\varphi & 0 & -\sinh\varphi \\ \sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is spacelike, then we have*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sinh\varphi & 0 & -\cosh\varphi \\ \cosh\varphi & 0 & -\sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Proposition 3 ([19])**.** *Assume that γ is a spacelike curve and B is a spacelike vector. Then γ^* is a spacelike curve and the following equations are available:*
i) *Let γ^* be a spacelike curve and B^* be a spacelike vector.*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sin\varphi & 0 & -\cos\varphi \\ -\cos\varphi & 0 & -\sin\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

ii) *Let γ^* be a spacelike curve and B^* be a timelike vector.*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sin\varphi & 0 & -\cos\varphi \\ \cos\varphi & 0 & \sin\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Proposition 4 ([19])**.** *Assume that γ is a spacelike curve and B is a timelike vector. In that case, γ^* is a spacelike curve and the following equations exist:*
i) *Suppose that γ^* is a spacelike curve and B^* is a spacelike vector.*
a) *If W Darboux vector is spacelike, then we have*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\sinh\varphi & 0 & \cosh\varphi \\ \cosh\varphi & 0 & -\sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is timelike, then we have*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cosh\varphi & 0 & \sinh\varphi \\ \sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

ii) *Suppose that γ^* is a spacelike curve and B^* is a timelike vector.*
a) *If W Darboux vector is spacelike, then we have*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sinh\varphi & 0 & -\cosh\varphi \\ \cosh\varphi & 0 & -\sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is timelike, then we have*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \cosh\varphi & 0 & -\sinh\varphi \\ \sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Definition 2 ([24]). *If $\gamma : I \to P$ is a unit speed curve, then $\gamma_B : I \to TP$ is known as the B-Lift curve and guarantees the following equation:*

$$\gamma_B(s) = (\gamma(s), B(s)) = B(s)|_{\gamma(s)}, \tag{1}$$

where $P \subset \mathbb{R}^3_1$ is a surface.

3. Involute Curves and B-Lift Curves in Minkowski 3-Space

Proposition 5. *Assume that γ is a timelike curve. Then, γ_B is a spacelike curve and B is spacelike or timelike.*

i) *Suppose that γ_B is a spacelike curve and B_B is timelike vector. The following equations are available:*

a) *If W Darboux vector is spacelike, we can write*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -\cosh\varphi & 0 & -\sinh\varphi \\ \sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is timelike, we can write*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -\sinh\varphi & 0 & -\cosh\varphi \\ \cosh\varphi & 0 & \sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

ii) *Assume that γ_B is a spacelike curve and B_B is spacelike vector. We are aware of the following equations connecting the Frenet frames $\{T_B, N_B, B_B\}$ and $\{T, N, B\}$ of curves γ_B and γ:*

a) *If W Darboux vector is spacelike, we know that*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ \cosh\varphi & 0 & \sinh\varphi \\ -\sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is timelike, we know that*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ \sinh\varphi & 0 & \cosh\varphi \\ \cosh\varphi & 0 & \sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Proposition 6. *Suppose that γ is a spacelike curve and B is a spacelike vector. Then, γ_B is a timelike curve. We know the following equations:*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ \cos\varphi & 0 & \sin\varphi \\ \sin\varphi & 0 & -\cos\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Proposition 7. *Suppose that γ is a spacelike curve and B is timelike vector. Then, γ_B is a spacelike curve and B_B is timelike or spacelike vector.*

i) *Let γ_B be a spacelike curve and B_B be a timelike vector.The following equations are available:*

a) *If W Darboux vector is spacelike, we have*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -\sinh\varphi & 0 & \cosh\varphi \\ \cosh\varphi & 0 & -\sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is timelike, we have*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -\cosh\varphi & 0 & \sinh\varphi \\ \sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

ii) *Let γ_B be a spacelike curve and B_B be spacelike vector. We have the following equations:*

a) *If W Darboux vector is spacelike, we have*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ \sinh\varphi & 0 & -\cosh\varphi \\ \cosh\varphi & 0 & -\sinh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

b) *If W Darboux vector is timelike, we have*

$$\begin{pmatrix} T_B \\ N_B \\ B_B \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ \cosh\varphi & 0 & \sinh\varphi \\ \sinh\varphi & 0 & -\cosh\varphi \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Corollary 1. *Assume that γ is a timelike curve. Then γ^* is a spacelike curve and B^* is spacelike vector.*

i) *If W Darboux vector is spacelike, then we get*

$$\begin{aligned} T^* &= -T_B, \\ N^* &= N_B, \\ B^* &= B_B. \end{aligned}$$

ii) *If W Darboux vector is timelike, then we get*

$$\begin{aligned} T^* &= -T_B, \\ N^* &= B_B, \\ B^* &= N_B. \end{aligned}$$

where $\{T_B, N_B, B_B\}$ is the Frenet frame of the curve γ_B.

Corollary 2. *Assume that γ is a timelike curve. Therefore γ^* is a spacelike curve and B^* is timelike vector.*
i) *If W Darboux vector is spacelike, then we get*

$$\begin{aligned} T^* &= -T_B, \\ N^* &= -B_B, \\ B^* &= N_B. \end{aligned}$$

ii) *If W Darboux vector is timelike, then we get*

$$\begin{aligned} T^* &= -T_B, \\ N^* &= N_B, \\ B^* &= -B_B, \end{aligned}$$

where $\{T_B, N_B, B_B\}$ is the Frenet frame of the curve γ_B.

Corollary 3. *Assume that γ is a spacelike curve and B is a spacelike vector. Then γ^* is a timelike curve.*
i) *If W Darboux vector is spacelike, then we have*

$$\begin{aligned} T^* &= -T_B, \\ N^* &= -B_B, \\ B^* &= N_B. \end{aligned}$$

ii) *If W Darboux vector is timelike, then we have*

$$\begin{aligned} T^* &= -T_B, \\ N^* &= N_B, \\ B^* &= -B_B, \end{aligned}$$

where $\{T_B, N_B, B_B\}$ is the Frenet frame of the curve γ_B.

Corollary 4. *Assume that γ is a spacelike curve and B is timelike vector. Then γ^* is a timelike curve.*

$$\begin{aligned} T^* &= -T_B, \\ N^* &= N_B, \\ B^* &= -B_B, \end{aligned}$$

where $\{T_B, N_B, B_B\}$ is the Frenet frame of the curve γ_B.

Corollary 5. *Assume that γ is a spacelike curve and B is spacelike vector.*
i) *If γ^* is spacelike curve and B^* is spacelike vector, hence we get*

$$\begin{aligned} T^* &= T_B, \\ N^* &= N_B, \\ B^* &= B_B. \end{aligned}$$

ii) *If γ^* is spacelike curve and B^* is timelike vector, hence we get*

$$\begin{aligned} T^* &= T_B, \\ N^* &= -N_B, \\ B^* &= B_B, \end{aligned}$$

where $\{T_B,\ N_B,\ B_B\}$ is the Frenet frame of the curve γ_B.

Corollary 6. *Assume that γ is a spacelike curve and B is timelike vector.*
i) *If γ and γ^* are spacelike curves with timelike binormal, then we get*

$$\begin{aligned} T^* &= T_B, \\ N^* &= -N_B, \\ B^* &= -B_B. \end{aligned}$$

ii) *If γ^* is spacelike curve and B^* is spacelike vector, then we get*

$$\begin{aligned} T^* &= T_B, \\ N^* &= N_B, \\ B^* &= B_B, \end{aligned}$$

where $\{T_B,\ N_B,\ B_B\}$ is the Frenet frame of the curve γ_B.

Corollary 7. *Let γ^* and γ_B be involute curve and B-Lift curve of a unit speed curve γ, respectively. Then, the sets $\{T^*, T_B\}$, $\{N^*, N_B\}$ and $\{B^*, B_B\}$ are linearly dependent.*

Example 1. *Suppose that the spacelike circular helix curve is given by*

$$\gamma(s) = (\frac{s}{\sqrt{3}}, 2\cos(\frac{s}{\sqrt{3}}), 2\sin(\frac{s}{\sqrt{3}})).$$

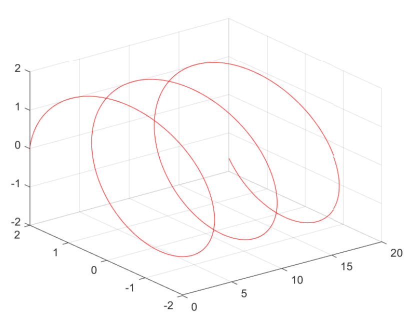

FIGURE 1. The spacelike helix curve $\gamma(s)$

For the spacelike helix curve γ, Frenet frames can be calculated by

$$\begin{aligned} T(s) &= (\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\sin(\frac{s}{\sqrt{3}}), \frac{2}{\sqrt{3}}\cos(\frac{s}{\sqrt{3}})), \\ N(s) &= (0, -\cos(\frac{s}{\sqrt{3}}), -\sin(\frac{s}{\sqrt{3}})), \\ B(s) &= (\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\sin(\frac{s}{\sqrt{3}}), \frac{1}{\sqrt{3}}\cos(\frac{s}{\sqrt{3}})). \end{aligned}$$

Then the B-lift curve is following as

$$\gamma_B(s) = (\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\sin(\frac{s}{\sqrt{3}}), \frac{1}{\sqrt{3}}\cos(\frac{s}{\sqrt{3}})).$$

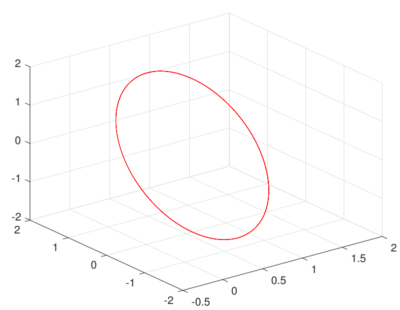

FIGURE 2. B-Lift curve of the curve $\gamma(s)$

For λ=$-\sqrt{3}$, the involute of the curve $\gamma(s)$ is given by

$$\begin{aligned} \gamma^*(s) &= \gamma(s) + \lambda.T(s) \\ &= (\frac{s}{\sqrt{3}}, 2\cos(\frac{s}{\sqrt{3}}), 2\sin(\frac{s}{\sqrt{3}})) + (-\sqrt{3}).(\frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\sin(\frac{s}{\sqrt{3}}), \frac{2}{\sqrt{3}}\cos(\frac{s}{\sqrt{3}})) \\ &= (\frac{s}{\sqrt{3}} - 1, 2(\cos(\frac{s}{\sqrt{3}}) + \sin(\frac{s}{\sqrt{3}})), 2(\sin(\frac{s}{\sqrt{3}}) - \cos(\frac{s}{\sqrt{3}}))) \end{aligned}$$

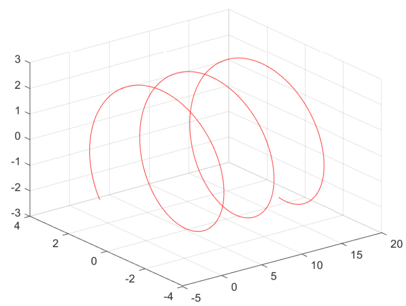

Figure 3. Involute curve of the curve $\gamma(s)$

4. Conclusions

In this study, the relations of a spacelike or timelike unit speed curve given in Minkowski-3 space with the B-Lift curve were examined. Furthermore, the equations relating the Frenet operators of the involute curve and the B-Lift curve were discovered. As a consequence, we may summarize the findings of this study as follows:

1. When the Frenet apparatus of the B-Lift curve of a unit speed curve are compared with the Frenet apparatus of the involute curve of a unit speed curve, it is shown that the Frenet vectors are similar; only their signs differ.

2. By giving an example, we obtained the B-Lift curve and the Frenet operators of the involute curve of a given curve and checked the results we found with the help of an example.

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