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Research Article

The Group of Transformations which Preserving Distance on Some Polyhedral Space

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Abstract

3-dimensional analytical space which is covered by a metric is called a Minkowski geometry. In the Minkowski geometries, the unit balls are symmetric, convex closed sets. So there are Minkowski geometries which unit spheres are rhombic triacontahedron, icosidodecahedron and disdyakis triacontahedron. One of the fundamental problems in geometry for a space with a metric is to determine the group of isometries. In this article we show that the group of isometries of the $3-$dimensional space covered by $RT-metric$, $ID-metric$ and $DT-metric$ are the semi-direct product of I_h and $T(3)$, where Icosahedral group I_h is the (Euclidean) symmetry group of the icosahedron and $T(3)$ is the group of all translations of the $3-$ dimensional space.

Keywords and 2020 Mathematics Subject Classification

Keywords: Catalan solids — Archimedean solids — isometry group — icosahedral symmetry — rhombic triacontahedron — icosidodecahedron — disdyakis triacontahedron

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1. Introduction

The history of man's interest in symmetry goes back many centuries. Symmetry is the primary matter of aesthetic thus it has been worked on, in various fields, for example in physics, chemistry, biology, art, architecture and of course in mathematics. Polyhedra have attracted the attention because of their symmetries. Consequently, polyhedra take place in many studies with respect to different fields [1, 2]. For each polyhedron, the faces of the polyhedron are the polygons that bound it; the edges are the line segments where the faces meet; the vertices are the points where edges meet. Just as regular polygons were the most "uniform" polygons possible, man wanted to find polyhedra that are as "uniform" as possible.

The idea of convexity is completely the same for polyhedra as for polygons. A polyhedron is convex if any two points in the polyhedron are joined by a line segment contained entirely in the polyhedron.

If a polygon has edges that have same lengths and all the angles are equal then it is said that it is regular. Similarly a convex polyhedron is a regular polyhedron if each face is regular polygon, all faces are identical and all vertices are identical, which means that all vertices are contained in the same number of faces. It is easy to see that in a regular polyhedron all edges must have the same length. Regular polyhedra are only five and they are called Platonic Solids, in the honor of ancient Greek philosopher Plato. If for a convex polyhedron each face is a regular polygon and all vertices are identical then it is called semi-regular polyhedron. Actually, it is obvious that not all faces need to be the same type of polygon. These polyhedra are called Archimedean Solids and they are thirteen.

Given a convex polyhedra, a new polyhedra can be formed, called its dual polyhedron as follows. First, for each face of the original polyhedron, a point in its interior is chosen (for example, the center of gravity of the face can be chosen). These chosen points are the vertices of the dual polyhedron, called dual vertices. Next, an edge in the original polyhedron is considered. This edge is contained in precisely two faces of the original polyhedron. Then an edge is put in the dual polyhedron joining two dual vertices that are contained in these two faces of the original polyhedron. Thus the edges of the dual polyhedron are obtained. Finally, a vertex in the original polyhedron is considered. This vertex is contained in some faces of the original polyhedron. Then a face is put in the dual polyhedron that has its vertices the dual vertices that are contained in these faces of the original polyhedron. Thus the faces of the dual polyhedron are obtained ([3]). The dual solids of Archimedean solids are called Catalan solids and they are thirteen just as Archimedean solids.

Minkowski geometry is a non-Euclidean geometry in a finite number of dimensions that is different from elliptic and hyperbolic geometry. In Minkowski geometry the linear structure is the same as the Euclidean one but distance is not uniform in all directions. Instead of the usual sphere in Euclidean space, the unit ball is a certain symmetric closed convex set ([4]). Each of the geometries induced by maximum, taxicab, Chinese Checkers, α_i and $\lambda-$distances is one of geometry of this type ([5–8]). Also there are Minkowski geometries which unit spheres are rhombic triacontahedron, icosidodecahedron and disdyakis triacontahedron [9, 10].

Transformations are detailed in Martin's book ([11]). A set of transformations is said to form a group if it contains the inverse of each and the product of any two (including the product of one with itself or with its inverse). The number of distinct transformations is called the order of the group (This may be either finite or infinite). Clearly the symmetry operations of any figure form a group. This is called the symmetry group of the figure. If the figure is completely irregular its symmetry group is of order one, consisting of identity alone ([12]).

Three essential methods geometric investigations; synthetic, metric and group approach. The group approach takes isometry groups of a geometry and convex sets plays an substantial role in indication of the group of isometries of geometries. Those properties are invariant under the group of motions and geometry studies those properties. There are a lot of studies about group of isometries of a plane or a space (See [13–24]). This problem enforce us to find group of isometries of spaces which unit spheres are rhombic triacontahedron, icosidodecahedron and disdyakis triacontahedron. Thus we show that the group of isometries of the 3−dimensional space covered by $RT-metric$, $ID-metric$ and $DT-metric$ are the semi-direct product of I_h and $T(3)$, where Icosahedral group I_h is the (Euclidean) symmetry group of the icosahedron and $T(3)$ is the group of all translations of the 3− dimensional space.

2. Preliminaries

Rhombic triacontahedron is a polyhedron which faces are rhombus shaped and the ratio of the long diagonal to the short diagonal of the each rhombus is exactly equal to the golden ratio φ. It has 30 faces, 60 edges and 32 vertices. Also rhombic triacontahedron is dual polyhedron of icosidodecahedron so the symmetry groups of them are the same. Icosidodecahedron is an Archimedean solid which has 32 faces, 30 vertices and 60 edges. Disdyakis triacontahedron is the solid which is a kleetope of rhombic triacontahedron and it has 120 faces each one is a scalene triangle. Thus, the symmetry groups of them are the same (see [25, 26]).

RT, ID and $DT-$metrics for 3−dimensional analytical space were introduced by authors in [9] and [10]. Now, the RT, ID and $DT-$metrics and some properties of them are given briefly by [9, 10]. Let $\mathbb{R}^3_{RT}$, $\mathbb{R}^3_{ID}$ and $\mathbb{R}^3_{DT}$ which are called rhombic triacontahedron, icosidodecahedron and disdyakis triacontahedron space denote 3-dimensional analytical space furnishing rhombic triacontahedron metric, icosidodecahedron metric and disdyakis triacontahedron metric, respectively. The rhombic triacontahedron 3-dimensional space $\mathbb{R}^3_{RT}$, the icosidodecahedron 3-dimensional space $\mathbb{R}^3_{ID}$ and the disdyakis triacontahedron 3-dimensional space $\mathbb{R}^3_{DT}$ are almost the same the Euclidean 3-dimensional space $\mathbb{R}^3$. The points, lines and planes are the same but the distance function is different. Each of them is a Minkowski geometry (for detailed information see [4]).

The taxicab (Manhattan) and the maximum (Chebyshev) norms are defined as $\|X\|_1 = |x| + |y| + |z|$ and $\|X\|_\infty = \max\{|x|, |y|, |z|\}$, respectively and they are special cases of l_p-norm; $\|X\|_p = (|x|^p + |y|^p + |z|^p)^{1/p}$, where $X = (x, y, z) \in \mathbb{R}^3$. Among l_p-metrics only crystalline metrics, i.e., metrics having polygonal unit balls are l_1- and $l_\infty-$ metrics [27]. First, we give some notions that will be used in the descriptions of distance functions we define. For $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$, M denotes $\|P_1 - P_2\|_\infty$ and S denotes $\|P_1 - P_2\|_1$. Moreover $X-Y-Z-X$ and $Z-Y-X-Z$ orientations are called positive (+) direction and negative (-) direction, respectively. M^+ and M^- express the next term in the respective direction according to M. For example, if $M = |x_1 - x_2|$, then $M^+ = |y_1 - y_2|$ and $M^- = |z_1 - z_2|$. The metrics for which the unit spheres are the rhombic triacontahedron, the icosidodecahedron and the disdyakis triacontahedron are defined as follows:

$$d_{RT}(P_1, P_2) = \max\left\{M \ , \ \frac{\varphi}{2}M + \frac{\varphi-1}{2}M^+ + \frac{1}{2}M^-\right\},$$

$$d_{ID}(P_1, P_2) = \max\left\{M + (\varphi-1)M^+ \ , \ M + (\varphi-1)^2 M^- \ , \ (\varphi-1)S\right\}$$

and

$$d_{DT}(P_1,P_2)=\max\left\{\begin{array}{l} M+\frac{4\varphi-5}{11}\left(M^{+}+M^{-}\right),\ \frac{5\varphi+2}{11}M+\frac{8\varphi-10}{11}M^{+}+\frac{3\varphi-1}{11}M^{-} \\ \frac{\varphi+7}{11}M+\frac{7\varphi-6}{11}M^{+}+\frac{4-\varphi}{11}M^{-},\ \frac{6\varphi-2}{11}M+\frac{3\varphi-1}{11}M^{+}+\frac{12-3\varphi}{11}M^{-} \\ \frac{10\varphi-7}{11}M+\frac{4\varphi-5}{11}M^{+}+\frac{2\varphi+3}{11}M^{-} \end{array}\right\},$$

where φ=$\frac{1+\sqrt{5}}{2}$ is golden ratio.

Readers who are wondering how metrics are found can refer to references [9] and [10]. Here, the metrics are adapted only to M, M^{+} and M^{-} notations, and they are as seem to different from according to references [9] and [10], indeed they are the same.

According to definition of $d_{RT}-$distance, there are two possible ways for the shortest paths between the points P_1 and P_2 as shown in Figure 1. Paths are determined according to which of the quantitatives in the relevant metric will be maximum. So, according to $d_{RT}-$distance, there are two possible paths. These paths are;

i. A line segment which is parallel to one of the coordinate axes,

ii. Union of three line segments which one is parallel to one of the coordinate axes and the two others making $\arctan(1/2)$ and $\arctan(\sqrt{5}/2)$ radians angle with one of the other coordinate axes.

Thus, the shortest $d_{RT}-$distance between P_1 and P_2 is either the Euclidean length of such a line segment or sum of the Euclidean length of such three line segments. Figure 1 shows the rhombic triacontahedron way from P_1 to P_2.

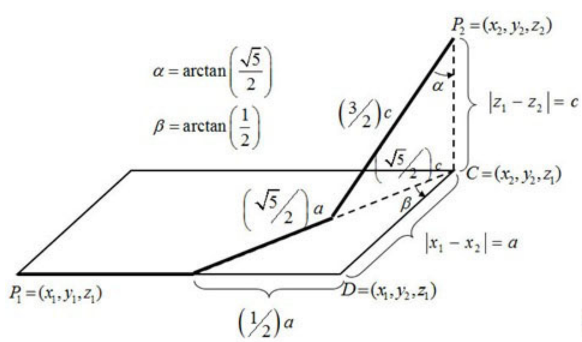


Fig. 1. The shortest paths between the points P_1 and P_2 according to $d_{RT}-$distance

By a similar discussion icosidodecahedron and disdyakis triacontahedron ways from P_1 to P_2 are would easily be considered. To the definition of $d_{ID}-$distance there are three possible ways for the shortest paths between the points P_1 and P_2 as shown in Figure 2.

i. Union of two line segments which one is parallel to one of the coordinate axis and the other making $\arctan(\sqrt{5}/2)$ radians angle with one of the coordinate axis.

ii. Union of two line segments which one is parallel to one of the coordinate axis and the other making $\arctan(1/2)$ radians angle with one of the coordinate axis.

iii. Union of three line segments which each one is parallel to a coordinate axis.

Thus, the shortest $d_{ID}-$distance between P_1 and P_2 is for parts i and ii sum of the Euclidean lengths of two line segments, for part iii $\frac{\sqrt{5}-1}{2}$ times of sum of the Euclidean lengths of mentioned three line segments.

To the definition of $d_{DT}-$distance there are five possible ways for the shortest paths between the points P_1 and P_2 as shown in Figure 3.

i. Union of three line segments which one is parallel to one of the coordinate axis and the other two are making $\arctan\left(\frac{10\sqrt{5}+18}{11}\right)$ radians angle with one of the coordinate axis.

ii. Union of three line segments which one is parallel to one of the coordinate axis and the two others making $\arctan\left(\frac{\sqrt{5}}{2}\right)$ and $\arctan\left(\frac{3\sqrt{5}}{2}\right)$ radians angle with one of the other coordinate axis.

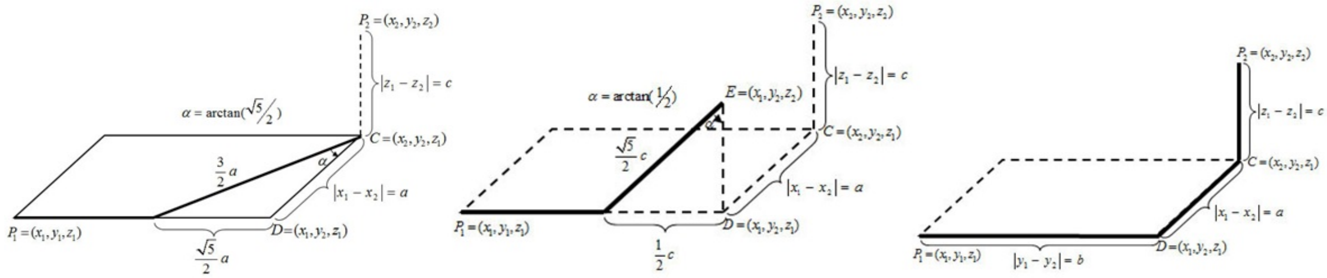


Fig. 2. The shortest paths between the points P_1 and P_2 according to $d_{ID}-$distance

iii. Union of three line segments which one is parallel to one of the coordinate axis and the two others making $\arctan\left(\frac{1}{2}\right)$ and $\arctan\left(\frac{5+9\sqrt{5}}{40}\right)$ radians angle with one of the other coordinate axis.

iv. Union of three line segments which one is parallel to one of the coordinate axis and the two others making $\arctan\left(\frac{3}{4}\right)$ and $\arctan\left(\frac{13-5\sqrt{5}}{24}\right)$ radians angle with one of the other coordinate axis.

v. Union of three line segments which one is parallel to one of the coordinate axis and the two others making $\arctan\left(\frac{10\sqrt{5}-18}{11}\right)$ and $\arctan\left(\frac{10+3\sqrt{5}}{11}\right)$ radians angle with one of the other coordinate axis.

Thus, the shortest $d_{DT}-$distance between P_1 and P_2 is for part i sum of the Euclidean lengths of three line segments, for part ii $\frac{9+5\sqrt{5}}{22}$ times of sum of the Euclidean lengths of mentioned three line segments, for part iii $\frac{15+\sqrt{5}}{22}$ times of sum of the Euclidean lengths of the three line segments, for part iv $\frac{1+3\sqrt{5}}{11}$ times of sum of the Euclidean lengths of the three line segments and for part v $\frac{5\sqrt{5}-2}{11}$ times of sum of the Euclidean lengths of the three line segments. Readers can also refer to reference [29] for metrics, paths, and more detailed information.

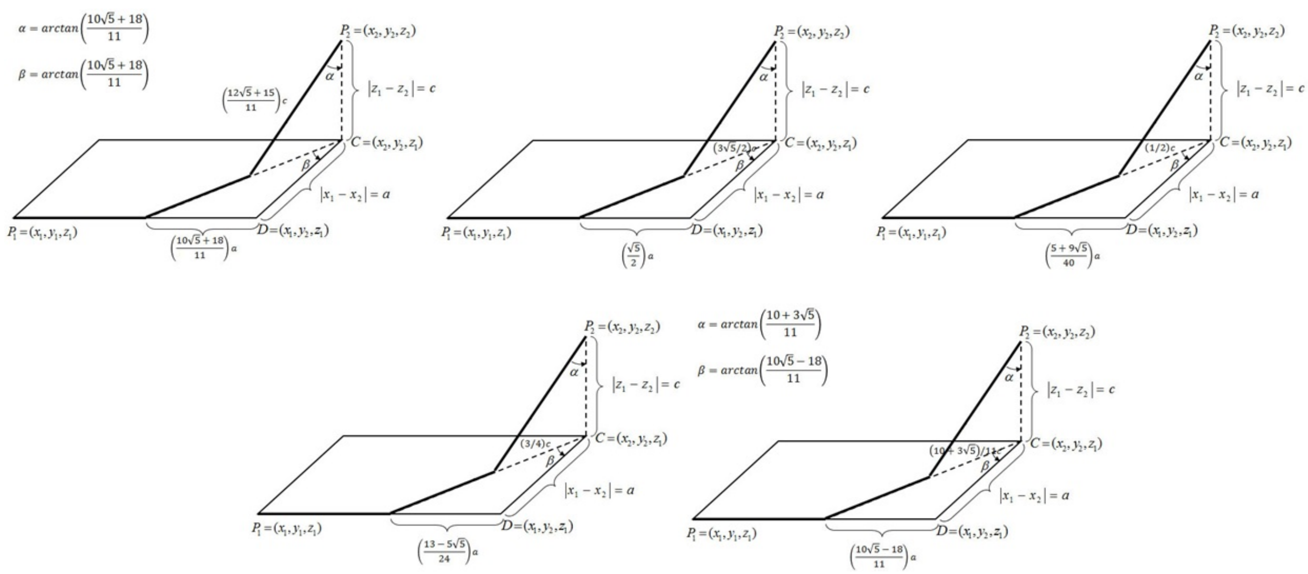
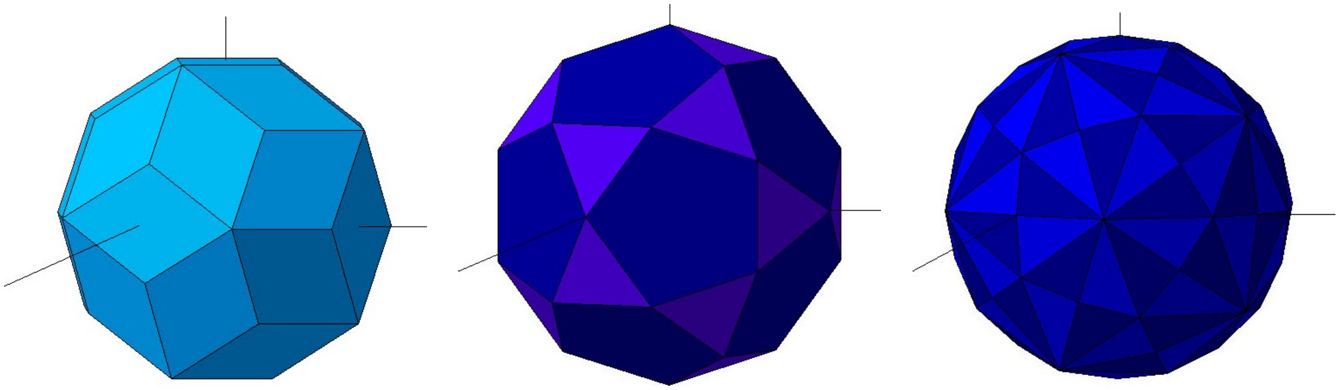


Fig. 3. The shortest paths between the points P_1 and P_2 according to $d_{DT}-$distance

Let $M_0=\|X-X_0\|_\infty$ for $X=(x,y,z)$ and $X_0=(x_0,y_0,z_0)$. A rhombic triacontahedron sphere with center (x_0,y_0,z_0) and radius r in $\mathbb{R}^3_{RT}$ is the set of points (x,y,z) in the 3$-$dimensional space satisfying the equation

$$\max\left\{M_0\ ,\ \frac{\varphi}{2}M_0+\frac{\varphi-1}{2}M_0^{+}+\frac{1}{2}M_0^{-}\right\}=r,$$



(a) (b) (c)

Fig. 4. a) The sphere with center O and radius r in $\mathbb{R}^3_{RT}$, b) The sphere with center O and radius r in $\mathbb{R}^3_{ID}$, c) The sphere with center O and radius r in $\mathbb{R}^3_{DT}$

which is a polyhedron with 30 faces and 32 vertices. Coordinates of the vertices are translations to (x_0, y_0, z_0) all possible $+/-$ signals of components of the points $((\varphi-1)r, 0, r)$, $(r, (\varphi-1)r, 0)$, $(0, r, (\varphi-1)r)$, $(0, (2-\varphi)r, r)$, $(r, 0, (2-\varphi)r)$, $((\varphi-1)r, (\varphi-1)r, (\varphi-1)r)$, $((2-\varphi)r, r, 0)$. Figure 4 (a) shows the $RT-$sphere centered at $O=(0,0,0)$.

Similarly, coordinates of the vertices of an icosidodecahedron sphere with radius r and center (x_0, y_0, z_0) in $\mathbb{R}^3_{ID}$ are translations to (x_0, y_0, z_0) all possible $+/-$ signals of components of the points, so its vertices are $(0,0,r)$, $(r,0,0)$, $(0,r,0)$, $\left(\frac{\varphi-1}{2}r, \frac{1}{2}r, \frac{\varphi}{2}r\right)$, $\left(\frac{\varphi}{2}r, \frac{\varphi-1}{2}r, \frac{1}{2}r\right)$, $\left(\frac{1}{2}r, \frac{\varphi}{2}r, \frac{\varphi-1}{2}r\right)$, and coordinates of the vertices of a disdyakis triacontahedron sphere with radius r and center (x_0, y_0, z_0) in $\mathbb{R}^3_{DT}$ are translations to (x_0, y_0, z_0) all possible $+/-$ signals of components of the points, so its vertices are $(0,0,r)$, $(r,0,0)$, $(0,r,0)$, $\left(0, \frac{5\varphi-7}{3}r, \frac{3\varphi-2}{3}r\right)$, $\left(\frac{3\varphi-2}{3}r, 0, \frac{5\varphi-7}{3}r\right)$, $\left(\frac{5\varphi-7}{3}r, \frac{3\varphi-2}{3}r, 0\right)$, $\left(\frac{3\varphi-2}{5}r, 0, \frac{\varphi+3}{5}r\right)$, $\left(\frac{\varphi+3}{5}r, \frac{3\varphi-2}{5}r, 0\right)$, $\left(0, \frac{\varphi+3}{5}r, \frac{3\varphi-2}{5}r\right)$, $\left(\frac{\varphi-1}{2}r, \frac{r}{2}, \frac{\varphi}{2}r\right)$, $\left(\frac{\varphi}{2}r, \frac{\varphi-1}{2}r, \frac{r}{2}\right)$, $\left(\frac{r}{2}, \frac{\varphi}{2}r, \frac{\varphi-1}{2}r\right)$, $\left(\frac{2\varphi-5}{3}r, \frac{2\varphi-5}{3}r, \frac{2\varphi-5}{3}r\right)$. Figure 4 (b)-(c) shows the $ID-$ sphere and $DT-$sphere centered at $O=(0,0,0)$, respectively.

The following lemma gives relations between d_{RT}, d_{ID} or d_{DT} and d_E. These relations are used by finding isometries of related spaces. For the proofs of the following lemma and its corollaries, one can see to [9, 10, 29].

Lemma 1. *Let $M_d = \|P\|_\infty$ and $S_d = \|P\|_1$ for $P=(p,q,r)$. Let l be the line through the points $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$ in $\mathbb{R}^3$ and d_E is Euclidean metric. If l has direction vector (p,q,r) then*

$$\begin{cases} d_{RT}(P_1,P_2) &= \mu_{RT}(P_1P_2)\, d_E(P_1,P_2), \\ d_{ID}(P_1,P_2) &= \mu_{ID}(P_1P_2)\, d_E(P_1,P_2), \\ d_{DT}(P_1,P_2) &= \mu_{DT}(P_1P_2)\, d_E(P_1,P_2), \end{cases}$$

where

$$\mu_{RT}(P_1P_2) = \frac{\max\left\{M_d \ , \ \frac{\varphi}{2}M_d + \frac{\varphi-1}{2}M_d^+ + \frac{1}{2}M_d^-\right\}}{\sqrt{p^2+q^2+r^2}},$$

$$\mu_{ID}(P_1P_2) = \frac{\max\left\{M_d + (\varphi-1)M_d^+ \ , \ M_d + (\varphi-1)^2 M_d^- \ , \ (\varphi-1)S_d\right\}}{\sqrt{p^2+q^2+r^2}}$$

and

$$\mu_{DT}(P_1P_2) = \frac{\max\left\{\begin{array}{l} M_d + \frac{4\varphi-5}{11}\left(M_d^+ + M_d^-\right) \ , \ \frac{5\varphi+2}{11}M_d + \frac{8\varphi-10}{11}M_d^+ + \frac{3\varphi-1}{11}M_d^- \\ \frac{\varphi+7}{11}M_d + \frac{7\varphi-6}{11}M_d^+ + \frac{4-\varphi}{11}M_d^- \ , \ \frac{6\varphi-2}{11}M_d + \frac{3\varphi-1}{11}M_d^+ + \frac{12-3\varphi}{11}M_d^- \\ \frac{10\varphi-7}{11}M_d + \frac{4\varphi-5}{11}M_d^+ + \frac{2\varphi+3}{11}M_d^- \end{array}\right\}}{\sqrt{p^2+q^2+r^2}}.$$

The above lemma states that d_{RT}, d_{ID} and $d_{DT}-$distances along any line are some positive constant multiple of Euclidean distance along the same line. Consequently one can reach the following corollaries:

Corollary 2. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then*

$$d_E(P_1,X) = d_E(P_2,X) \iff d_{RT}(P_1,X) = d_{RT}(P_2,X),$$

$$d_E(P_1,X) = d_E(P_2,X) \iff d_{ID}(P_1,X) = d_{ID}(P_2,X)$$

and

$$d_E(P_1,X) = d_E(P_2,X) \iff d_{DT}(P_1,X) = d_{DT}(P_2,X).$$

Corollary 3. *If P_1, P_2 and X are any three distinct collinear points in $\mathbb{R}^3$, then*

$$\frac{d_E(X,P_1)}{d_E(X,P_2)} = \frac{d_{RT}(X,P_1)}{d_{RT}(X,P_2)} = \frac{d_{ID}(X,P_1)}{d_{ID}(X,P_2)} = \frac{d_{DT}(X,P_1)}{d_{DT}(X,P_2)}.$$

Corollary 3 means that the ratios of Euclidean and d_{RT}, d_{ID}, $d_{DT}-$distances along a line are the same.

In the following part of this work, we will study the isometries of $\mathbb{R}^3_{RT}$, $\mathbb{R}^3_{ID}$ and $\mathbb{R}^3_{DT}$ and determine their groups of isometries.

3. Isometries of spaces $\mathbb{R}^3_{RT}$, $\mathbb{R}^3_{ID}$ and $\mathbb{R}^3_{DT}$

One of the fundamental question in geometry for S, which is a space with d metric, is to define the G group of isometries If S is Euclidean 3-dimensional space with usual metric then it is obviously known that G consists of translations, rotations, reflections, glide reflections and screw of the 3- dimensional space.

We need following definitions which are quoted from Martin ([11]):

i. A *transformation* is one to one equivalence from the set of points in space onto itself. If $d(X,Y) = d(\alpha(X), \alpha(Y))$ for every point X and Y, then α transformation is named an *isometry.*

ii. For all points X, if $\iota(X) = X$, then ι is called *identity.*

iii. If an isometry α fixes some set of points then α is called a symmetry for that set of points.

iv. For Δ plane, If $\sigma_\Delta(X) = X$ for point X on Δ and if $\sigma_\Delta(X) = Y$ for point X off Δ and Δ is perpendicular bisector of XY line segment, then σ_Δ, which is mapping on the points in $\mathbb{R}_{TH}$, is called reflection.

v. $\sigma_\Delta \sigma_\Gamma$ is defined *a rotation* about axis *l,* if Γ and Δ are two intersecting planes at line *l.*

vi. $\sigma_\Pi \sigma_\Delta \sigma_\Gamma$ is defined *a rotary reflection* about the common point to Γ, Δ and Π if Γ, Δ ,which each one perpendicular to Π, are intersecting planes.

vii. If $\sigma_N(X) = Y$ for every X points and N is midpoint of X and Y, then σ_N inversion about N is called *a transformation*. At the same time, σ_N is defined *a point reflection.*

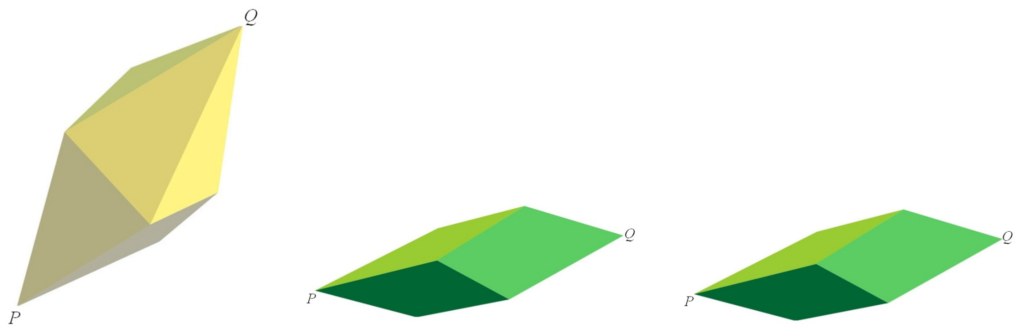
Three essential methods geometric investigations; synthetic, metric and group approach. The group approach takes isometry groups of a geometry and convex sets plays an substantial role in indication of the group of isometries of geometries. Those properties are invariant under the group of motions and geometry studies those properties. There are a lot of studies about group of isometries of a space (see [13–24]).

It is mentioned in introduction section that in Minkowski geometry the linear structure is the same as the Euclidean one but distance is not uniform in all directions. Instead of the usual sphere in Euclidean space, the unit ball is a certain symmetric closed convex set. In [28], the author gives the following theorem:

Theorem 4. *If the unit ball C of $(V, \|\|)$ does not intersect a two-plane in an ellipse, then the group $I(3)$ of isometries of $(V, \|\|)$ is isomorphic to the semi-direct product of the translation group $T(3)$ of $\mathbb{R}^3$ with a finite subgroup of the group of linear transformations with determinant* ± 1.

After this theorem remains a single question. This question is that what is the relevant subgroup?

Now we will show that all isometries of the $\mathbb{R}^3_{RT}$, $\mathbb{R}^3_{ID}$ and $\mathbb{R}^3_{DT}$ are in $T(3).G(RT)$, $T(3).G(ID)$ and $T(3).G(DT)$ respectively. Here, $G(RT)$, $G(ID)$ and $G(DT)$ denote the symmetry groups of rhombic triacontahedron, icosidodecahedron and disdyakis triacontahedron, respectively. In the rest of article, we take $\triangle = RT$, $\triangle = ID$ or $\triangle = DT$. That is, $\triangle \in \{RT, ID, DT\}$.



(a) (b) (c)

Fig. 5. a) The minimum distance set of $\mathbb{R}^3_{RT}$, b) The minimum distance set of $\mathbb{R}^3_{ID}$, c) The minimum distance set of $\mathbb{R}^3_{DT}$

Definition 5. *Let P, Q be two points in $\mathbb{R}^3_{\triangle}$. The minimum distance set of P, Q is defined by*

$$\left\{X \mid d_{\triangle}(P,X) + d_{\triangle}(Q,X) = d_{\triangle}(P,Q)\right\}$$

and denoted by $[PQ]$.

The minimum distance set, which has a very simple definition as the set of points whose sum of distances to P and Q points is equal to the distance between P and Q, can be easily found with the help of various computer programs, although it seems a bit difficult to determine this set when complex metrics are used in 3-dimensional space. In general, $[PQ]$ stands for an octahedron which is not necessary uniform in $\mathbb{R}^3_{RT}$ as shown in Figure 5 (a). $[PQ]$ stands for a parallelepiped in $\mathbb{R}^3_{ID}$ and $\mathbb{R}^3_{DT}$ with diagonal PQ as shown in Figure 5 (b), (c), respectively.

Proposition 6. *Let $\phi : \mathbb{R}^3_{\triangle} \to \mathbb{R}^3_{\triangle}$ be an isometry and let $[PQ]$ be the minimum distance set of P, Q. Then $\phi([PQ]) = [\phi(P)\phi(Q)]$.*

Proof. Let $Y \in \phi([PQ])$. Then,

$$\begin{aligned} Y \in \phi([PQ]) \quad &\Leftrightarrow \quad \exists X \in [PQ] \ni Y = \phi(X) \\ &\Leftrightarrow \quad d_{\triangle}(P,X) + d_{\triangle}(Q,X) = d_{\triangle}(P,Q) \\ &\Leftrightarrow \quad d_{\triangle}(\phi(P),\phi(X)) + d_{\triangle}(\phi(Q),\phi(X)) = d_{\triangle}(\phi(P),\phi(Q)) \\ &\Leftrightarrow \quad Y = \phi(X) \in [\phi(P)\phi(Q)]. \end{aligned}$$

■

Corollary 7. *Let $\phi : \mathbb{R}^3_{\triangle} \to \mathbb{R}^3_{\triangle}$ be an isometry and $[PQ]$ be the minimum distance set. Then ϕ maps vertices to vertices and preserves the lengths of the edges of $[PQ]$.*

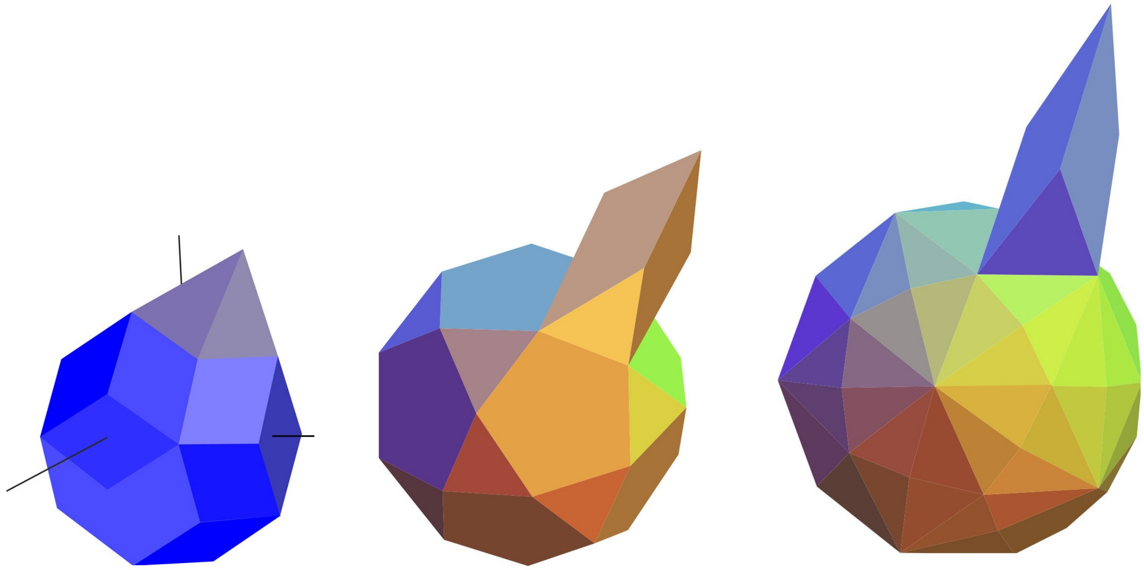
Proposition 8. *Let $\phi : \mathbb{R}^3_{\triangle} \to \mathbb{R}^3_{\triangle}$ be an isometry such that $\phi(O) = O$. Then $\phi \in G(\triangle)$.*

Proof. Since $\triangle \in \{RT, ID, DT\}$, there are three possibility for $\triangle$. Let $\triangle = RT$, and let $V_1 = (\varphi - 1, 0, 1)$, $V_9 = (0, 1, \varphi - 1)$, $V_{13} = (0, 2 - \varphi, 1)$, $V_{25} = (\varphi - 1, \varphi - 1, \varphi - 1)$ and $R = (\varphi - 1, 1, \varphi + 1)$ be five points in $\mathbb{R}^3_{RT}$. Consider $[OR]$ which is an octahedron in Figure 6 (a).

Also points V_1, V_9, V_{13}, V_{25} lie on minimum distance set $[OR]$ and unit sphere with center at the origin. Moreover these four points are the corner points of a rhombic triacontahedron's face which is a rhombus. ϕ maps points V_1, V_9, V_{13}, V_{25} to the vertices of a rhombic triacontahedron by Corollary 7. Since ϕ preserve the lengths of the edges, V_iV_l, V_jV_k and V_jV_l are four edges of the rhombic triacontahedron and $f(V_1) = V_i$, $f(V_9) = V_j$, $f(V_{13}) = V_k$, $f(V_{25}) = V_l$ such that $i, j \in \{0, 1, ..., 12\}$ and $k, l \in \{13, 14, ..., 31\}$. Since rhombic triacontahedron has 30 rhombus faces, there are 30 possibilities to points which they can map, and also there are four possibilities to points which they can map on the face of rhombic triacontahedron. Therefore total number of possibilities are 120. Some of these cases can be seen as follows:

i. If $\phi(V_1) = V_1$, $\phi(V_9) = V_3$, $\phi(V_{13}) = V_{13}$ and $\phi(V_{25}) = V_{15}$, then $\phi = \sigma_{\Delta}$ is the reflection about the plane $\Delta : -x - \varphi y + (\varphi - 1)z = 0$.

ii. If $\phi(V_1) = V_1$, $\phi(V_9) = V_3$, $\phi(V_{13}) = V_{15}$ and $\phi(V_{25}) = V_{13}$, then $\phi = r_{\frac{2\pi}{5}}$ is the rotation with rotation axis $\parallel \left(\sqrt{(\varphi + 2)/5}, \sqrt{(3 - \varphi)/5}, 0\right)$.



(a) (b) (c)

Fig. 6. a) The unit sphere with center O and $[OR]$ in $\mathbb{R}^3_{RT}$, b) The unit sphere with center O and $[OR]$ in $\mathbb{R}^3_{ID}$, c) The unit sphere with center O and $[OR]$ in $\mathbb{R}^3_{DT}$

iii. If $\phi(V_1) = V_1$, $\phi(V_9) = V_9$, $\phi(V_{13}) = V_{13}$ and $\phi(V_{25}) = V_{25}$ then ϕ is the identity.

The remaining cases can be similarly given.

Let $\triangle$=ID, and let V_7=$\left(\frac{\varphi-1}{2}, \frac{1}{2}, \frac{\varphi}{2}\right)$, V_{15}=$\left(\frac{\varphi}{2}, \frac{\varphi-1}{2}, \frac{1}{2}\right)$, V_{23}=$\left(\frac{1}{2}, \frac{\varphi}{2}, \frac{\varphi-1}{2}\right)$ and R=$(\varphi, \varphi, \varphi)$ be four points in $\mathbb{R}^3_{ID}$. Consider $[OR]$ that is the parallelepiped with diagonal OR (Figure 6 (b)). Also points V_7, V_{15}, V_{23} lie on minimum distance set $[OR]$ and unit sphere with center at the origin. Moreover, these three points are the corner points of an icosidodecahedron's face which is a equilateral triangle. ϕ maps points V_7, V_{15}, V_{23} to the vertices of an icosidodecahedron by Corollary 7. Since ϕ preserve the lengths of the edges, $\phi(V_7) = A_i$, $\phi(V_{15}) = A_j$ and $\phi(V_{23}) = A_k$ such that $i, j, k \in \{1, 2, ..., 30\}$. Since icosidodecahedron have 20 equilateral triangle faces, there are 20 possibilities to points which they can map, and also there are six possibilities to points which they can map on the face of icosidodecahedron. Therefore total number of possibilities are 120. Some of these cases can be seen as follows:

i. If $\phi(V_7) = V_1$, $\phi(V_{15}) = V_9$ and $\phi(V_{23}) = V_{13}$, then $f = \sigma_\Delta$ is the reflection about $\Delta : -x - \varphi y + (\varphi - 1)z = 0$.

ii. If $\phi(V_7) = V_1$, $\phi(V_{15}) = V_{13}$ and $\phi(V_{23}) = V_9$, then $\phi = r_{\frac{4\pi}{3}}$ is the rotation with rotation axis $\parallel \left(0, (\varphi-1)/\sqrt{3}, \varphi/\sqrt{3}\right)$.

iii. If $\phi(V_7) = V_1$, $\phi(V_{15}) = V_{11}$ and $\phi(V_{23}) = V_7$ then $\phi = \sigma_O r_{\frac{6\pi}{5}}$ is the rotary inversion with rotation axis $\parallel$ $\left(\sqrt{(\varphi+2)/5}, -\sqrt{(3-\varphi)/5}, 0\right)$

The remaining cases can be similarly given.

Let $\triangle = DT$, and let $V_{31} = \left(\frac{\varphi-1}{2}, \frac{1}{2}, \frac{\varphi}{2}\right)$, $V_{55} = \left(\frac{5-2\varphi}{3}, \frac{5-2\varphi}{3}, \frac{5-2\varphi}{3}\right)$, $V_{27} = \left(0, \frac{\varphi+3}{5}, \frac{3\varphi-2}{5}\right)$ and $R = \left(\frac{\varphi-1}{2}, \frac{2\varphi+11}{10}, \frac{11\varphi-4}{10}\right)$ be four points in $\mathbb{R}^3_{DT}$. Consider $[OR]$ that is the parallelepiped with diagonal OR (Figure 6 (c)) Also points V_{31}, V_{55}, V_{27} lie on minimum distance set $[OR]$ and unit sphere with center at the origin. Moreover these three points are the corner points of a disdyakis triacontahedron's face which is a scalene triangle. ϕ maps points V_{31}, V_{55}, V_{27} to the vertices of an disdyakis triacontahedron by Corollary 7. Since ϕ preserve the lengths of the edges, $\phi(V_{31}) = A_i$, $\phi(V_{55}) = A_j$ and $\phi(V_{27}) = A_k$ such that $i \in \{1, 2, ..., 6, 31, 32, ..., 54\}$, $j \in \{7, 8, ..., 18, 55, 56, ..., 62\}$, $k \in \{19, 20, ..., 30\}$. Since icosidodecahedron have 120 scalene triangle faces, there are 120 possibilities to points which they can map, and also there are only one possibility to points which they can map on the face of disdyakis dodecahedron. Therefore total number of possibilities are 120. Some of these cases can be seen as follows:

i. If $\phi(V_{31}) = V_1$, $\phi(V_{55}) = V_9$ and $\phi(V_{27}) = V_{21}$, then $\phi = \sigma_\Delta$ is the reflection about the plane $\Delta : -x - \varphi y + (\varphi - 1)z = 0$.

ii. If $\phi(V_{31}) = V_1$, $\phi(V_{55}) = V_9$ and $\phi(V_{27}) = V_{19}$, then $\phi = r_{\frac{4\pi}{3}}$ is the rotation with rotation axis $\parallel \left(0, (\varphi-1)/\sqrt{3}, \varphi/\sqrt{3}\right)$.

iii. If $\phi(V_{31}) = V_1$, $\phi(V_{55}) = V_{17}$ and $\phi(V_{27}) = V_{19}$ then $\phi = \sigma_O r_{\frac{6\pi}{5}}$ is the rotary inversion with rotation axis $\| \left(\sqrt{(\varphi+2)/5}, -\sqrt{(3-\varphi)/5}, 0\right)$.

The remaining cases can be similarly given.

Only three possible examples are given here for each of the three cases. It is necessary to show each of the 120 possibilities that are actually possible, one by one. But this is a rather tedious and very long list for a research article. Readers who want to observe each of these situations can refer to [29] for a detailed review of all situations, one by one. ■

Theorem 9. *Let $\phi : \mathbb{R}^3_\triangle \to \mathbb{R}^3_\triangle$ be an isometry. Then there exists a unique $T_A \in T(3)$ and $\psi \in G(\triangle)$ where $\phi = T_A \circ \psi$.*

Proof. Let $\phi(O) = A$ such that $A = (a_1, a_2, a_3)$. Define $\psi = T_{-A} \circ \phi$. We know that $\psi(O) = O$ and ψ is an isometry. Thereby, $\psi \in G(\triangle)$ and $\phi = T_A \circ \psi$ by Proposition 8. The proof of uniqueness is trivial. ■

4. Conclusions

In this paper, the spaces of which their sphere are rhombic triacontahedron, icosidodecahedron and disdyakis triacontahedron are introduced and some properties of metrics which are used setting up these spaces are given. Also, isometry groups of these spaces are given. So each of the groups of isometries of the spaces covering with rhombic triacontahedron, icosidodecahedron and disdyakis triacontahedron metrics is the semi-direct product of the icosahedral group I_h and $T(3)$, where I_h is the (Euclidean) symmetry group of the icosahedron and $T(3)$ is the group of all translations of the 3-dimensional space. In the future works, handled solids in this paper are Catalan and Archimedean solids, the new metric space by considering different solids from these solids can be constructed and investigate their some properties which are related to metrics.

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