SAKARYA ÜNİVERSİTESİ FEN BİLİMLERİ ENSTİTÜSÜ DERGİSİ
SAKARYA UNIVERSITY JOURNAL OF SCIENCE
e-ISSN: 2147-835X
Dergi sayfası:http://dergipark.gov.tr/saufenbilder

Geliş/Received
24-04-2017
Kabul/Accepted
01-08-2017

Doi
10.16984/saufenbilder.308097

Stability and boundedness of solutions of nonlinear fourth order differential equations with bounded delay

Erdal Korkmaz *[1]

ABSTRACT

In this paper, we determine sufficient conditions for the boundedness, uniformly asymtotically stability of the solutions to a certain fourth-order non-autonomous differential equations with bounded delay by considering second method of Lyapunov. The results obtain essentially improve, include and complement the consequences in the current literature.

Keywords: Stability, Boundedness, Lyapunov functional, Delay differential equations, Fourth order.

Dördüncü mertebeden sınırlı gecikmeli nonlineer diferansiyel denklemlerin çözümlerinin kararlılığı ve sınırlılığı

ÖZ

Bu makalede Lyapunov'un ikinci metodu kullanılarak dördüncü mertebeden otonom olmayan değişken gecikmeli diferansiyel denklemlerin çözümlerinin düzgün asimptotik kararlılığı ve sınırlılığı için yeterli şartları veririz. Elde edilen sonuçlar literatürdeki sonuçları tamamlar, kapsar ve geliştirir.

Anahtar Kelimeler: Kararlılık, Sınırlılık, Lyapunov fonksiyonu, Gecikmeli diferansiyel denklemler, Dördüncü mertebe.

[1] Muş Alparslan Üniversitesi, Fen Edebiyat Fakültesi, Matematik Bölümü, korkmazerdal36@hotmail.com

1. INTRODUCTION

Differential equations with higher-order have been widely used in mechanics, vibration theory, electromechanical systems of physics and engineering. Solutions of the boundedness and stability problem assocaited to differential equation in fourth-order is one of the most prominent issue and it has been found hihgly remarkable for many authors. Very interesting results related to the solutions have been obtained. Particularly, majority of these results were obtained using the second method to the Lyapunov, which is thought as the most result-oriented and secured methods (see, Lyapunov [13] and Yoshizawa [28]). However, [4,5,16] include such a useful content about the qualitative behaviors of differential equations without or with delay. To gain much better perspective on the boundedness and stability, see the papers of Ezeilo [6,7], Hara [8], Harrow [9,10], Tunç [22,23,24,25,26], Remili et al. [15,17,18], Wu and Xiong [27] and others and theirs references. As motive from references, we obtain some new consequences on the uniformly asymtotically stability and boundedness of the solutions by means of the Lyapunov's functional approach. Our results differ from that obtained in the literature (see, [1]-[28] and the references therein). By this way, this paper enrich to the current literature and contribute future studies by presenting useful information for the solutions of higher-order functional differential equation's qualitative behaviors. In view of all the mentioned information, it can be checked the novelty and originality of the current paper.

In this paper, we seek sufficient condition to obtain the uniformly asymptotically stability of the solutions for $p(t,x,x',x'',x''')\equiv 0$ and boundedness of solutions to the fourth order nonlinear differential equation with bounded veriable delay

$$\begin{aligned}&\big(g(x(t))x''(t)\big)'' + a(t)\big(k(x(t))x''(t)\big)'\\&+b(t)\big(q(x(t))x'(t)\big)' + c(t)f(x(t))x'(t)\\&+d(t)h(x(t-r(t))) = p(t,x,x',x'',x''').\end{aligned}\tag{1}$$

For convenience, we let

$$\theta_1(t)=\frac{g'(x(t))}{g^2(x(t))}x'(t),\quad \theta_2(t)=\frac{k'(x(t))}{g^2(x(t))}x'(t)$$

and

$$\theta_3(t)=\frac{q'(x(t))}{g^2(x(t))}x'(t),\quad \theta_4(t)=\frac{f'(x(t))}{g^2(x(t))}x'(t).$$

We write (1) in the system form

$$\begin{aligned}x'&=y,\\ y'&=\frac{1}{g(x)}z,\\ z'&=w,\\ w'&=-a(t)\frac{k(x)}{g(x)}w\\&+\left(a(t)k(x)\theta_1(t)-b(t)\frac{q(x)}{g(x)}-a(t)g(x)\theta_2(t)\right)z\\&-\big(b(t)g^2(x)\theta_3(t)+c(t)f(x)\big)y-d(t)h(x)\\&+d(t)\int_{t-r(t)}^{t}h'(x(\eta))y(\eta)d\eta+p(t,x,y,z,w),\end{aligned}\tag{2}$$

where $r(t)$ is a bounded delay, $0\le r(t)\le\Omega$, $r'(t)\le\lambda$, $0<\lambda<1$, λ and Ω some positive constants, Ω which will be determined later, the functions a,b,c,d are continuously differentiable functions and the functions f,h,g,q,k and p are continuous functions depending only on the arguments shown. Also derivatives $g'(x), g''(x), k'(x), q'(x), f'(x)$ and $h'(x)$ exist and are continuous. The continuity of the functions $a,b,c,d,g,g',g'',k,k',q,q',f,p$ and h guarantees the existence of the solutions of equation (1). If the right-hand side of the system (2) satisfies a Lipchitz condition in $x(t),y(t),z(t),w(t)$ and $x(t-r(t))$ and exists of solutions of system (2), then it is unique solution of system (2).

Assuming $a_0,b_0,c_0,d_0,f_0,g_0,q_0,k_0,a_1,b_1,c_1,d_1,f_1,g_1,$ $q_1,k_1,m,M,$ and δ are constants then, following assumptions hold:

(A1) $0<a_0\le a(t)\le a_1$;
$0<b_0\le b(t)\le b_1$; $0<c_0\le c(t)\le c_1$;
$0<d_0\le d(t)\le d_1$ for $t\ge 0$.

(A2) $0<f_0\le f(x)\le f_1$;
$0<g_0\le g(x)\le g_1$; $0<k_0\le k(x)\le k_1$;
$0<q_0\le q(x)\le q_1$ for $x\in R$ and
$0<m<\min\{f_0,k_0,g_0,1\}$,
$M>\max\{f_1,g_1,k_1,1\}$.

(A3) $\frac{h(x)}{x} \geq \delta > 0$ for $x \neq 0$, $h(0) = 0$.

(A4) $|p(t,x,y,z,w)| \leq |e(t)|$.

2. PRELIMINARIES

We also consider the functional differential equation

$$\dot{x} = f(t, x_t), \quad x_t(\theta) = x(t+\theta), \quad -r \leq \theta \leq 0, \quad t \geq 0. \quad (3)$$

where $f : IxC_H \to \mathrm{R}^n$ is a continuous mapping, $f(t,0) = 0$, $C_H := \{\phi \in (C[-r,0], \mathrm{R}^n) : \|\phi\| \leq H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(t,\phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Theorem 2.1. Let $V(t,\phi) : IxC_H \to \mathrm{R}$ be a continuous functional satisfying a local Lipchitz condition, $V(t,0) = 0$, and wedges W_i such that :

1) $W_1(\|\phi\|) \leq V(t,\phi) \leq W_2(\|\phi\|)$.

2) $V'_{(3)}(t,\phi) \leq -W_3(\|\phi\|)$.

Then, it implies that the equation (3) is uniformly asymptotically stable for the zero solution (Burton [4]).

3. MAIN RESULTS

Lemma 3.1. Let $h(0) = 0$, $xh(x) > 0$ $(x \neq 0)$ and $\delta(t) - h'(x) \geq 0$, $(\delta(t) > 0)$, then $2\delta(t)H(x) \geq h^2(x)$, where $H(x) = \int_0^x h(s)ds$ (Hara [8])

Theorem 3.1. Besides to the fundamental assumptions imposed on the functions a, b, c, d, g, k, q, f and h let we suppose that there exists non-negative constants $h_0, \delta_0, \upsilon_1, \upsilon_2, \eta_1, \eta_2, \eta_3$ and η_4 so that the following statements are hold:

i. $\frac{h_0}{m} - \frac{a_0 m \delta_0}{d_1} \leq h'(x) \leq \frac{h_0}{2M}$, $|g'(x)| < \eta_4$ for $x \in R$.

ii. $b_0 q_0 > \max\{\upsilon_1, \upsilon_2\}$ where

$$\begin{cases} \upsilon_1 = \frac{a_1 h_0 d_1 M^2}{c_0 m^3} + \frac{M^3(c_1+\delta_0)}{a_0 m^2} + a_0 a_1 m(M-1) \\ \upsilon_2 = \frac{2d_1 h_0 a_0}{c_0(M-1)}\left(\frac{1}{m} - \frac{1}{M}\right)^2 + \frac{2c_0 M}{a_0} \\ \qquad + \frac{2a_1 d_1 h_0 M}{c_0 m^3} + \frac{c_0 c_1 (M^2+2)mM}{d_1 h_0} \end{cases}$$

iii. $\int_0^\infty \left(|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|\right)dt < \eta_1$.

iv. $\int_{-\infty}^{+\infty} \left(|g'(s)| + |k'(s)| + |q'(s)| + |f'(s)|\right)ds < \eta_2$.

v. $\int_0^\infty |e(t)|\, dt < \eta_3$.

Then any solution $x(t)$ equation (1) are bounded and trival solution of equation (1) for $p(t,x,x',x'',x''') \equiv 0$ is uniformly asymtotically stability, if

$$\Omega < \frac{2(1-\lambda)}{d_1 h_1} \min\left\{ \frac{\varepsilon c_0 m}{\alpha + \beta(2-\lambda) + 1}, \frac{\varepsilon a_0 m}{M\alpha(1-\lambda)}, \frac{m^2(b_0 q_0 - \upsilon_1) - \varepsilon M^2(a_1 + c_1 m M)}{Mm^2} \right\}.$$

Proof We take a Lyapunov functional for the usage of basic tool for the proof,

$$W = W(t,x,y,z,w) = e^{\frac{-1}{\eta}\int_0^t \gamma(s)ds} V, \quad (4)$$

where

$$\gamma(t) = |a'(t)| + |b'(t)| + |c'(t)| + |d'(t)| + |\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)| + |\theta_4(t)|,$$

and

$$\begin{aligned} 2V = {} & 2\beta d(t)H(x) + c(t)g(x)f(x)y^2 \\ & + \alpha b(t)\frac{q(x)}{g(x)}z^2 + a(t)\frac{k(x)}{g(x)}z^2 \\ & + 2\beta a(t)\frac{k(x)}{g(x)}yz + [\beta b(t)q(x) - \alpha h_0 d(t)]y^2 \\ & - \beta\frac{1}{g(x)}z^2 + \alpha w^2 + 2d(t)g(x)h(x)y \\ & + 2\alpha d(t)h(x)z + 2\alpha c(t)f(x)yz \\ & + 2\beta yw + 2zw + \sigma\int_{-r(t)}^0 \int_{t+s}^t y^2(\gamma)d\gamma ds \end{aligned}$$

with $H(x) = \int_0^x h(s)ds$, $\alpha = \frac{M}{a_0 m} + \varepsilon$, $\beta = \frac{d_1 h_0}{c_0 m} + \varepsilon$, and η are non-negative constants to be described later. We can rewrite it in the form $2V$ as

$$2V = a(t)k(x)\left[\frac{w}{a(t)k(x)} + z + \beta\frac{1}{g(x)}y\right]^2 + c(t)f(x)\left[\frac{d(t)h(x)}{c(t)f(x)} + y + \alpha z\right]^2 + c(t)f(x)\left[(g(x)-1)y + \frac{d(t)h(x)}{c(t)f(x)}\right]^2 + 2\varepsilon d(t)H(x) + \sigma\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\gamma)d\gamma ds + L_1 + L_2 + L_{3,}$$

where

$$L_1 = 2d(t)\int_0^x h(s)\left[\frac{d_1h_0}{c_0m} - 2\frac{d(t)}{c(t)f(x)}h'(s)\right]ds,$$

$$L_2 = \left[\alpha b(t)\frac{q(x)}{g(x)} - \beta\frac{1}{g(x)} - \alpha^2 c(t)f(x) + a(t)k(x)\left(\frac{1}{g(x)} - 1\right)\right]z^2,$$

$$L_3 = \left[\beta b(t)q(x) - \alpha h_0 d(t) - \beta^2 a(t)\frac{k(x)}{g^2(x)} - c(t)f(x)\left(g^2(x) - 3g(x) + 2\right)\right]y^2 + \left[\alpha - \frac{1}{a(t)k(x)}\right]w^2 + 2\beta\left(1 - \frac{1}{g(x)}\right)yw.$$

Let

$$\varepsilon < \min\left\{\frac{M}{a_0m}, \frac{d_1h_0}{c_0m}, \frac{m^2(b_0q_0 - \upsilon_1)}{M^2(a_1 + mMc_1)}\right\} \tag{5}$$

then

$$\frac{M}{a_0m} < \alpha < \frac{2M}{a_0m}, \quad \frac{d_1h_0}{c_0m} < \beta < 2\frac{d_1h_0}{c_0m}. \tag{6}$$

Considering conditions (A1)-(A3), (i)-(ii) and inequalities (5), (6) we have

$$L_1 \geq 4d(t)\frac{d_1}{c_0m}\int_0^x h(s)\left[\frac{h_0}{2M} - h'(s)\right]ds \geq 0,$$

$$L_2 \geq \alpha\left(\frac{b_0q_0}{M} - \left(\frac{d_1h_0}{c_0m} + \varepsilon\right)\frac{a_1}{m} - \left(\frac{M}{a_0m} + \varepsilon\right)c_1m - \frac{a_1a_0m}{M}(M-1)\right)z^2 + \beta\left(\alpha\frac{a_0}{M} - \frac{1}{m}\right)z^2$$

$$\geq \alpha\left(\frac{b_0q_0}{M} - \frac{d_1h_0a_1}{c_0m^2} - \frac{c_1M^2}{a_0m} - a_1\frac{a_0m}{M}(M-1) - \frac{\varepsilon}{m}(a_1 + c_1mM)\right)z^2,$$

$$\geq \frac{\alpha}{Mm}\left(m(b_0q_0 - \upsilon_1) - \varepsilon M(a_1 + c_1mM)\right)z^2 \geq 0,$$

and

$$L_3 \geq \beta\left(b_0q_0 - \frac{\alpha}{\beta}h_0d_1 - \beta a_1\frac{M}{g^2(x)} - \frac{c_1M(M^2+2)}{\beta}\right)y^2 + \left(\frac{M-1}{a_0m}\right)w^2 + 2\beta\left(1 - \frac{1}{g(x)}\right)yw$$

$$\geq \beta\left(b_0q_0 - 2\frac{Mc_0}{a_0} - 2a_1\frac{d_1h_0M}{c_0m^3} - \frac{c_0c_1(M^2+2)mM}{d_1h_0}\right)y^2 + \left(\frac{M-1}{a_0m}\right)w^2 + 2\beta\left(1 - \frac{1}{g(x)}\right)yw$$

$$\geq \beta\frac{2d_1h_0a_0}{c_0(M-1)}\left(\frac{1}{m} - \frac{1}{M}\right)^2 y^2 + \left(\frac{M-1}{a_0m}\right)w^2 + 2\beta\left(1 - \frac{1}{g(x)}\right)yw,$$

and by calculating the discriminant, we obtain

$$\Delta = \beta^2\left(1 - \frac{1}{g(x)}\right)^2 - \beta\frac{2d_1h_0}{c_0m}\left(\frac{1}{m} - \frac{1}{M}\right)^2$$

$$\Delta \leq \beta\left[\frac{2d_1h_0}{c_0m}\left(\frac{1}{m} - \frac{1}{M}\right)^2 - \frac{2d_1h_0}{c_0m}\left(\frac{1}{m} - \frac{1}{M}\right)^2\right] = 0.$$

Thus

$$L_3 \geq 0.$$

From the above inequalities, there exists non-negative constant D_0 so that

$$2V \geq D_0(y^2 + z^2 + w^2 + H(x)). \tag{7}$$

Considering Lemma 3.1, (A3) and (i), we find a positive constant D_1 such that

$$2V \geq D_1(x^2 + y^2 + z^2 + w^2) \tag{8}$$

In this way V is positive definite. In consideration of (A1)-(A3), we can have a positive constant U_1 such that

$$V \le U_1(x^2 + y^2 + z^2 + w^2). \qquad (9)$$

Considering the condition (iv), we write

$$\begin{aligned}\int_0^t \sum_{i=1}^{4} |\theta_i(s)|\, ds &= \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{|g'(u)| + |k'(u)|}{g^2(u)} du \\ &+ \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{|q'(u)| + |f'(u)|}{g^2(u)} du \\ &\le \frac{1}{m^2}\int_{-\infty}^{+\infty} (|g'(u)| + |k'(u)|) du \qquad (10) \\ &+ \frac{1}{m^2}\int_{-\infty}^{+\infty} (|q'(u)| + |f'(u)|) du \\ &< \frac{\eta_2}{m^2} < \infty\end{aligned}$$

where $\alpha_1(t) = \min\{x(0), x(t)\}$ and $\alpha_2(t) = \max\{x(0), x(t)\}$. From inequalities (5), (9) and (10), it follows that

$$W \ge D_2(x^2 + y^2 + z^2 + w^2) \qquad (11)$$

where $D_2 = \frac{D_1}{2} e^{-\frac{1}{\eta}\left(\eta_1 + \frac{\eta_2}{m^2}\right)}$.

Also, it is easy to see that there is a positive constant U_2 such that

$$W \le U_2(x^2 + y^2 + z^2 + w^2) \qquad (12)$$

for all x, y, z, w and all $t \ge 0$.

Now our goal is to show that $\dot{W}$ is negative definite function. For the function V taking derivative with respect to t yields to obtain following statement along any solution $(x(t), y(t), z(t), w(t))$ of the system (2)

$$\begin{aligned}2\dot{V}_{(2)} &= -2\varepsilon c(t) f(x) y^2 + L_4 + L_5 + L_6 + L_7 \\ &+ L_8 + L_9 + 2(\beta y + z + \alpha w) p(t, x, y, z, w)\end{aligned}$$

where

$$\begin{aligned}L_4 &= -2\left(\frac{d_1 h_0}{c_0 m} c(t) f(x) - d(t) g(x) h'(x)\right) y^2 \\ &- 2\alpha d(t)\left(\frac{h_0}{g(x)} - h'(x)\right) yz,\end{aligned}$$

$$L_5 = -2\left(\frac{b(t)q(x)}{g(x)} - \alpha c(t)\frac{f(x)}{g(x)} - \beta a(t)\frac{k(x)}{g^2(x)}\right) z^2,$$

$$L_6 = -2\left(\alpha \frac{a(t)k(x)}{g(x)} - 1\right) w^2,$$

$$\begin{aligned}L_7 &= 2\alpha d(t) w \int_{t-r(t)}^{t} h'(x(\eta)) y(\eta) d\eta \\ &+ 2\beta d(t) y(t) \int_{t-r(t)}^{t} h'(x(\eta)) y(\eta) d\eta \\ &+ 2d(t) z(t) \int_{t-r(t)}^{t} h'(x(\eta)) y(\eta) d\eta \\ &+ \sigma r y^2(t) - \sigma(1 - r'(t)) \int_{t-r(t)}^{t} y^2(\eta) d\eta\end{aligned}$$

$$\begin{aligned}L_8 &= \theta_1 \big(a(t)k(x)z^2 - \alpha b(t) q(x) z^2 \\ &+ c(t) f(x) g^2(x) y^2 + \beta z^2 \\ &+ 2d(t) g^2(x) h(x) y + 2\alpha a(t) k(x) zw\big) \\ &- b(t)\theta_3 g(x)\big(\alpha z^2 + 2\alpha g(x) zw + \beta g(x) y^2 \\ &+ 2g(x) yz\big) - a(t)\theta_2 g(x)\big(z^2 + 2\alpha zw\big) \\ &+ \theta_4 \big(c(t) g^3(x) y^2 + 2\alpha c(t) g^2(x) yz\big),\end{aligned}$$

$$\begin{aligned}L_9 &= d'(t)\big[2\beta H(x) - \alpha h_0 y^2 + 2g(x)h(x) y + 2\alpha h(x) z\big] \\ &+ c'(t)\big[g(x) f(x) y^2 + 2\alpha f(x) yz\big] \\ &+ b'(t)\left[\alpha \frac{q(x)}{g(x)} z^2 + \beta q(x) y^2\right] \\ &+ a'(t)\left[\frac{k(x)}{g(x)} z^2 + 2\beta \frac{k(x)}{g(x)} yz\right].\end{aligned}$$

By regarding conditions (A1), (A2), (i), (ii) and inequalty (6), (7) we have the following

$$\begin{aligned}L_4 &\le -2d(t) g(x)\left[\frac{h_0}{g(x)} - h'(x)\right] y^2 \\ &- 2\alpha d(t)\left[\frac{h_0}{g(x)} - h'(x)\right] yz \\ &\le 2d(t) m\left[\frac{h_0}{g(x)} - h'(x)\right]\left[\left(y + \frac{\alpha}{2m} z\right)^2 - \left(\frac{\alpha}{2m} z\right)^2\right] \\ &\le \frac{\alpha^2}{2m} d(t)\left[\frac{h_0}{m} - h'(x)\right] z^2.\end{aligned}$$

In that case,

$$L_4 + L_5 \le -2\left[\frac{b_0 q_0}{M} - \left(\frac{M}{a_0 m} + \varepsilon\right)\frac{c_1 M}{m}\right.$$

$$-\left(\frac{d_1h_0}{c_0m}+\varepsilon\right)\frac{a_1M}{m^2}-\frac{\alpha^2}{4m}\left(a_0\delta_0\right)\Bigg]z^2$$

$$\leq -2\left[\frac{b_0q_0}{M}-\frac{M^2}{a_0m^2}c_1-\frac{d_1h_0a_1M}{c_0m^3}\right.$$
$$\left.-\frac{M^2\delta_0}{a_0m^2}-\varepsilon\frac{M}{m}\left(\frac{a_1}{m}+c_1\right)\right]z^2$$
$$\leq -\frac{2}{Mm^2}\left[m^2\left(b_0q_0-\upsilon_1\right)-\varepsilon M^2\left(a_1+c_1m\right)\right]z^2\leq 0,$$

and

$$L_6\leq -2\left[\alpha\frac{a_0m}{M}-1\right]w^2=-2\varepsilon\frac{a_0m}{M}w^2\leq 0.$$

By taking $h_1=\max\left\{\left|\frac{h_0}{m}-\frac{a_0m\delta_0}{d_1}\right|,\frac{h_0}{2M}\right\}$, we get

$$L_7\leq d_1h_1r(t)(\alpha w^2+\beta y^2+z^2)+\sigma r(t)y^2$$
$$+\left[d_1h_1(\alpha+\beta+1)-\sigma(1-\lambda)\right]\int_{t-r(t)}^{t}y^2(s)ds$$

If we choose $\sigma=\frac{d_1h_1}{(1-\lambda)}(\alpha+\beta+1)$, we get

$$L_7\leq\frac{d_1h_1r(t)}{1-\lambda}\left[(1-\lambda)\alpha w^2\right.$$
$$\left.+(\alpha+\beta(2-\lambda)+1)y^2+z^2\right].$$

Thus, there exists a positive constant D_3 such that

$$-2\varepsilon c(t)f(x)y^2+L_4+L_5+L_6+L_7$$
$$\leq -2D_3(y^2+z^2+w^2).$$

From (7), and the Cauchy Schwartz inequality, we obtain

$$L_8\leq|\theta_1|\Big(a(t)k(x)z^2+\alpha b(t)q(x)z^2$$
$$+c(t)f(x)g^2(x)y^2+\beta z^2$$
$$+d(t)g^2(x)(h^2(x)+y^2)+\alpha a(t)k(x)(z^2+w^2)\Big)$$
$$+|\theta_4|\left(c(t)g^3(x)y^2+\alpha c(t)g^2(x)(y^2+z^2)\right)$$
$$+a(t)|\theta_2|\,g(x)\left(z^2+\alpha(z^2+w^2)\right)$$
$$+b(t)|\theta_3|\,g(x)\Big(\alpha z^2+\alpha g(x)(z^2+w^2)$$
$$+\beta g(x)y^2+g(x)(y^2+z^2)\Big)$$
$$\leq\lambda_1\left(|\theta_1|+|\theta_2|+|\theta_3|+|\theta_4|\right)\left(y^2+z^2+w^2+H(x)\right)$$
$$\leq 2\frac{\lambda_1}{D_0}\left(|\theta_1|+|\theta_2|+|\theta_3|+|\theta_4|\right)V,$$

where

$$\lambda_1=\max\{d_1h_0M,b_1M(\alpha+\alpha M+M+\beta M),$$
$$\beta+M(a_1+\alpha b_1+\alpha a_1),M^2(d_1+\alpha c_1+c_1M)\}.$$

Using condition (i) and Lemma 3.1, we can write

$$|L_9|\leq|d'(t)|\left[2\beta H(x)+\alpha h_0y^2\right.$$
$$\left.+g(x)(h^2(x)+y^2)+\alpha\left(h^2(x)+z^2\right)\right]$$
$$+|c'(t)|\,f(x)\left[g(x)y^2+\alpha\left(y^2+z^2\right)\right]$$
$$+|b'(t)|\,q(x)\left[\alpha\frac{1}{g(x)}z^2+\beta y^2\right]$$
$$+|a'(t)|\frac{k(x)}{g(x)}\left[z^2+\beta\left(y^2+z^2\right)\right]$$
$$\leq\lambda_2\left[|a'(t)|+|b'(t)|+|c'(t)|+|d'(t)|\right]\left(y^2+z^2\right.$$
$$\left.+w^2+H(x)\right)$$
$$\leq 2\frac{\lambda_2}{D_0}\left[|a'(t)|+|b'(t)|+|c'(t)|+|d'(t)|\right]V,$$

such that

$$\lambda_2=\max\{2\beta+h_0(1+\tfrac{\alpha}{M}),\alpha h_0+M(\alpha+M+1),$$
$$\tfrac{M}{m}(\alpha+\beta+1)\}.$$

By taking $\frac{1}{\eta}=\frac{1}{D_0}\max\{\lambda_1,\lambda_2\}$, we have

$$\dot{V}_{(2)}\leq -D_3\left(y^2+z^2+w^2\right)$$
$$+\left(\beta y+z+\alpha w\right)p(t,x,y,z,w)$$
$$+\frac{1}{\eta}\left(|a'(t)|+|b'(t)|+|c'(t)|\right.\tag{13}$$
$$\left.+|d'(t)|+|\theta_1|+|\theta_2|+|\theta_3|+|\theta_4|\right)V.$$

From (A4), (iii), (iv), (10), (11), (13) and the Cauchy Schwartz inequality, we get

$$\dot{W}_{(2)}=\left(\dot{V}_{(2)}-\frac{1}{\eta}\gamma(t)V\right)e^{-\frac{1}{\eta}\int_0^t\gamma(s)ds}$$
$$\leq\left(-D_3\left(y^2+z^2+w^2\right)+\left(\beta y+z\right.\right.$$
$$\left.\left.+\alpha w\right)p(t,x,y,z,w)\right)e^{-\frac{1}{\eta}\int_0^t\gamma(s)ds}$$
$$\leq\left(\beta|y|+|z|+\alpha|w|\right)|p(t,x,y,z,w)|\tag{14}$$
$$\leq D_4\left(|y|+|z|+|w|\right)|e(t)|$$
$$\leq D_4\left(3+y^2+z^2+w^2\right)|e(t)|$$
$$\leq D_4\left(3+\frac{1}{D_2}W\right)|e(t)|$$
$$\leq 3D_4|e(t)|+\frac{D_4}{D_2}W|e(t)|\tag{15}$$

where $D_4 = \max\{\alpha, \beta, 1\}$. Integrating (15) from 0 to t and using the condition (v) and the Gronwall inequality, we have

$$W \le W(0, x(0), y(0), z(0), w(0)) + 3D_4\eta_3$$
$$+\frac{D_4}{D_2}\int_0^t W(s, x(s), y(s), z(s), w(s))|e(s)|\,ds$$
$$\le (W(0, x(0), y(0), z(0), w(0)) + 3D_4\eta_3)e^{\frac{D_4}{D_2}\int_0^t |e(s)|\,ds}$$
$$\le (W(0, x(0), y(0), z(0), w(0)) + 3D_4\eta_3)e^{\frac{D_4}{D_2}\eta_3} \quad (16)$$
$$= K_1 < \infty$$

Because of inequalities (11) and (16), we write

$$(x^2 + y^2 + z^2 + w^2) \le \frac{1}{D_2}W \le K_2, \quad (17)$$

where $K_2 = \frac{K_1}{D_2}$. Clearly (17) imlies that

$$|x(t)| \le \sqrt{K_2},\ |y(t)| \le \sqrt{K_2},$$
$$|z(t)| \le \sqrt{K_2},\ |w(t)| \le \sqrt{K_2} \text{ for all } t \ge 0.$$

Thus, by using conditions (A2), (i) and (17) with the system (2) we have

$$|x(t)| \le \sqrt{K_2},\ |x'(t)| \le \sqrt{K_2},$$
$$|x''(t)| \le |y'(t)| = \left|\frac{1}{g(x)}\right||z(t)| \le \frac{1}{g_0}\sqrt{K_2},$$
$$|x'''(t)| \le \frac{1}{g(x)}|w(t)| + \frac{|g'(x)|}{|g^2(x)|}|y(t)||z(t)| \quad (18)$$
$$\le \frac{1}{g_0}\sqrt{K_2} + \frac{\eta_4}{g_0{}^2}K_2 \ \ for\ all\ t \ge 0.$$

In this case $x(t), x'(t), x''(t)$ and $x'''(t)$ are bounded.

By taking $p(t, x, y, z, w) \equiv 0$ in the inequality (14) obtained

$$\dot{W}_{(2)} = \left(\dot{V}_{(2)} - \frac{1}{\eta}\gamma(t)V\right)e^{-\frac{1}{\eta}\int_0^t \gamma(s)ds}$$
$$\le -D_3(y^2 + z^2 + w^2)e^{-\frac{1}{\eta}\int_0^t \gamma(s)ds}$$
$$\le -\mu(y^2 + z^2 + w^2),$$

where $\mu = D_3 e^{-\frac{\eta_1+\eta_2}{\eta}}$. It can also be observed that the only solution of system (2) for which $\dot{W}_{(2)}(t, x, y, z, w) = 0$ is the solution $x = y = z = w = 0$. In this way, trival solution of equation (1) is uniformly asymptotically stable and are bounded solutions of equation (1).

ACKNOWLEDGMENTS

The author would like to present his sincere thanks to the referee(s) for the detailed read and valuable input on the manuscript.

REFERENCES

[1] A. M. A. Abou-Ela-Ela and A. I. Sadek, "A stability result for certain fourth order differential equations," *Ann. Differential Equations*, vol. 1, no. 6, pp. 1-9, 1990.

[2] O. A. Adesina and B. S. Ogundare, "Some new stability and boundedness results on a certainfourth order nonlinear differential equation.," *Nonlinear Stud*, vol. 3, no. 19, pp. 359-369, 2012.

[3] H. Bereketoğlu, "Asymptotic stability in a fourth order delay differential equation.," *Dynam. Systems Appl*, vol. 1, no. 7, pp. 105-115, 1998.

[4] T. A. Burton, Stability and periodic solutions of ordinary and functional differential equations. Mathematics in science and engineering, 1985.

[5] L. Elsgolts, Introduction to the Theory of Differential Equations with Deviating Arguments. Translated from the Russian by Robert J. McLaughlin Holden-Day, Inc. San Francisco Calif.-London-Amsterdam, 1966.

[6] J. O. C. Ezeilo and H. O. Tejumola, "On the boundedness and the stability properties of solutions of certain fourth order differential equations," *Ann. Mat. Pura Appl.*, vol. 4, no. 95, pp. 131-145, 1973.

[7] J. O. C. Ezeilo, "Stability results for the solutions of some third and fourth order differential equations," *Ann. Mat. Pura Appl.*, vol. 4, no. 66, pp. 233-249, 1964.

[8] T. Hara, "On the asymptotic behavior of the solutions of some third and fourth order nonautonomous differential equations," *Publ. RIMS, Kyoto Univ.*, no. 9, pp. 649-673, 1974.

[9] M. Harrow, "Further results on the boundedness and the stability of solutions of some differential equations of the fourth

order," *SIAM J. Math. Anal.*, no. 1, pp. 189-194, 1970.

[10] M. Harrow, "A Stability result for Solutions of a Certain Fourth Order Homogeneous Differential Equations," *J. London Math. Soc.*, no. 42, pp. 51-56, 1967.

[11] Z. Körpınar. "Kesirli Klein-Gordon denklemi için residual power seri metodu. " *SAÜ Fen Bilimleri Enstitüsü Dergisi*, 21 (3), 285-293, 2017.

[12] N. N. Krasovskii, "On the stability in the large of the solution of a nonlinear system of differential equations (Russian)," *Prikl. Mat. Meh.*, no. 18, pp. 735-737, 1954.

[13] A. M. Lyapunov, The general problem of the stability of motion. Translated from Edouard Davaux's French translation (1907) of the 1892 Russian original and edited by A. T. Fuller., London: Taylor & Francis, Ltd., 1992.

[14] MM Al Qurashi, ZS Korpinar, M. Inc, "Approximate solutions of bright and dark optical solitons in birefrigent fibers, " *Optik-International Journal for Light and Electron Optics*, 140, 45-61, 2017.

[15] M. Rahmane and M. Remili, "On Stability and Boundedness of Solutions of Certain Non Autonomous Fourth-Order Delay Differential Equations," *Acta Universitatis Matthiae Belii, series Mathematics Issue*, no. 17-30, 2015.

[16] R. Reissig, G. Sansone and R. Conti, Non-linear Differential Equations of Higher Order. Translated from the German., Leyden: Noordhoff International Publishing, 1974.

[17] M. Remili and M. Rahmane, "Boundedness and square integrability of solutions of nonlinear fourth-order differential equations," *Nonlinear Dynamics and Systems Theory*, vol. 2, no. 16, pp. 192-205, 2016.

[18] M. Remili and M. Rahmane, "Sufficient conditions for the boundedness and square integrability of solutions of fourth-order differential equations," *Proyecciones Journal of Mathematics*., vol. 1, no. 35, pp. 41-61, 2016.

[19] A. Shair, "Asymptotic properties of linear fourth order differential equations," *American mathematical society, vol.* 1, no. 59, pp. 45-51, 1976.

[20] A. S. C. Sinha, "On stability of solutions of some third and fourth order delay-differential equations," *Information and Control*, no. 23, pp. 165-172, 1973.

[21] H. O. Tejumola, "Further results on the boundedness and the stability of certain fourth order differential equations," *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.*, vol. 8, no. 52, pp. 16-23, 1972.

[22] C. Tunç, "Some remarks on the stability and boundedness of solutions of certain differential equations of fourth-order," *Computational and Applied Mathematics*, vol. 1, no. 26, pp. 1-17, 2007.

[23] C. Tunç, "Some stability and boundedness results for the solutions of certain fourth order differential equations," *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.*, no. 44, pp. 161-171, 2005.

[24] C. Tunç, "Some stability results for the solutions of certain fourth order delay differential equations," *Differential Equations and Applications*, no. 4, pp. 165-174, 2004.

[25] C. Tunç, "Stability and boundedness of solutions to certain fourth order differential equations," Electronic *Journal of Differential Equations*, no. 35, pp. 1-10, 2006.

[26] C. Tunç, "Stability and boundedness of solutions to certain fourth order differential equations," *Electron. J. Differential Equations*, no. 35, p. 10pp., 2006.

[27] X. Wu and K. Xiong, "Remarks on stability results for the solutions of certain fourth-order autonomous differential equations," *Internat. J. Control*, vol. 2, no. 69, p. 353, 1998.

[28] T. Yoshizawa, Stability Theory by Liapunovs Second Method, Tokyo,: The Mathematical Society of Japan, 1966.