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EXACT SOLUTION OF THE SCHRÖDINGER EQUATION IN TOPOLOGICALLY MASSIVE SPACETIME

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Abstract. We study exact solutions of the Schrödinger equation in a topologically massive space-time. Exact solutions are obtained in terms of the hypergeometric functions. We also obtained the momentum quantization with the help of the condition of the wave function to be bounded. The investigation is performed in the framework of rainbow formalism of the General Relativity Theory (RGT). The quantized momentum is evaluated for different choices of the rainbow functions.

1. Introduction

In last decades, it has been a crucial area to study exact solutions of the non-relativistic and relativistic wave equations that present precious data concerning to the quantum mechanical systems. In this circumstances, the Schrödinger (for spinless and non-relativistic massive particles), the Klein-Gordon (KG) (spin-0 particles, e.g., pions), the Dirac (for spin-1/2, e.g., electrons) and the Duffin-Kemmer-Petiau (DKP) (for spin-1, e.g., $W^{\pm}$, Z^0 bosons and photons) equations are the most examined equations [1, 2, 3, 4, 5, 6, 7, 8]. Except the Schrödinger equation, the rest are the fundamental single particle equations of the relativistic quantum mechanics.

The Schrödinger equation defines the non-relativistic quantum mechanical character of an isolated physical system by evolution of a wave function over the time. Its solutions given for the presence of external electromagnetical fields have fundamental applications used in technology, engineering, electro-mechanics, particle physics, medical physics, and so on. Compared to the other wave equations the Schrödinger equation has much less been studied in curved geometry. Examining the Schrödinger equation in curved space-time is a way of finding the effective low-energy characterization of a quantum particle in a curved geometry.

Recently, a new approach to the Einstein's General Relativity (EGG), which is called "Doubly General Relativity" (DGR) is introduced [9], thereafter called as Rainbow Gravity (RG) to study the quantum effects of the gravitation in the smallest accessible regions, namely the Planck scale. The idea behind the RG

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approach of gravitation is that at ultra-high energy regimes the geometry of classical space-time alters by the probing particles that have different energies [9, 10, 11]. Thus, the standard metric is deformed and this phenomenon is represented in space-time metric with rainbow functions. Because of this modified perspective, the rainbow version of a metric can be written by the replacements $dx^0 \to \frac{dx^0}{f}$ for the time coordinate and $dx^i \to \frac{dx^i}{g}$ for the spatial coordinates. As the particle moves in geometry, it will perceive gravity differently for each energy it has, as the way a prism acts on light.

The structure of the paper will be as follows. In section 2, we give a brief theoretical set-up of the problem and in section 3, we will solve the Schrödinger equation for the considered rainbow space-time. In section 4, by obtaining approximate solutions, the quantization condition of the momentum will be derived. Finally, section 5 is devoted to the discussion of results.

2. Preliminaries

For the investigation of our problem, we will study in the RG formalism and discuss the dynamics of particle by the topologically massive space-time given by,

$$ds^2 = d\theta^2 + d\phi^2 + 2\cos(\nu\theta)d\psi d\phi + d\psi^2 \tag{2.1}$$

which is basically a de Sitter space with the polar angle suffering a conic defect. This metric has been offered by Aliev et.al. [12] and it has obvious importance for the gauge field theories. Here, the term ν is the topological mass and it is related to the cosmological constant as $\lambda = \frac{\nu^2}{4}$. The metric can be diagonalized by introducing new variables as $\varphi = \psi + \phi$ and $\chi = \psi - \phi$. Therefore the above metric takes the following form,

$$ds^2 = d\theta^2 + \cos^2\frac{\nu\theta}{2}d\varphi^2 + \sin^2\frac{\nu\theta}{2}d\chi^2 \tag{2.2}$$

In the modified perspective, the rainbow counterpart of the above metric can be written as

$$ds^2 = \frac{1}{g^2(\varepsilon)}\left[d\theta^2 + \cos^2\frac{\nu\theta}{2}d\varphi^2 + \sin^2\frac{\nu\theta}{2}d\chi^2\right] \tag{2.3}$$

where $g(\varepsilon)$ is the energy-dependent rainbow function, $\varepsilon = \frac{E}{E_{Pl.}}$, E is the energy of the probing particle and $E_{Pl.}$ is the Planck energy.

3. Exact solution of the Schrödinger equation

The covariant form of the Schrödinger equation in curved space is given as follows [13],

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{-\hbar^2}{2m}\left[\frac{1}{\sqrt{detg_{\mu\nu}}}\frac{\partial}{\partial x^\mu}(\sqrt{detg_{\mu\nu}}g^{\mu\nu}\frac{\partial\Psi}{\partial x^\nu})\right] - \frac{\hbar^2}{6}R\Psi \tag{3.1}$$

where m is the mass of particle, $\hbar$ is the Planck constant, $g^{\mu\nu}$ is the metric given by Eq.(2.3) and R is the scalar curvature of the space which is calculated by the contradiction of the Ricci tensor and given as

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}\left(\partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\mu\Gamma^\alpha_{\alpha\nu} + \Gamma^\alpha_{\alpha\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\alpha_{\mu\rho}\Gamma^\rho_{\alpha\nu}\right) \tag{3.2}$$

where

$$\Gamma^{\alpha}_{\nu\lambda} = \frac{1}{2} g^{\alpha\beta}\left(\partial_\nu g_{\lambda\beta} + \partial_\lambda g_{\beta\nu} - \partial_\beta g_{\nu\lambda}\right) \tag{3.3}$$

are the Christoffel symbols [14] and obtained as follows

$$\Gamma^{1}_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\nu}{4}\sin(\nu\theta) & 0 \\ 0 & 0 & -\frac{\nu}{4}\sin(\nu\theta) \end{pmatrix}, \tag{3.4}$$

$$\Gamma^{2}_{ij} = \begin{pmatrix} 0 & -\frac{\nu}{2}\tan(\nu\theta) & 0 \\ -\frac{\nu}{2}\tan(\nu\theta) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.5}$$

and

$$\Gamma^{2}_{ij} = \begin{pmatrix} 0 & 0 & \frac{\nu}{2}\cot(\nu\theta) \\ 0 & 0 & 0 \\ \frac{\nu}{2}\cot(\nu\theta) & 0 & 0 \end{pmatrix}. \tag{3.6}$$

By using the line element given by Eq.(2.3) and (3.4, 3.5, 3.6) in Eq.(3.2), the scalar curvature is obtained as $R = \frac{3g^2(\varepsilon)\nu^2}{2}$. With the help of these results and reminding that $\sqrt{detg_{\mu\nu}} = \frac{\sin(\nu\theta)}{2g^3}$, the Schrödinger equation (3.1) reduces to the following form,

$$f^{''}(\theta) + 2\cot(2y)f^{'}(\theta) + \left[c - \left(\frac{a}{\cos^2(y)} + \frac{b}{\sin^2(y)}\right)\right] f(\theta) = 0 \tag{3.7}$$

where the definitions $\Psi = e^{i(\alpha\varphi+\beta\chi-Et)}f(\theta)$, $a = \frac{4\alpha^2}{\nu^2}$, $b = \frac{4\beta^2}{\nu^2}$, $c = \frac{8m}{\nu^2 g^2}\left(E + \frac{\nu^2 g^2}{4}\right)$ and $y = \frac{\nu\theta}{2}$ were made. If the variable is changed as $\cos^2 y = \frac{1}{u}$, Eq.(3.7) is transformed to into the below form,

$$4u^2(u-1)^2 f^{''}(u) + 4u^2(u-1)f^{'}(u) - \left[au^2 - Bu + c\right] f(u) = 0 \tag{3.8}$$

where $B = a - b + c$.

If we define the wave function as $f(u) = u^{p-1}(u-1)^q\Omega(u)$, Eq.(3.8) can be written as

(3.9)

$$u(u-1)\Omega^{''} + [u(2p+2q+1) + 2(1-p)]\Omega^{'} + \left[2pq + \frac{p^2(u-1) - 2pu + u^3 p}{u} + \frac{qu(q-2) + 2q + \frac{uK}{4} + \frac{L}{4}}{u-1} + \frac{M}{4u(u-1)}\right]\Omega = 0$$

where $K = 4 - a$, $L = B - 12$ and $M = 8 - c$.

For the choices of $p^2 - 3p + \frac{M}{4} = 0$, $4q^2 + K + L + M = 0$ and definitions $2(p-1) = \gamma$, $2(p+q)+1 = P+Q+1$, $2pq + p^2 - 2(p+q) - \frac{M+L}{4} = PQ$, we obtain

$$u(u-1)\Omega^{''}(u) + [(P+Q+1)u - \gamma]\Omega^{'}(u) + PQ\Omega(u) = 0 \tag{3.10}$$

which has the form of hypergeometric differential equation [15]. Solutions of this equation are given by

$$\Omega(u) = {}_2F_1\left(P, Q, \gamma; u\right) \tag{3.11}$$

and

$$\Omega(u) = u^{1-\gamma}{}_2F_1\left(P+1-\gamma, Q+1-\gamma, 2-\gamma; u\right) \tag{3.12}$$

where ${}_2F_1$ are hypergeometric functions. Thence, exact solutions of the Schrödinger equation is obtained as

$$\Psi = e^{i(\alpha\varphi+\beta\chi-Et)}\left[\cos(\frac{\nu\theta}{2})\right]^{2(1-p)}\left[\cos(\frac{\nu\theta}{2})\right]^{-2q}{}_2F_1\left(P,Q,\gamma;\cos^{-2}(\frac{\nu\theta}{2})\right) \tag{3.13}$$

For the specific discussions of our general results, one can use various scenarios introduced in literature for the rainbow functions. We give a few well-known proposals of the rainbow functions in TABLE I.

Table 1. Most studied proposals of the rainbow functions. Here, c_1, c_2, c_3, c_4 and t are arbitrary parameters.

f	g	Reference
1	$\sqrt{1-c_1\chi^t}$	[16]
$(c_2\chi)^{-1}(\exp[c_2\chi]-1)$	1	[16]
$(1-c_3\chi)^{-1}$	$(1-c_3\chi)^{-1}$	[16]
$(1-c_4\chi)^{-1}$	1	[17]
$\exp[-\frac{\chi^2}{2}]$	1	[18]
1	$1+\frac{\chi}{2}$	[19]
$1+\frac{\chi}{2}$	$1+(2\chi)^{-1}$	[19]
1	$1+\chi^t$	[20]

4. Asymptotic solution of the Schrödinger equation

For the small value of the argument, namely $y=\frac{\nu\theta}{2}\ll 1$, Eq. (3.7) transforms into

$$y^2f^{''}(y)+2yf^{'}(y)+[(c-a)y^2-b]f(y)=0 \tag{4.1}$$

This is the Bessel differential equation and solution is given by [15]

$$f(y)=\frac{1}{\sqrt{y}}Z_{\sqrt{b+\frac{1}{4}}}(\sqrt{c-a}y) \tag{4.2}$$

where $Z_\mu(\zeta)$ are the cylindrical functions and can be written in terms of the Bessel functions as

$$Z_\mu = c_1J_\mu + c_2Y_\mu \tag{4.3}$$

where J_μ is first type and Y_μ is second type of Bessel functions that are related to Kummer functions, $M(a,b,z)$, as following [15]

$$J_\mu=\frac{(\frac{y}{2})^\mu}{\Gamma(\mu+1)}e^{-iy}M(\mu+\frac{1}{2},2\mu+1,2iy) \tag{4.4}$$

and

$$Y_\mu=\frac{J_\mu\cos(\mu\pi)-J_{-\mu}}{\sin(\mu\pi)} \tag{4.5}$$

In order Kummer functions to be finite, we require the bound condition of the Kummer functions as $\mu+\frac{1}{2}=-n$ [15]. Therefore, we obtain the quantized momentum of the Schrödinger particle in terms of the topological mass as in the follow,

$$\beta=\frac{\nu}{2}\sqrt{n(n+1)} \tag{4.6}$$

which is the momentum in the χ-direction.

5. Conclusion

In this study, we have analyzed the Schrödinger equation in a modified rainbow background. In the process of obtaining the solutions we used the separation of variables method. Both the wave function and momentum of the Schrödinger particle are obtained depending on the topological mass. The topological massive $(3+0)$-space is hard to study for the relativistic higher spinning particles. So, this study may have the potential of providing insights into the relativistic spinning particles as well. This is going to be a further study in the corresponding space. One of the interesting finding of this study is that although the dynamics of the particle depends on the angular variables, topological mass term and energy of the probing particle, the quantized momentum depends on only the topological mass.

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