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NUMERICAL RADIUS AND p-SCHATTEN NORM INEQUALITIES FOR POWER SERIES OF OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. Assume that the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is convergent on the open disk $D(0,R)$, $f_a(z) := \sum_{k=0}^{\infty} |a_k| z^k$ that has the same radius of convergence R and A, B, $C \in B(H)$ with $\|A\| < R$, then we have the following Schwarz type inequality

$$|\langle C^* A f(A) Bx, y\rangle| \le f_a(\|A\|) \left\langle \left|\left|A\right|^{\alpha} B\right|^2 x, x\right\rangle^{1/2} \left\langle \left|\left|A^*\right|^{1-\alpha} C\right|^2 y, y\right\rangle^{1/2}$$

for $\alpha \in [0,1]$ and x, $y \in H$. Some natural applications for *numerical radius* and *p-Schatten norm* are also provided.

1. Introduction

The *numerical radius* $\omega(T)$ of an operator T on H is given by

$$\omega(T) = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}. \tag{1}$$

Obviously, by (1), for any $x \in H$ one has

$$|\langle Tx, x\rangle| \le \omega(T) \|x\|^2. \tag{2}$$

It is well known that $\omega(\cdot)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators $T : H \to H$, i.e.,

(i) $\omega(T) \ge 0$ for any $T \in B(H)$ and $\omega(T) = 0$ if and only if $T = 0$;
(ii) $\omega(\lambda T) = |\lambda| \omega(T)$ for any $\lambda \in \mathbb{C}$ and $T \in B(H)$;

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(iii) $\omega(T+V) \leq \omega(T) + \omega(V)$ for any $T, V \in B(H)$.

This norm is equivalent with the operator norm. In fact, the following more precise result holds:

$$\omega(T) \leq \|T\| \leq 2\omega(T) \tag{3}$$

for any $T \in B(H)$.

F. Kittaneh, in 2003 [7], showed that for any operator $T \in B(H)$ we have the following refinement of the first inequality in (3):

$$\omega(T) \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{1/2}\right). \tag{4}$$

Utilizing the Cartesian decomposition for operators, F. Kittaneh in [8] improved the inequality (3) as follows:

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\| \tag{5}$$

for any operator $T \in B(H)$.

For powers of the absolute value of operators, one can state the following results obtained by El-Haddad & Kittaneh in 2007, [5]:

If for an operator $T \in B(H)$ we denote $|T| := (T^*T)^{1/2}$, then

$$\omega^r(T) \leq \frac{1}{2}\left\||T|^{2\alpha r} + |T^*|^{2(1-\alpha)r}\right\| \tag{6}$$

and

$$\omega^{2r}(T) \leq \left\|\alpha |T|^{2r} + (1-\alpha)|T^*|^{2r}\right\|, \tag{7}$$

where $\alpha \in (0,1)$ and $r \geq 1$.

If we take $\alpha = \frac{1}{2}$ and $r = 1$ we get from (6) that

$$\omega(T) \leq \frac{1}{2}\||T| + |T^*|\| \tag{8}$$

and from (7) that

$$\omega^2(T) \leq \frac{1}{2}\left\||T|^2 + |T^*|^2\right\|. \tag{9}$$

For more related results, see the recent books on inequalities for numerical radii [3] and [1].

Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{10}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{11}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (11) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 1. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{12}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\ \ \textit{and}\ \ |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{13}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_1(H)$.

For a large number of results concerning trace inequalities, see the recent survey paper [4].

An operator $A\in\mathcal{B}(H)$ is said to belong to the *von Neumann-Schatten class* $\mathcal{B}_p(H)$, $1\le p<\infty$ if the *p-Schatten norm* is finite [12, p. 60-64]

$$\|A\|_p:=[\operatorname{tr}(|A|^p)]^{1/p}=\left(\sum_{i\in I}\langle|A|^p e_i,e_i\rangle\right)^{1/p}<\infty.$$

For $1<p<q<\infty$ we have that

$$\mathcal{B}_1(H)\subset\mathcal{B}_p(H)\subset\mathcal{B}_q(H)\subset\mathcal{B}(H) \tag{14}$$

and

$$\|A\|_1\ge\|A\|_p\ge\|A\|_q\ge\|A\|. \tag{15}$$

For $p\ge1$ the functional $\|\cdot\|_p$ is a *norm* on the $*$-ideal $\mathcal{B}_p(H)$ and $\left(\mathcal{B}_p(H),\|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [12, p. 60-64],

$$\|A\|_p=\|A^*\|_p,\ A\in\mathcal{B}_p(H) \tag{16}$$

$$\|AB\|_p\le\|A\|_p\|B\|_p,\ A,B\in\mathcal{B}_p(H) \tag{17}$$

and

$$\|AB\|_p\le\|A\|_p\|B\|,\ \ \|BA\|_p\le\|B\|\|A\|_p,\ A\in\mathcal{B}_p(H),\ B\in\mathcal{B}(H). \tag{18}$$

This implies that

$$\|CAB\|_p\le\|C\|\|A\|_p\|B\|,\ A\in\mathcal{B}_p(H),\ B,\ C\in\mathcal{B}(H). \tag{19}$$

In terms of *p-Schatten norm* we have the *Hölder inequality* for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$(|\operatorname{tr}(AB)|\le)\,\|AB\|_1\le\|A\|_p\,\|B\|_q\,,\ A\in\mathcal{B}_p(H)\,,\ B\in\mathcal{B}_q(H)\,. \tag{20}$$

For the theory of trace functionals and their applications the reader is referred to [10] and [12].

For $\mathcal{E}:=\{e_i\}_{i\in I}$ an orthonormal basis of H we define for $A\in\mathcal{B}_p(H)\,,\ p\ge 1$

$$\|A\|_{\mathcal{E},p}:=\left(\sum_{i\in I}|\langle Ae_i,e_i\rangle|^p\right)^{1/p}.$$

We observe that $\|\cdot\|_{\mathcal{E},p}$ is a norm on $\mathcal{B}_p(H)$ and

$$\|A\|_{\mathcal{E},p}\le\|A\|_p\ \text{ for }A\in\mathcal{B}_p(H)\,.$$

Further, we can take the supremum over all orthonormal basis in H we can also define, for $A\in\mathcal{B}_p(H)\,,$ that

$$\omega_p(A):=\sup_{\mathcal{E}}\|A\|_{\mathcal{E},p}\le\|A\|_p\,,$$

which is a *norm* on $\mathcal{B}_p(H)\,.$

It is also known that, if $\mathcal{E}=\{e_i\}_{i\in I}$ and $\mathcal{F}=\{f_i\}_{i\in I}$ are orthonormal basis, then [11]

$$\sup_{\mathcal{E},\mathcal{F}}\sum_{i\in I}|\langle Te_i,f_i\rangle|^s=\|T\|_s^s\ \text{ for }s\ge 1. \tag{21}$$

2. Vector Inequalities

In 1988 F. Kittaneh [6, Corollary 7] obtained the following Schwarz type inequality for powers of operators:

Lemma 1. *Let $A\in B(H)$ and $\alpha\in[0,1]$. Then for $n\ge 1$ we have*

$$|\langle A^nx,y\rangle|^2\le\|A\|^{2n-2}\left\langle|A|^{2\alpha}x,x\right\rangle\left\langle|A^*|^{2(1-\alpha)}y,y\right\rangle \tag{22}$$

for all $x,\ y\in H$.

We can state the following result as well:

Corollary 1. *Let $A,\ B,\ C\in B(H)$ and $\alpha\in[0,1]$. Then for $n\ge 1$ we have*

$$|\langle C^*A^nBx,y\rangle|^2\le\|A\|^{2n-2}\left\langle||A|^{\alpha}B|^2x,x\right\rangle\left\langle\left||A^*|^{1-\alpha}C\right|^2y,y\right\rangle \tag{23}$$

for all $x,\ y\in H$.

Proof. If we replace x by Bx and y by Cy in (22), then we get

$$|\langle C^*A^nBx,y\rangle|^2\le\|A\|^{2n-2}\left\langle B^*|A|^{2\alpha}Bx,x\right\rangle\left\langle C^*|A^*|^{2(1-\alpha)}Cy,y\right\rangle. \tag{24}$$

Observe that $B^* |A|^{2\alpha} B = ||A|^{\alpha} B|^2$ and $C^* |A^*|^{2(1-\alpha)} C = \left||A^*|^{1-\alpha} C\right|^2$, then by (24) we get (23). □

We consider the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ with $a_k \in \mathbb{C}$ for $k \in \mathbb{N} := \{0, 1, ...\}$. We assume that this power series in convergent on the open disk $D(0, R) := \{z \in \mathbb{C} | \ z < R\}$. If $R = \infty$ then $D(0, R) = \mathbb{C}$. We define $f_a(z) := \sum_{k=0}^{\infty} |a_k| z^k$ which has the same radius of convergence R.

As some natural examples that are useful for applications, we can point out that, if

$$f(\lambda) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \lambda^n = \ln \frac{1}{1+\lambda}, \ \lambda \in D(0,1); \tag{25}$$

$$g(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \lambda^{2n} = \cos \lambda, \ \lambda \in \mathbb{C};$$

$$h(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \lambda^{2n+1} = \sin \lambda, \ \lambda \in \mathbb{C};$$

$$l(\lambda) = \sum_{n=0}^{\infty} (-1)^n \lambda^n = \frac{1}{1+\lambda}, \ \lambda \in D(0,1);$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$f_a(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n} \lambda^n = \ln \frac{1}{1-\lambda}, \ \lambda \in D(0,1); \tag{26}$$

$$g_a(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \lambda^{2n} = \cosh \lambda, \ \lambda \in \mathbb{C};$$

$$h_a(\lambda) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \lambda^{2n+1} = \sinh \lambda, \ \lambda \in \mathbb{C};$$

$$l_a(\lambda) = \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda}, \ \lambda \in D(0,1).$$

Other important examples of functions as power series representations with non-negative coefficients are:

$$\exp(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^n \qquad \lambda \in \mathbb{C}, \tag{27}$$

$$\frac{1}{2} \ln \left(\frac{1+\lambda}{1-\lambda} \right) = \sum_{n=1}^{\infty} \frac{1}{2n-1} \lambda^{2n-1}, \qquad \lambda \in D(0,1);$$

$$\sin^{-1}(\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}(2n+1)n!}\lambda^{2n+1}, \qquad \lambda \in D(0,1);$$

$$\tanh^{-1}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{2n-1}\lambda^{2n-1}, \qquad \lambda \in D(0,1)$$

$$_2F_1(\alpha,\beta,\gamma,\lambda) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)\Gamma(\gamma)}{n!\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\gamma)}\lambda^n, \alpha,\beta,\gamma > 0,$$
$$\lambda \in D(0,1);$$

where Γ is *Gamma function.*

The following result is of interest:

Theorem 2. *Assume that the power series with complex coefficients* $f(z) := \sum_{k=0}^{\infty} a_k z^k$ *is convergent on* $D(0,R)$ *and* A, B, $C \in B(H)$ *with* $\|A\| < R$, *then*

$$|\langle C^*Af(A)Bx,y\rangle|^2 \le f_a^2(\|A\|)\left\langle \left||A|^{\alpha}B\right|^2 x,x\right\rangle \left\langle \left||A^*|^{1-\alpha}C\right|^2 y,y\right\rangle \tag{28}$$

for $\alpha \in [0,1]$ *and* x, $y \in H$.

In particular,

$$|\langle C^*Af(A)Bx,y\rangle|^2 \le f_a^2(\|A\|)\left\langle \left||A|^{1/2}B\right|^2 x,x\right\rangle \left\langle \left||A^*|^{1/2}C\right|^2 y,y\right\rangle \tag{29}$$

for x, $y \in H$.

Proof. If we take $n = k+1$, $k \in \mathbb{N}$ in (23) and take the square root, then we get

$$\left|\langle C^*AA^kBx,y\rangle\right| \le \|A\|^k \left\langle \left||A|^{\alpha}B\right|^2 x,x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha}C\right|^2 y,y\right\rangle^{1/2}$$

for all x, $y \in H$.

Further, if we multiply by $|a_k| \ge 0$, $k \in \{0,1,...\}$ and sum over k from 0 to m, then we get

$$\begin{aligned} &\left|\left\langle C^*A\sum_{k=0}^{m} a_k A^k Bx,y\right\rangle\right| \qquad (30)\\ &= \left|\sum_{k=0}^{m} a_k \langle C^*AA^kBx,y\rangle\right| \le \sum_{k=0}^{m} |a_k| \left|\langle C^*AA^kBx,y\rangle\right| \\ &\le \sum_{k=0}^{m} |a_k| \|A\|^k \left\langle \left||A|^{\alpha}B\right|^2 x,x\right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha}C\right|^2 y,y\right\rangle^{1/2} \end{aligned}$$

for all x, $y \in H$.

Since $\|A\| < R$ then series $\sum_{k=0}^{\infty} a_k A^k$ and $\sum_{k=0}^{\infty} |a_k| \|A\|^k$ are convergent and

$$\sum_{k=0}^{\infty} a_k A^k = f(A) \text{ and } \sum_{k=0}^{\infty} |a_k| \|A\|^k = f_a(\|A\|).$$

By taking now the limit over $m \to \infty$ in (30) we deduce the desired result (28). □

Remark 1. *If A, B, $C \in B(H)$ with $\|A\| < 1$, then for $\alpha \in [0,1]$*

$$\left|\left\langle C^* A (I \pm A)^{-1} Bx, y\right\rangle\right|^2 \le (1 - \|A\|)^{-2} \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \tag{31}$$

and

$$|\langle C^* A \ln(I \pm A) Bx, y\rangle|^2 \le [\ln(1 - \|A\|)]^2 \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \tag{32}$$

for all x, $y \in H$.

For $\alpha = 1/2$ in (31) and (32) we obtain

$$\left|\left\langle C^* A (I \pm A)^{-1} Bx, y\right\rangle\right|^2 \le (1 - \|A\|)^{-2} \langle B^* |A| Bx, x\rangle \langle C^* |A^*| Cy, y\rangle \tag{33}$$

and

$$|\langle C^* A \ln(I \pm A) Bx, y\rangle|^2 \le [\ln(1 - \|A\|)]^2 \langle B^* |A| Bx, x\rangle \langle C^* |A^*| Cy, y\rangle \tag{34}$$

for all x, $y \in H$.

If A, B, $C \in B(H)$ and $\alpha \in [0,1]$, then

$$|\langle C^* A \sin(A) Bx, y\rangle|^2 \le [\sinh(\|A\|)]^2 \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \tag{35}$$

and

$$|\langle C^* A \cos(A) Bx, y\rangle|^2 \le [\cosh(\|A\|)]^2 \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle \tag{36}$$

for all x, $y \in H$.

For $\alpha = 1/2$ in (35) and (36) we obtain

$$|\langle C^* A \sin(A) Bx, y\rangle|^2 \le [\sinh(\|A\|)]^2 \langle B^* |A| Bx, x\rangle \langle C^* |A^*| Cy, y\rangle \tag{37}$$

and

$$|\langle C^* A \cos(A) Bx, y\rangle|^2 \le [\cosh(\|A\|)]^2 \langle B^* |A| Bx, x\rangle \langle C^* |A^*| Cy, y\rangle \tag{38}$$

for all x, $y \in H$.

Also, if A, B, $C \in B(H)$ and $\alpha \in [0,1]$, then

$$|\langle C^* A \exp(A) Bx, y\rangle|^2 \le \exp(2\|A\|) \left\langle \left||A|^\alpha B\right|^2 x, x\right\rangle \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y\right\rangle, \tag{39}$$

$$\begin{aligned}&\left|\left\langle C^{*}A\sinh\left(A\right)Bx,y\right\rangle\right|^{2} \\ &\leq\left[\sinh\left(\left\|A\right\|\right)\right]^{2}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}y,y\right\rangle\end{aligned} \tag{40}$$

and

$$\begin{aligned}&\left|\left\langle C^{*}A\cosh\left(A\right)Bx,y\right\rangle\right|^{2} \\ &\leq\left[\cosh\left(\left\|A\right\|\right)\right]^{2}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}y,y\right\rangle\end{aligned} \tag{41}$$

for all x, $y\in H$.

For $\alpha=1/2$ *in (39)-(41) we obtain some simpler inequalities. We omit the details.*

3. Norm and Numerical Radius Inequalities

The following vector inequality for positive operators $A\geq 0$, obtained by C. A. McCarthy in [9] is well known,

$$\left\langle Ax,x\right\rangle^{p}\leq\left\langle A^{p}x,x\right\rangle,\ p\geq 1$$

for $x\in H$, $\|x\|=1$.

Buzano's inequality [2],

$$\frac{1}{2}\left[\|x\|\,\|y\|+|\langle x,y\rangle|\right]\geq\left|\langle x,e\rangle\,\langle e,y\rangle\right| \tag{42}$$

that holds for any x, y, $e\in H$ with $\|e\|=1$ will also be used in the sequel.

Our first main result is as follows:

Theorem 3. *Assume that the power series with complex coefficients* $f(z):=\sum_{k=0}^{\infty}a_{k}z^{k}$ *is convergent on* $D(0,R)$, $\alpha\in[0,1]$ *and* A, B, $C\in B(H)$ *with* $\|A\|<R$, *then we have the norm inequality*

$$\left\|C^{*}Af\left(A\right)B\right\|\leq f_{a}\left(\left\|A\right\|\right)\left\|\left|A\right|^{\alpha}B\right\|\left\|\left|A^{*}\right|^{1-\alpha}C\right\|. \tag{43}$$

We also have the numerical radius inequalities

$$\omega\left(C^{*}Af\left(A\right)B\right)\leq\frac{1}{2}f_{a}\left(\left\|A\right\|\right)\left\|\left|\left|A\right|^{\alpha}B\right|^{2}+\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\| \tag{44}$$

and

$$\begin{aligned}&\omega^{2}\left(C^{*}Af\left(A\right)B\right) \\ &\leq\frac{1}{2}f_{a}^{2}\left(\left\|A\right\|\right)\left[\left\|\left|A\right|^{\alpha}B\right\|^{2}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|^{2}+\omega\left(\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right)\right].\end{aligned} \tag{45}$$

Proof. We have from (28), by taking the supremum over $\|x\|=\|y\|=1$, that

$$\left\|C^{*}Af\left(A\right)B\right\|^{2}=\sup_{\|x\|=\|y\|=1}\left|\left\langle C^{*}Af\left(A\right)Bx,y\right\rangle\right|^{2}$$

$$\leq f_a^2(\|A\|) \sup_{\|x\|=1} \left\langle \left||A|^{\alpha} B\right|^2 x, x \right\rangle \sup_{\|y\|=1} \left\langle \left||A^*|^{1-\alpha} C\right|^2 y, y \right\rangle$$
$$= f_a^2(\|A\|) \left\|\left||A|^{\alpha} B\right|^2\right\| \left\|\left||A^*|^{1-\alpha} C\right|^2\right\|$$
$$= f_a^2(\|A\|) \left\||A|^{\alpha} B\right\|^2 \left\||A^*|^{1-\alpha} C\right\|^2,$$

which gives (43).

From (28) we get, by taking $y = x$, the square root and using the *A-G-mean inequality*, that

$$|\langle C^* A f(A) B x, x\rangle| \tag{46}$$
$$\leq f_a(\|A\|) \left\langle \left||A|^{\alpha} B\right|^2 x, x \right\rangle^{1/2} \left\langle \left||A^*|^{1-\alpha} C\right|^2 x, x \right\rangle^{1/2}$$
$$\leq \frac{1}{2} f_a(\|A\|) \left(\left\langle \left||A|^{\alpha} B\right|^2 x, x \right\rangle + \left\langle \left||A^*|^{1-\alpha} C\right|^2 x, x \right\rangle \right)$$
$$= \frac{1}{2} f_a(\|A\|) \left\langle \left(\left||A|^{\alpha} B\right|^2 + \left||A^*|^{1-\alpha} C\right|^2 \right) x, x \right\rangle$$

for all $x \in H$.

By taking the supremum over $\|x\| = 1$ in (46) we get that

$$\omega(C^* A f(A) B)$$
$$= \sup_{\|x\|=1} |\langle C^* A f(A) B x, x\rangle|$$
$$\leq \frac{1}{2} f_a(\|A\|) \sup_{\|x\|=1} \left\langle \left(\left||A|^{\alpha} B\right|^2 + \left||A^*|^{1-\alpha} C\right|^2 \right) x, x \right\rangle$$
$$= \frac{1}{2} f_a(\|A\|) \left\| \left||A|^{\alpha} B\right|^2 + \left||A^*|^{1-\alpha} C\right|^2 \right\|,$$

which proves (44).

From (28) for $y = x$ and Buzano's inequality we derive that

$$|\langle C^* A f(A) B x, x\rangle|^2 \tag{47}$$
$$\leq f_a^2(\|A\|) \left\langle \left||A|^{\alpha} B\right|^2 x, x \right\rangle \left\langle x, \left||A^*|^{1-\alpha} C\right|^2 x \right\rangle$$
$$\leq \frac{1}{2} f_a^2(\|A\|)$$
$$\times \left[\left\| \left||A|^{\alpha} B\right|^2 x \right\| \left\| \left||A^*|^{1-\alpha} C\right|^2 x \right\| + \left| \left\langle \left||A|^{\alpha} B\right|^2 x, \left||A^*|^{1-\alpha} C\right|^2 x \right\rangle \right| \right]$$
$$= \frac{1}{2} f_a^2(\|A\|)$$

$$\times\left[\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|\right]$$

for all $x\in H,\|x\|=1$.

By taking the supremum over $\|x\|=1$ in (47) we get that

$$\begin{aligned}
&\omega^{2}\left(C^{*}Af\left(A\right)B\right)\\
&=\sup_{\|x\|=1}\left|\left\langle C^{*}Af\left(A\right)Bx,x\right\rangle\right|^{2}\\
&\leq\frac{1}{2}f_{a}^{2}\left(\|A\|\right)\\
&\times\sup_{\|x\|=1}\left[\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|\right]\\
&\leq\frac{1}{2}f_{a}^{2}\left(\|A\|\right)\\
&\times\left[\sup_{\|x\|=1}\left\{\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|\right\}\right.\\
&\left.+\sup_{\|x\|=1}\left|\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|\right]\\
&\leq\frac{1}{2}f_{a}^{2}\left(\|A\|\right)\\
&\times\left[\sup_{\|x\|=1}\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|\sup_{\|x\|=1}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|\right.\\
&\left.+\sup_{\|x\|=1}\left|\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|\right]\\
&=\frac{1}{2}f_{a}^{2}\left(\|A\|\right)\left[\left\|\left|\left|A\right|^{\alpha}B\right|^{2}\right\|\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\|+\omega\left(\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right)\right]\\
&=\frac{1}{2}f_{a}^{2}\left(\|A\|\right)\left[\left\|\left|A\right|^{\alpha}B\right\|^{2}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|^{2}+\omega\left(\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right)\right],
\end{aligned}$$

which proves (45). □

Remark 2. *If we take $\alpha=1/2$ in Theorem 3, then we get the norm inequality*

$$\left\|C^{*}Af\left(A\right)B\right\|\leq f_{a}\left(\|A\|\right)\left\|\left|A\right|^{1/2}B\right\|\left\|\left|A^{*}\right|^{1/2}C\right\| \tag{48}$$

and the numerical radius inequalities

$$\omega\left(C^{*}Af\left(A\right)B\right)\leq\frac{1}{2}f_{a}\left(\|A\|\right)\left\|\left|\left|A\right|^{1/2}B\right|^{2}+\left|\left|A^{*}\right|^{1/2}C\right|^{2}\right\| \tag{49}$$

and

$$\omega^2 (C^* A f(A) B) \leq \frac{1}{2} f_a^2 (\|A\|) \left[\left\| |A|^{1/2} B \right\|^2 \left\| |A^*|^{1/2} C \right\|^2 + \omega \left(\left| |A^*|^{1/2} C \right|^2 \left| |A|^{1/2} B \right|^2 \right) \right]. \tag{50}$$

The second main result is as follows:

Theorem 4. *Assume that the conditions of Theorem 3 are satisfied. If* $\alpha \in [0,1]$, $r > 0$, p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* pr, $qr \geq 1$, *then*

$$\omega^{2r} (C^* A f(A) B) \leq f_a^{2r} (\|A\|) \left\| \frac{1}{p} \left| |A|^\alpha B \right|^{2rp} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2rq} \right\|. \tag{51}$$

If $r \geq 1$, *then*

$$\omega^{2r} (C^* A f(A) B) \leq \frac{1}{2} f_a^{2r} (\|A\|) \left[\left\| |A|^\alpha B \right\|^{2r} \left\| |A^*|^{1-\alpha} C \right\|^{2r} + \omega^r \left(\left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^\alpha B \right|^2 \right) \right]. \tag{52}$$

If $r \geq 1$, p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* pr, $qr \geq 2$, *then also*

$$\omega^{2r} (C^* A f(A) B) \leq \frac{1}{2} f_a^{2r} (\|A\|) \left(\left\| \frac{1}{p} \left| |A|^\alpha B \right|^{2pr} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2qr} \right\| + \omega^r \left(\left| |A^*|^{1-\alpha} C \right|^2 \left| |A|^\alpha B \right|^2 \right) \right). \tag{53}$$

Proof. If we take the power $r > 0$ in (28) written for $y = x$ then we get, by Young and McCarthy inequalities that

$$\begin{aligned}
&|\langle C^* A f(A) Bx, x \rangle|^{2r} \\
&\leq f_a^{2r} (\|A\|) \left\langle \left| |A|^\alpha B \right|^2 x, x \right\rangle^r \left\langle \left| |A^*|^{1-\alpha} C \right|^2 x, x \right\rangle^r \\
&\leq f_a^{2r} (\|A\|) \left[\frac{1}{p} \left\langle \left| |A|^\alpha B \right|^2 x, x \right\rangle^{rp} + \frac{1}{q} \left\langle \left| |A^*|^{1-\alpha} C \right|^2 x, x \right\rangle^{rq} \right] \\
&\leq f_a^{2r} (\|A\|) \left[\frac{1}{p} \left\langle \left| |A|^\alpha B \right|^{2rp} x, x \right\rangle + \frac{1}{q} \left\langle \left| |A^*|^{1-\alpha} C \right|^{2rq} x, x \right\rangle \right] \\
&= f_a^{2r} (\|A\|) \left[\left\langle \frac{1}{p} \left| |A|^\alpha B \right|^{2rp} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2rq} x, x \right\rangle \right]
\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, then we get that

$$\begin{aligned}
&\omega^{2r} (C^* A f(A) B) \\
&= \sup_{\|x\|=1} |\langle C^* A f(A) Bx, x \rangle|^{2r}
\end{aligned}$$

$$\leq f_a^{2r}(\|A\|) \sup_{\|x\|=1}\left[\left\langle\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2rp}+\frac{1}{q}\left|\left|A^*\right|^{1-\alpha}C\right|^{2rq}\right)x,x\right\rangle\right]$$
$$= f_a^{2r}(\|A\|)\left\|\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2rp}+\frac{1}{q}\left|\left|A^*\right|^{1-\alpha}C\right|^{2rq}\right\|,$$

which proves (51).

If we take the power $r \geq 1$ in (47) and by using the convexity of the power function, we get

$$\left|\left\langle C^*Af(A)Bx,x\right\rangle\right|^{2r} \tag{54}$$
$$= f_a^{2r}(\|A\|)$$
$$\times\left[\frac{\left\|\left|\left|A\right|^{\alpha}B\right|^2x\right\|\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\|+\left|\left\langle\left|\left|A^*\right|^{1-\alpha}C\right|^2\left|\left|A\right|^{\alpha}B\right|^2x,x\right\rangle\right|}{2}\right]^r$$
$$\leq f_a^{2r}(\|A\|)$$
$$\times\frac{\left\|\left|\left|A\right|^{\alpha}B\right|^2x\right\|^r\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\|^r+\left|\left\langle\left|\left|A^*\right|^{1-\alpha}C\right|^2\left|\left|A\right|^{\alpha}B\right|^2x,x\right\rangle\right|^r}{2}$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, then we get that

$$\omega^{2r}(C^*Af(A)B)$$
$$\leq f_a^{2r}(\|A\|)$$
$$\times\frac{\left\|\left|\left|A\right|^{\alpha}B\right|^2\right\|^r\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^2\right\|^r+\omega^r\left(\left|\left|A^*\right|^{1-\alpha}C\right|^2\left|\left|A\right|^{\alpha}B\right|^2\right)}{2}$$
$$= f_a^{2r}(\|A\|)$$
$$\times\frac{\left\|\left|A\right|^{\alpha}B\right\|^{2r}\left\|\left|A^*\right|^{1-\alpha}C\right\|^{2r}+\omega^r\left(\left|\left|A^*\right|^{1-\alpha}C\right|^2\left|\left|A\right|^{\alpha}B\right|^2\right)}{2},$$

which proves (52).

Also, observe that

$$\left\|\left|\left|A\right|^{\alpha}B\right|^2x\right\|^r\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\|^r$$
$$\leq\frac{1}{p}\left\|\left|\left|A\right|^{\alpha}B\right|^2x\right\|^{pr}+\frac{1}{q}\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\|^{qr}$$
$$=\frac{1}{p}\left\|\left|\left|A\right|^{\alpha}B\right|^2x\right\|^{2\frac{pr}{2}}+\frac{1}{q}\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\|^{2\frac{qr}{2}}$$

$$= \frac{1}{p}\left\langle \left|\left|A\right|^{\alpha}B\right|^{4}x,x\right\rangle^{\frac{pr}{2}} + \frac{1}{q}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}x,x\right\rangle^{\frac{qr}{2}}$$
$$\leq \frac{1}{p}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2pr}x,x\right\rangle + \frac{1}{q}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}x,x\right\rangle$$
$$= \left\langle \left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr} + \frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle,$$

for $x \in H$ with $\|x\| = 1$. Then

$$\frac{\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|^{r}\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|^{r} + \left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|^{r}}{2}$$
$$\leq \frac{1}{2}\left[\left\langle \left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr} + \frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle + \left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|^{r}\right]$$

and by (54)

$$\left|\left\langle C^{*}Af\left(A\right)Bx,x\right\rangle\right|^{2r}$$
$$\leq \frac{1}{2}f_{a}^{2r}\left(\|A\|\right)\left[\left\langle \left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr} + \frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle + \left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|^{r}\right]$$

for $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$, we derive (53). □

Remark 3. *If we take $r = 1$ and $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ in (51), then we obtain*

$$\omega^{2}\left(C^{*}Af\left(A\right)B\right) \leq f_{a}^{2}\left(\|A\|\right)\left\|\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2p} + \frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2q}\right\|, \tag{55}$$

which for $p = q = 2$ gives

$$\omega^{2}\left(C^{*}Af\left(A\right)B\right) \leq \frac{1}{2}f_{a}^{2}\left(\|A\|\right)\left\|\left|\left|A\right|^{\alpha}B\right|^{4} + \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}\right\|. \tag{56}$$

If we take $r = 1$ and $p = q = 2$ in (53), then we get

$$\omega^{2}\left(C^{*}Af\left(A\right)B\right) \leq \frac{1}{2}f_{a}^{2}\left(\|A\|\right)\left(\frac{1}{2}\left\|\left|\left|A\right|^{\alpha}B\right|^{4} + \left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}\right\| + \omega\left(\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}\right)\right). \tag{57}$$

If we take $r=2$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ in (53), then we get

$$\omega^4\left(C^*Af\left(A\right)B\right)\leq\frac{1}{2}f_a^4\left(\|A\|\right)\left(\left\|\frac{1}{p}\left||A|^{\alpha}B\right|^{4p}+\frac{1}{q}\left|\left|A^*\right|^{1-\alpha}C\right|^{4q}\right\|\right.\tag{58}$$
$$\left.+\omega^2\left(\left|\left|A^*\right|^{1-\alpha}C\right|^2\left||A|^{\alpha}B\right|^2\right)\right).$$

We also have:

Theorem 5. *With the assumptions of Theorem 3, we have for $r\geq1,\ \lambda\in[0,1]$ that*

$$\omega^2\left(C^*Af\left(A\right)B\right)\leq f_a^2\left(\|A\|\right)\left\|\left(1-\lambda\right)\left||A|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right\|^{1/r}\tag{59}$$
$$\times\left\||A|^{\alpha}B\right\|^{2\lambda}\left\|\left|A^*\right|^{1-\alpha}C\right\|^{2(1-\lambda)}$$

for all $\alpha\in[0,1]$.

Also, we have

$$\omega^2\left(C^*Af\left(A\right)B\right)\leq f_a^2\left(\|A\|\right)\left\|\left(1-\lambda\right)\left||A|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right\|^{1/r}\tag{60}$$
$$\times\left\|\lambda\left||A|^{\alpha}B\right|^{2r}+\left(1-\lambda\right)\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right\|^{1/r}$$

for all $\alpha\in[0,1]$ and $r\geq1$.

Proof. From the first part of (47) we have

$$\begin{aligned}&\left|\left\langle C^*Af\left(A\right)Bx,x\right\rangle\right|^2\\&\leq f_a^2\left(\|A\|\right)\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle\\&=f_a^2\left(\|A\|\right)\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle^{1-\lambda}\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle^{\lambda}\\&\times\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle^{\alpha}\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle^{1-\lambda}\\&\leq f_a^2\left(\|A\|\right)\left[\left(1-\lambda\right)\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle+\lambda\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle\right]\\&\times\left\langle\left||A|^{\alpha}B\right|^2x,x\right\rangle^{\lambda}\left\langle x,\left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\rangle^{1-\lambda}\end{aligned}$$

for all $x\in H,\ \|x\|=1.$

If we take the power $r\geq1$, then we get by the convexity of power r that

$$\left|\left\langle C^*Af\left(A\right)Bx,x\right\rangle\right|^{2r}\tag{61}$$

$$\leq f_a^{2r}(\|A\|)\left[(1-\lambda)\left\langle \left||A|^{\alpha}B\right|^2 x,x\right\rangle+\lambda\left\langle x,\left||A^*|^{1-\alpha}C\right|^2 x\right\rangle\right]^r$$
$$\times\left\langle \left||A|^{\alpha}B\right|^2 x,x\right\rangle^{r\lambda}\left\langle x,\left||A^*|^{1-\alpha}C\right|^2 x\right\rangle^{r(1-\lambda)}$$
$$\leq f_a^{2r}(\|A\|)\left[(1-\lambda)\left\langle \left||A|^{\alpha}B\right|^2 x,x\right\rangle^r+\lambda\left\langle x,\left||A^*|^{1-\alpha}C\right|^2 x\right\rangle^r\right]$$
$$\times\left\langle \left||A|^{\alpha}B\right|^2 x,x\right\rangle^{r\lambda}\left\langle x,\left||A^*|^{1-\alpha}C\right|^2 x\right\rangle^{r(1-\lambda)}$$

for all $x\in H$, $\|x\|=1$.

If we use McCarthy inequality for power $r\geq 1$, then we get

$$(1-\lambda)\left\langle \left||A|^{\alpha}B\right|^2 x,x\right\rangle^r+\lambda\left\langle x,\left||A^*|^{1-\alpha}C\right|^2 x\right\rangle^r$$
$$\leq(1-\lambda)\left\langle \left||A|^{\alpha}B\right|^{2r} x,x\right\rangle+\lambda\left\langle x,\left||A^*|^{1-\alpha}C\right|^{2r} x\right\rangle$$
$$=\left\langle\left[(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^*|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle$$

and by (61)

$$\begin{aligned}&|\langle C^*Af(A)Bx,x\rangle|^{2r}\\&\leq f_a^{2r}(\|A\|)\left[\left\langle\left[(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^*|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]\\&\times\left\langle \left||A|^{\alpha}B\right|^2 x,x\right\rangle^{r\lambda}\left\langle x,\left||A^*|^{1-\alpha}C\right|^2 x\right\rangle^{r(1-\lambda)}\end{aligned}\tag{62}$$

for all $x\in H$, $\|x\|=1$.

If we take the supremum over $\|x\|=1$, then we get

$$\omega^{2r}(C^*Af(A)B)$$
$$=\sup_{\|x\|=1}|\langle C^*Af(A)Bx,x\rangle|^{2r}$$
$$\leq f_a^{2r}(\|A\|)\sup_{\|x\|=1}\left[\left\langle\left[(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^*|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]$$
$$\times\sup_{\|x\|=1}\left\langle \left||A|^{\alpha}B\right|^2 x,x\right\rangle^{r\lambda}\sup_{\|x\|=1}\left\langle x,\left||A^*|^{1-\alpha}C\right|^2 x\right\rangle^{r(1-\lambda)}$$
$$=f_a^{2r}(\|A\|)\left\|(1-\lambda)\left||A|^{\alpha}B\right|^{2r}+\lambda\left||A^*|^{1-\alpha}C\right|^{2r}\right\|$$
$$\times\left\||A|^{\alpha}B\right\|^{2r\lambda}\left\||A^*|^{1-\alpha}C\right\|^{2r(1-\lambda)},$$

which gives (59).

We also have

$$\begin{aligned}
&\left|\langle C^{*}Af\left(A\right)Bx,x\rangle\right|^{2r}\\
&\leq f_{a}^{2r}\left(\|A\|\right)\left[\left\langle\left[\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]\\
&\times\left[\left\langle\left[\lambda\left|\left|A\right|^{\alpha}B\right|^{2r}+\left(1-\lambda\right)\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right]x,x\right\rangle\right]
\end{aligned}$$

for all $x\in H$, $\|x\|=1$, which proves (60). □

Remark 4. *If we take $r=1$ in Theorem 5, then we get*

$$\begin{aligned}
\omega^{2}\left(C^{*}Af\left(A\right)B\right)\leq f_{a}^{2}\left(\|A\|\right)&\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\|\\
&\times\left\|\left|A\right|^{\alpha}B\right\|^{2\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|^{2(1-\lambda)}
\end{aligned}\tag{63}$$

and

$$\begin{aligned}
\omega^{2}\left(C^{*}Af\left(A\right)B\right)\leq f_{a}^{2}\left(\|A\|\right)&\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\|\\
&\times\left\|\lambda\left|\left|A\right|^{\alpha}B\right|^{2}+\left(1-\lambda\right)\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\|
\end{aligned}\tag{64}$$

for all α, $\lambda\in[0,1]$.

If we take $\lambda=1/2$ in (63), then we obtain

$$\begin{aligned}
&\omega^{2}\left(C^{*}Af\left(A\right)B\right)\\
&\leq\frac{1}{2}f_{a}^{2}\left(\|A\|\right)\left\|\left|\left|A\right|^{\alpha}B\right|^{2}+\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\|\left\|\left|A\right|^{\alpha}B\right\|\left\|\left|A^{*}\right|^{1-\alpha}C\right\|
\end{aligned}\tag{65}$$

If we take $r=2$ in Theorem 5, then we get

$$\begin{aligned}
\omega^{2}\left(C^{*}Af\left(A\right)B\right)\leq f_{a}^{2}\left(\|A\|\right)&\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{4}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}\right\|^{1/2}\\
&\times\left\|\left|A\right|^{\alpha}B\right\|^{2\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|^{2(1-\lambda)}
\end{aligned}\tag{66}$$

and

$$\begin{aligned}
\omega^{2}\left(C^{*}Af\left(A\right)B\right)\leq f_{a}^{2}\left(\|A\|\right)&\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{4}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}\right\|^{1/2}\\
&\times\left\|\lambda\left|\left|A\right|^{\alpha}B\right|^{4}+\left(1-\lambda\right)\left|\left|A^{*}\right|^{1-\alpha}C\right|^{4}\right\|^{1/2}
\end{aligned}\tag{67}$$

for all α, $\lambda\in[0,1]$.

If we take $\lambda = 1/2$ in (66), then we obtain

$$\omega^2 \left(C^* A f(A) B\right) \le \frac{\sqrt{2}}{2} f_a^2 (\|A\|) \left\| \left||A|^\alpha B\right|^4 + \left||A^*|^{1-\alpha} C\right|^4 \right\|^{1/2} \left\||A|^\alpha B\right\| \left\||A^*|^{1-\alpha} C\right\|. \tag{68}$$

4. Inequalities for Trace of Operators

We have the following result for trace of operators:

Theorem 6. *Let $r \ge 1/2$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and pr, $qr \ge 1$. Assume that the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is convergent on $D(0, R)$ and A, B, $C \in B(H)$ with $\|A\| < R$. If $|A|^\alpha B \in \mathcal{B}_{2pr}(H)$ and $|A^*|^{1-\alpha} C \in \mathcal{B}_{2qr}(H)$ for $\alpha \in [0, 1]$, then $C^* A f(A) B \in \mathcal{B}_{2r}(H)$ and*

$$\|C^* A f(A) B\|_{2r} \le f_a(\|A\|) \left\||A|^\alpha B\right\|_{2pr} \left\||A^*|^{1-\alpha} C\right\|_{2qr}. \tag{69}$$

In particular,

$$\|C^* A f(A) B\|_{2r} \le f_a(\|A\|) \left\||A|^{1/2} B\right\|_{2pr} \left\||A^*|^{1/2} C\right\|_{2qr} \tag{70}$$

for $|A|^{1/2} B \in \mathcal{B}_{2pr}(H)$ and $|A^*|^{1/2} C \in \mathcal{B}_{2qr}(H)$.

Proof. If we take in (28) the power $r > 0$ and $x = e_i$, $y = f_i$ where $\mathcal{E} = \{e_i\}_{i \in I}$ and $\mathcal{F} = \{f_i\}_{i \in I}$ are orthonormal basis and sum, then we get

$$\sum_{i \in I} \left|\langle C^* A f(A) B e_i, f_i\rangle\right|^{2r} \le f_a^{2r}(\|A\|) \sum_{i \in I} \left\langle \left||A|^\alpha B\right|^2 e_i, e_i \right\rangle^r \left\langle \left||A^*|^{1-\alpha} C\right|^2 f_i, f_i \right\rangle^r. \tag{71}$$

If we use the Hölder's inequality for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we get

$$\sum_{i \in I} \left\langle \left||A|^\alpha B\right|^2 e_i, e_i \right\rangle^r \left\langle \left||A^*|^{1-\alpha} C\right|^2 f_i, f_i \right\rangle^r \le \left(\sum_{i \in I} \left\langle \left||A|^\alpha B\right|^2 e_i, e_i \right\rangle^{pr} \right)^{1/p} \left(\sum_{i \in I} \left\langle \left||A^*|^{1-\alpha} C\right|^2 f_i, f_i \right\rangle^{qr} \right)^{1/q} \tag{72}$$

By the McCarthy inequality for pr, $qr \ge 1$, we have

$$\sum_{i \in I} \left\langle \left||A|^\alpha B\right|^2 e_i, e_i \right\rangle^{pr} \le \sum_{i \in I} \left\langle \left||A|^\alpha B\right|^{2pr} e_i, e_i \right\rangle$$

and

$$\sum_{i \in I} \left\langle \left||A^*|^{1-\alpha} C\right|^2 f_i, f_i \right\rangle^{qr} \le \sum_{i \in I} \left\langle \left||A^*|^{1-\alpha} C\right|^{2qr} f_i, f_i \right\rangle,$$

therefore

$$\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}e_{i},e_{i}\right\rangle^{pr}\right)^{1/p}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{qr}\right)^{1/q}$$

$$\leq\left(\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2pr}e_{i},e_{i}\right\rangle\right)^{1/p}\left(\sum_{i\in I}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}f_{i},f_{i}\right\rangle^{qr}\right)^{1/q}$$

$$=\left(\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2pr}\right)^{1/p}\left(\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2qr}\right)^{1/q}=\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2r}.$$

By (71) and (72) we derive

$$\sum_{i\in I}\left|\left\langle C^{*}Af\left(A\right)Be_{i},f_{i}\right\rangle\right|^{2r}\leq f_{a}^{2r}\left(\left\|A\right\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2r}. \tag{73}$$

Now, if we take the supremum over $\mathcal{E}$ and $\mathcal{F}$ in (30), then by (21) we get

$$\left\|C^{*}Af\left(A\right)B\right\|_{2r}^{2r}\leq f_{a}^{2r}\left(\left\|A\right\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2pr}^{2r}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2qr}^{2r}$$

and the inequality (69) is obtained. □

Remark 5. *If we take $r=1/2$ and $p=q=2$, then by (69) we get*

$$\left\|C^{*}Af\left(A\right)B\right\|_{1}\leq f_{a}\left(\left\|A\right\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2} \tag{74}$$

provided that $\left|A\right|^{\alpha}B\in\mathcal{B}_{2}\left(H\right)$ and $\left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_{2}\left(H\right)$ for $\alpha\in[0,1]$.

Also, if $r=1$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by (69) we get

$$\left\|C^{*}Af\left(A\right)B\right\|_{2}\leq f_{a}\left(\left\|A\right\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2p}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2q} \tag{75}$$

provided that $\left|A\right|^{\alpha}B\in\mathcal{B}_{2p}\left(H\right)$ and $\left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_{2q}\left(H\right)$ for $\alpha\in[0,1]$.

We also have:

Theorem 7. *Let $r\geq 1/2$, $p,\ q\geq 1$ with $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. Assume that the power series with complex coefficients $f\left(z\right):=\sum_{k=0}^{\infty}a_{k}z^{k}$ is convergent on $D(0,R)$ and $A,\ B,\ C\in B\left(H\right)$ with $\left\|A\right\|<R$. If $\left|A\right|^{\alpha}B\in\mathcal{B}_{2p}\left(H\right)$ and $\left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_{2q}\left(H\right)$ for $\alpha\in[0,1]$, then $C^{*}Af\left(A\right)B\in\mathcal{B}_{2r}\left(H\right)$ and*

$$\left\|C^{*}Af\left(A\right)B\right\|_{2r}\leq f_{a}\left(\left\|A\right\|\right)\left\|\left|A\right|^{\alpha}B\right\|_{2p}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2q}. \tag{76}$$

In particular,

$$\left\|C^{*}Af\left(A\right)B\right\|_{2r}\leq f_{a}\left(\left\|A\right\|\right)\left\|\left|A\right|^{1/2}B\right\|_{2p}\left\|\left|A^{*}\right|^{1/2}C\right\|_{2q} \tag{77}$$

for $\left|A\right|^{1/2}B\in\mathcal{B}_{2p}\left(H\right)$ and $\left|A^{*}\right|^{1/2}C\in\mathcal{B}_{2q}\left(H\right)$.

Proof. Assume that $\mathcal{E} = \{e_i\}_{i\in I}$ and $\mathcal{F} = \{f_i\}_{i\in I}$ are orthonormal basis in H. Observe that we have $\frac{1}{\frac{p}{r}} + \frac{1}{\frac{q}{r}} = 1$ and by Hölder's inequality for $\frac{p}{r}$ and $\frac{q}{r}$ we have

$$\sum_{i\in I} \left\langle \left|\left|A\right|^{\alpha} B\right|^2 e_i, e_i \right\rangle^r \left\langle \left|\left|A^*\right|^{1-\alpha} C\right|^2 f_i, f_i \right\rangle^r \tag{78}$$
$$= \sum_{i\in I} \left[\left\langle \left|\left|A\right|^{\alpha} B\right|^2 e_i, e_i \right\rangle^p\right]^{\frac{r}{p}} \left[\left\langle \left|\left|A^*\right|^{1-\alpha} C\right|^2 f_i, f_i \right\rangle^q\right]^{\frac{r}{q}}$$
$$\le \left(\sum_{i\in I} \left\langle \left|\left|A\right|^{\alpha} B\right|^2 e_i, e_i \right\rangle^p\right)^{r/p} \left(\sum_{i\in I} \left\langle \left|\left|A^*\right|^{1-\alpha} C\right|^2 f_i, f_i \right\rangle^q\right)^{r/q}.$$

By McCarthy inequality for p, $q > 1$ we get

$$\sum_{i\in I} \left\langle \left|\left|A\right|^{\alpha} B\right|^2 e_i, e_i \right\rangle^p \le \sum_{i\in I} \left\langle \left|\left|A\right|^{\alpha} B\right|^{2p} e_i, e_i \right\rangle$$

and

$$\sum_{i\in I} \left\langle \left|\left|A^*\right|^{1-\alpha} C\right|^2 f_i, f_i \right\rangle^q \le \sum_{i\in I} \left\langle \left|\left|A^*\right|^{1-\alpha} C\right|^{2q} f_i, f_i \right\rangle$$

and by (78)

$$\sum_{i\in I} \left\langle \left|\left|A\right|^{\alpha} B\right|^2 e_i, e_i \right\rangle^r \left\langle \left|\left|A^*\right|^{1-\alpha} C\right|^2 f_i, f_i \right\rangle^r \tag{79}$$
$$\le \left(\sum_{i\in I} \left\langle \left|\left|A\right|^{\alpha} B\right|^{2p} e_i, e_i \right\rangle\right)^{r/p} \left(\sum_{i\in I} \left\langle \left|\left|A^*\right|^{1-\alpha} C\right|^{2q} f_i, f_i \right\rangle\right)^{r/q}$$
$$= \left\|\left|A\right|^{\alpha} B\right\|_{2p}^{2r} \left\|\left|A^*\right|^{1-\alpha} C\right\|_{2q}^{2r}.$$

By (71) and (79) we get

$$\sum_{i\in I} \left|\left\langle C^* A f\left(A\right) B e_i, f_i \right\rangle\right|^{2r} \le f_a^{2r}\left(\|A\|\right) \left\|\left|A\right|^{\alpha} B\right\|_{2p}^{2r} \left\|\left|A^*\right|^{1-\alpha} C\right\|_{2q}^{2r}. \tag{80}$$

Now, if we take the supremum over $\mathcal{E}$ and $\mathcal{F}$ in (80) we get

$$\left\|C^* A f\left(A\right) B\right\|_{2r}^{2r} \le f_a^{2r}\left(\|A\|\right) \left\|\left|A\right|^{\alpha} B\right\|_{2p}^{2r} \left\|\left|A^*\right|^{1-\alpha} C\right\|_{2q}^{2r}$$

and the inequality (76) is thus proved. □

Remark 6. *If we take $p = q = 2r = s \ge 1$, then by (76) we get*

$$\left\|C^* A f\left(A\right) B\right\|_{s} \le f_a\left(\|A\|\right) \left\|\left|A\right|^{\alpha} B\right\|_{2s} \left\|\left|A^*\right|^{1-\alpha} C\right\|_{2s} \tag{81}$$

provided that $\left|A\right|^{\alpha} B \in \mathcal{B}_{2s}\left(H\right)$ and $\left|A^*\right|^{1-\alpha} C \in \mathcal{B}_{2s}\left(H\right)$ for $\alpha \in [0,1]$.

For $\alpha = 1/2$ we have

$$\|C^* A f(A) B\|_s \le f_a(\|A\|) \left\| |A|^{1/2} B \right\|_{2s} \left\| |A^*|^{1/2} C \right\|_{2s} \tag{82}$$

provided that $|A|^{1/2} B \in \mathcal{B}_{2s}(H)$ and $|A^*|^{1/2} C \in \mathcal{B}_{2s}(H)$.

If $r = 2$ and $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, then

$$\|C^* A f(A) B\|_4 \le f_a(\|A\|) \| |A|^\alpha B \|_{2p} \left\| |A^*|^{1-\alpha} C \right\|_{2q} \tag{83}$$

provided that $|A|^\alpha B \in \mathcal{B}_{2p}(H)$ and $|A^*|^{1-\alpha} C \in \mathcal{B}_{2q}(H)$ for $\alpha \in [0,1]$.

In particular,

$$\|C^* A f(A) B\|_4 \le f_a(\|A\|) \left\| |A|^{1/2} B \right\|_{2p} \left\| |A^*|^{1/2} C \right\|_{2q} \tag{84}$$

for $|A|^{1/2} B \in \mathcal{B}_{2p}(H)$ and $|A^*|^{1/2} C \in \mathcal{B}_{2q}(H)$.

Theorem 8. *Assume that the power series with complex coefficients $f(z) := \sum_{k=0}^{\infty} a_k z^k$ is convergent on $D(0,R)$, $A,\ B,\ C \in B(H)$ with $\|A\| < R$.*

If $r \ge 1/2$, $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $pr,\ qr \ge 1$ and $||A|^\alpha B|^{2pr}$, $\left| |A^*|^{1-\alpha} C \right|^{2qr} \in \mathcal{B}_1(H)$, then $C^* A f(A) B \in \mathcal{B}_{2r}(H)$ and

$$\omega_{2r}^{2r}(C^* A f(A) B) \le f_a^{2r}(\|A\|) \operatorname{tr}\left(\frac{1}{p} ||A|^\alpha B|^{2pr} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2qr} \right). \tag{85}$$

If $r \ge 1$ and $|A|^\alpha B, |A^*|^{1-\alpha} C \in \mathcal{B}_{4r}(H)$, then $C^* A f(A) B \in \mathcal{B}_{2r}(H)$ and

$$\begin{aligned} &\omega_{2r}^{2r}(C^* A f(A) B) \\ &\le \frac{1}{2} f_a^{2r}(\|A\|) \left(\| |A|^\alpha B \|_{4r}^{2r} \left\| |A^*|^{1-\alpha} C \right\|_{4r}^{2r} + \omega_r^r \left(\left| |A^*|^{1-\alpha} C \right|^2 ||A|^\alpha B|^2 \right) \right) \\ &\le \frac{1}{2} f_a^{2r}(\|A\|) \left(\| |A|^\alpha B \|_{4r}^{2r} \left\| |A^*|^{1-\alpha} C \right\|_{4r}^{2r} + \left\| \left| |A^*|^{1-\alpha} C \right|^2 ||A|^\alpha B|^2 \right\|_r^r \right). \end{aligned} \tag{86}$$

If $r \ge 1$, $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $pr, qr \ge 2$, then

$$\begin{aligned} \omega_{2r}^{2r}(C^* A f(A) B) &\le \frac{1}{2} f_a^{2r}(\|A\|) \left[\operatorname{tr}\left(\frac{1}{p} ||A|^\alpha B|^{2pr} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2qr} \right) \right. \\ &\quad \left. + \omega_r^r \left(\left| |A^*|^{1-\alpha} C \right|^2 ||A|^\alpha B|^2 \right) \right] \\ &\le \frac{1}{2} f_a^{2r}(\|A\|) \left[\operatorname{tr}\left(\frac{1}{p} ||A|^\alpha B|^{2pr} + \frac{1}{q} \left| |A^*|^{1-\alpha} C \right|^{2qr} \right) \right. \\ &\quad \left. + \left\| \left| |A^*|^{1-\alpha} C \right|^2 ||A|^\alpha B|^2 \right\|_r^r \right]. \end{aligned} \tag{87}$$

Proof. From (28) for $y = x$ we have that

$$\left|\left\langle C^{*}Af\left(A\right)Bx,x\right\rangle\right|^{2}\leq f_{a}^{2}\left(\left\|A\right\|\right)\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x,x\right\rangle \tag{88}$$

for $x \in H$ with $\|x\| = 1$.

If we take the power $r > 0$, we get, by Young and McCarthy inequalities, that

$$\begin{aligned}
&\left|\left\langle C^{*}Af\left(A\right)Bx,x\right\rangle\right|^{2r}\\
&\leq f_{a}^{2r}\left(\left\|A\right\|\right)\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x,x\right\rangle^{r}\\
&\leq f_{a}^{2r}\left(\left\|A\right\|\right)\left[\frac{1}{p}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{pr}+\frac{1}{q}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x,x\right\rangle^{qr}\right]\\
&\leq f_{a}^{2r}\left(\left\|A\right\|\right)\left[\frac{1}{p}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2pr}x,x\right\rangle+\frac{1}{q}\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}x,x\right\rangle\right]\\
&= f_{a}^{2r}\left(\left\|A\right\|\right)\left\langle \left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle
\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

If $\mathcal{E} = \{e_i\}_{i\in I}$ is an orthonormal basis, then by taking $x = e_i$ and summing over $i \in I$ we get

$$\begin{aligned}
&\left\|C^{*}Af\left(A\right)B\right\|_{\mathcal{E},2r}^{2r}\\
&=\sum_{i\in I}\left|\left\langle C^{*}Af\left(A\right)Be_{i},e_{i}\right\rangle\right|^{2r}\\
&\leq f_{a}^{2r}\left(\left\|A\right\|\right)\sum_{i\in I}\left\langle \left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right)e_{i},e_{i}\right\rangle\\
&= f_{a}^{2r}\left(\left\|A\right\|\right)\operatorname{tr}\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2qr}\right),
\end{aligned}$$

which, by taking the supremum over $\mathcal{E}$, proves (85).

By Buzano's inequality we have

$$\begin{aligned}
&\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle\\
&\leq\frac{1}{2}\left[\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle\right|\right]\\
&=\frac{1}{2}\left[\left\|\left|\left|A\right|^{\alpha}B\right|^{2}x\right\|\left\|\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\|+\left|\left\langle \left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle\right|\right]
\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

If we take the power $r \geq 1$ and use the convexity of power function, then we get

$$\left\langle \left|\left|A\right|^{\alpha}B\right|^{2}x,x\right\rangle^{r}\left\langle x,\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}x\right\rangle^{r}$$

$$\leq \left[\frac{\left\| \left| |A|^{\alpha} B \right|^{2} x \right\| \left\| \left| |A^{*}|^{1-\alpha} C \right|^{2} x \right\| + \left| \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} x, x \right\rangle \right|}{2} \right]^{r}$$

$$\leq \frac{\left\| \left| |A|^{\alpha} B \right|^{2} x \right\|^{r} \left\| \left| |A^{*}|^{1-\alpha} C \right|^{2} x \right\|^{r} + \left| \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} x, x \right\rangle \right|^{r}}{2}$$

$$= \frac{\left\| \left| |A|^{\alpha} B \right|^{2} x \right\|^{2\frac{r}{2}} \left\| \left| |A^{*}|^{1-\alpha} C \right|^{2} x \right\|^{2\frac{r}{2}} + \left| \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} x, x \right\rangle \right|^{r}}{2}$$

$$= \frac{\left\langle \left| |A|^{\alpha} B \right|^{4} x, x \right\rangle^{\frac{r}{2}} \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{4} x, x \right\rangle^{\frac{r}{2}} + \left| \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} x, x \right\rangle \right|^{r}}{2}$$

for $x \in H$ with $\|x\| = 1$.

Therefore

$$\begin{aligned} & \|C^{*} A f(A) B\|_{\mathcal{E},2r}^{2r} \qquad (89) \\ & = \sum_{i \in I} \left| \langle C^{*} A f(A) B e_i, e_i \rangle \right|^{2r} \\ & \leq f_a^{2r}(\|A\|) \sum_{i \in I} \left\langle \left| |A|^{\alpha} B \right|^{2} e_i, e_i \right\rangle^{r} \left\langle e_i, \left| |A^{*}|^{1-\alpha} C \right|^{2} e_i \right\rangle^{r} \\ & \leq \frac{1}{2} f_a^{2r}(\|A\|) \left[\sum_{i \in I} \left\langle \left| |A|^{\alpha} B \right|^{4} e_i, e_i \right\rangle^{\frac{r}{2}} \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{4} e_i, e_i \right\rangle^{\frac{r}{2}} \right. \\ & \left. + \sum_{i \in I} \left| \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{2} \left| |A|^{\alpha} B \right|^{2} e_i, e_i \right\rangle \right|^{r} \right] \end{aligned}$$

Using Cauchy-Schwarz inequality we have

$$\begin{aligned} & \sum_{i \in I} \left\langle \left| |A|^{\alpha} B \right|^{4} e_i, e_i \right\rangle^{\frac{r}{2}} \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{4} e_i, e_i \right\rangle^{\frac{r}{2}} \\ & \leq \left(\sum_{i \in I} \left\langle \left| |A|^{\alpha} B \right|^{4} e_i, e_i \right\rangle^{r} \right)^{1/2} \left(\sum_{i \in I} \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{4} e_i, e_i \right\rangle^{r} \right)^{1/2} \\ & \leq \left(\sum_{i \in I} \left\langle \left| |A|^{\alpha} B \right|^{4r} e_i, e_i \right\rangle \right)^{1/2} \left(\sum_{i \in I} \left\langle \left| |A^{*}|^{1-\alpha} C \right|^{4r} e_i, e_i \right\rangle \right)^{1/2} \\ & = \left\| |A|^{\alpha} B \right\|_{4r}^{2r} \left\| |A^{*}|^{1-\alpha} C \right\|_{4r}^{2r}, \end{aligned}$$

where for the last inequality we used McCarthy's result for $r \geq 1$. This proves (86).

Further, if we use Young's inequality for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$ab\le \frac{1}{p}a^p+\frac{1}{q}b^q,\ a,b\ge 0,$$

then we get

$$\begin{aligned}\left\| \left|\left|A\right|^{\alpha}B\right|^2x\right\|^r\left\| \left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\|^r &\le \frac{1}{p}\left\| \left|\left|A\right|^{\alpha}B\right|^2x\right\|^{pr}+\frac{1}{q}\left\| \left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\|^{qr}\\ &=\frac{1}{p}\left\| \left|\left|A\right|^{\alpha}B\right|^2x\right\|^{2\frac{pr}{2}}+\frac{1}{q}\left\| \left|\left|A^*\right|^{1-\alpha}C\right|^2x\right\|^{2\frac{qr}{2}}\\ &=\frac{1}{p}\left\langle \left|\left|A\right|^{\alpha}B\right|^4x,x\right\rangle^{\frac{pr}{2}}+\frac{1}{q}\left\langle \left|\left|A^*\right|^{1-\alpha}C\right|^4x,x\right\rangle^{\frac{qr}{2}}\\ &\le \frac{1}{p}\left\langle \left|\left|A\right|^{\alpha}B\right|^{2pr}x,x\right\rangle+\frac{1}{q}\left\langle \left|\left|A^*\right|^{1-\alpha}C\right|^{2qr}x,x\right\rangle\\ &=\left\langle \left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^*\right|^{1-\alpha}C\right|^{2qr}\right)x,x\right\rangle\end{aligned}$$

for $x\in H$ with $\|x\|=1$.

Therefore

$$\begin{aligned}\left\|C^*Af\left(A\right)B\right\|_{\mathcal{E},2r}^{2r}&=\sum_{i\in I}\left|\left\langle C^*Af\left(A\right)Be_i,e_i\right\rangle\right|^{2r}\\ &\le f_a^{2r}\left(\|A\|\right)\sum_{i\in I}\left\langle \left|\left|A\right|^{\alpha}B\right|^2e_i,e_i\right\rangle^r\left\langle e_i,\left|\left|A^*\right|^{1-\alpha}C\right|^2e_i\right\rangle^r\\ &\le \frac{1}{2}f_a^{2r}\left(\|A\|\right)\left[\sum_{i\in I}\left\langle \left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^*\right|^{1-\alpha}C\right|^{2qr}\right)e_i,e_i\right\rangle\right.\\ &\left.+\sum_{i\in I}\left|\left\langle \left|\left|A^*\right|^{1-\alpha}C\right|^2\left|\left|A\right|^{\alpha}B\right|^2e_i,e_i\right\rangle\right|^r\right]\\ &=\frac{1}{2}f_a^{2r}\left(\|A\|\right)\left[\operatorname{tr}\left(\frac{1}{p}\left|\left|A\right|^{\alpha}B\right|^{2pr}+\frac{1}{q}\left|\left|A^*\right|^{1-\alpha}C\right|^{2qr}\right)\right.\\ &\left.+\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^2\left|\left|A\right|^{\alpha}B\right|^2\right\|_{\mathcal{E},r}^r\right],\end{aligned}$$

which proves, by taking the supremum over $\mathcal{E}$, the desired inequality (87). □

Remark 7. *Let $\alpha\in[0,1]$. If $r=1/2$, $p,\ q=2$ and $\left|\left|A\right|^{\alpha}B\right|^2,\ \left|\left|A^*\right|^{1-\alpha}C\right|^2\in\mathcal{B}_1\left(H\right)$, then $C^*Af\left(A\right)B\in\mathcal{B}_1\left(H\right)$ and by (85) we get*

$$\omega_1\left(C^*Af\left(A\right)B\right)\le \frac{1}{2}f_a\left(\|A\|\right)\operatorname{tr}\left(\left|\left|A\right|^{\alpha}B\right|^2+\left|\left|A^*\right|^{1-\alpha}C\right|^2\right). \tag{90}$$

If $r=1$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by (85) we obtain

$$\omega_2^2\left(C^*Af\left(A\right)B\right)\le f_a^2\left(\|A\|\right)\operatorname{tr}\left(\frac{1}{p}\left||A|^\alpha B\right|^{2p}+\frac{1}{q}\left|\left|A^*\right|^{1-\alpha}C\right|^{2q}\right), \qquad (91)$$

provided that $\left||A|^\alpha B\right|^{2p},\ \left|\left|A^*\right|^{1-\alpha}C\right|^{2q}\in\mathcal{B}_1\left(H\right).$

If we take $r=1$ in (86), then we get

$$\begin{aligned}&\omega_2^2\left(C^*Af\left(A\right)B\right) &(92)\\ &\le\frac{1}{2}f_a^2\left(\|A\|\right)\left(\left\||A|^\alpha B\right\|_4^2\left\|\left|A^*\right|^{1-\alpha}C\right\|_4^2+\omega_1\left(\left|\left|A^*\right|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2\right)\right)\\ &\le\frac{1}{2}f_a^2\left(\|A\|\right)\left(\left\||A|^\alpha B\right\|_4^2\left\|\left|A^*\right|^{1-\alpha}C\right\|_4^2+\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2\right\|_1\right),\end{aligned}$$

provided that $|A|^\alpha B,\ \left|A^*\right|^{1-\alpha}C\in\mathcal{B}_4\left(H\right).$

If $r=1$ and $p=q=2$ in (87), then we get for $\left||A|^\alpha B\right|^{2p},\ \left|\left|A^*\right|^{1-\alpha}C\right|^{2q}\in\mathcal{B}_1\left(H\right)$ *that*

$$\begin{aligned}\omega_2^2\left(C^*Af\left(A\right)B\right)&\le\frac{1}{4}f_a^2\left(\|A\|\right)\left[\operatorname{tr}\left(\left||A|^\alpha B\right|^{2p}+\left|\left|A^*\right|^{1-\alpha}C\right|^{2q}\right)\right. &(93)\\ &\quad+\frac{1}{2}f_a^2\left(\|A\|\right)\omega_1\left(\left|\left|A^*\right|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2\right)\\ &\le\frac{1}{4}f_a^2\left(\|A\|\right)\operatorname{tr}\left(\left||A|^\alpha B\right|^{2p}+\left|\left|A^*\right|^{1-\alpha}C\right|^{2q}\right)\\ &\quad+\frac{1}{2}f_a^2\left(\|A\|\right)\left\|\left|\left|A^*\right|^{1-\alpha}C\right|^2\left||A|^\alpha B\right|^2\right\|_1.\end{aligned}$$

We also have:

Theorem 9. *With the assumptions of Theorem 8, we have for $r\ge1,\ \lambda\in[0,1]$ that*

$$\begin{aligned}\omega_{2r}^{2r}\left(C^*Af\left(A\right)B\right)&\le f_a^{2r}\left(\|A\|\right)\left\|\left(1-\lambda\right)\left||A|^\alpha B\right|^{2r}+\lambda\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right\| &(94)\\ &\quad\times\left\||A|^\alpha B\right\|_{2r}^{2r\lambda}\left\|\left|A^*\right|^{1-\alpha}C\right\|_{2r}^{2r(1-\lambda)},\end{aligned}$$

provided that $|A|^\alpha B,\ \left|A^*\right|^{1-\alpha}C\in\mathcal{B}_{2r}\left(H\right).$

In particular,

$$\begin{aligned}\omega_{2r}^{2r}\left(C^*Af\left(A\right)B\right)&\le\frac{1}{2}f_a^{2r}\left(\|A\|\right)\left\|\left||A|^\alpha B\right|^{2r}+\left|\left|A^*\right|^{1-\alpha}C\right|^{2r}\right\| &(95)\\ &\quad\times\left\||A|^\alpha B\right\|_{2r}^{r}\left\|\left|A^*\right|^{1-\alpha}C\right\|_{2r}^{r}.\end{aligned}$$

Proof. If $\mathcal{E}=\{e_i\}_{i\in I}$ is an orthonormal basis, then by taking $x=e_i$ in (62) and summing over $i\in I$ we get

$$\begin{aligned}
&\sum_{i\in I}\left|\left\langle C^{*}Af\left(A\right)Be_i,e_i\right\rangle\right|^{2r} \qquad (96)\\
&\leq f_a^{2r}\left(\left\|A\right\|\right)\sum_{i\in I}\left[\left\langle\left[\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right]e_i,e_i\right\rangle\right]\\
&\times\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{r\lambda}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_i,e_i\right\rangle^{r(1-\lambda)}\\
&\leq f_a^{2r}\left(\left\|A\right\|\right)\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2r}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}\right\|\\
&\times\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{r\lambda}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_i,e_i\right\rangle^{r(1-\lambda)}.
\end{aligned}$$

If we use Hölder's inequality for $p=\frac{1}{\lambda}$, $q=\frac{1}{1-\lambda}$, then we have

$$\begin{aligned}
&\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{r\lambda}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_i,e_i\right\rangle^{r(1-\lambda)}\\
&\leq\left(\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2}e_i,e_i\right\rangle^{r}\right)^{\lambda}\left(\sum_{i\in I}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}e_i,e_i\right\rangle^{r}\right)^{1-\lambda}\\
&\leq\left(\sum_{i\in I}\left\langle\left|\left|A\right|^{\alpha}B\right|^{2r}e_i,e_i\right\rangle\right)^{\lambda}\left(\sum_{i\in I}\left\langle\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2r}e_i,e_i\right\rangle\right)^{1-\lambda}\\
&=\left\|\left|A\right|^{\alpha}B\right\|_{2r}^{2r\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_{2r}^{2r(1-\lambda)},
\end{aligned}$$

which proves (94). □

Remark 8. *If we take $r=1$ in Theorem 9, then we get for $\alpha\in[0,1]$ that*

$$\begin{aligned}
\omega_2^2\left(C^{*}Af\left(A\right)B\right)&\leq f_a^2\left(\left\|A\right\|\right)\left\|\left(1-\lambda\right)\left|\left|A\right|^{\alpha}B\right|^{2}+\lambda\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\| \qquad (97)\\
&\times\left\|\left|A\right|^{\alpha}B\right\|_2^{2\lambda}\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_2^{2(1-\lambda)},
\end{aligned}$$

provided that $\left|A\right|^{\alpha}B,\ \left|A^{*}\right|^{1-\alpha}C\in\mathcal{B}_2\left(H\right).$

In particular,

$$\begin{aligned}
\omega_2^2\left(C^{*}Af\left(A\right)B\right)&\leq\frac{1}{2}f_a^2\left(\left\|A\right\|\right)\left\|\left|\left|A\right|^{\alpha}B\right|^{2}+\left|\left|A^{*}\right|^{1-\alpha}C\right|^{2}\right\| \qquad (98)\\
&\times\left\|\left|A\right|^{\alpha}B\right\|_2\left\|\left|A^{*}\right|^{1-\alpha}C\right\|_2.
\end{aligned}$$

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