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Analyses of the SIR Epidemic Model Including Treatment and Immigration

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Abstract

This paper aims to examine the dynamics of a variation of a nonlinear SIR epidemic model. We analyze the complex dynamic nature of the discrete-time SIR epidemic model by discretizing a continuous SIR epidemic model subject to treatment and immigration effects with the Euler method. First of all, we show the existence of equilibrium points in the model by reducing the three-dimensional system to the two-dimensional system. Next, we show the stability conditions of the obtained positive equilibrium point and the visibility of flip bifurcation. A feedback control strategy is applied to control the chaos occurring in the system after a certain period of time. We also perform numerical simulations to support analytical results. We do all these analyses for models with and without immigration and show the effect of immigration on dynamics.

1. Introduction

Mathematical models describing epidemics affecting population dynamics are often expressed with differential equations or difference equations [1–3]. The models of differential equations are used to describe situations where change is continuous. Analysis of continuous-time epidemic models has been studied by many researchers [4–9]. If the change is discrete, it would be more appropriate to use difference equations for modelling. Moreover, these equations provide a more realistic approach to describe events with different characteristic processes, while retaining the essential properties of the corresponding continuous time models, [10–29]. For this purpose, we provide more recent articles as references [30–32]. When a parameter of the model is changed, the stability behavior of the model may change. New stable points may emerge or existing points may disappear. Changes in the topological or qualitative structure of a dynamic system are determined using bifurcation theory [33, 34]. Sometimes, the existence of bifurcation behavior may be detected without the need for deep analysis [14, 22].
Wang [4] analyzed the following model, and showed that there is bifurcation depending on the size of the treatment capacity:

$$\begin{aligned} \frac{dS}{dt} &= A - dS - \lambda SI \\ \frac{dI}{dt} &= \lambda SI - (d+\gamma+\varepsilon)I - T(I) \\ \frac{dR}{dt} &= \gamma I + T(I) - dR. \end{aligned} \tag{1.1}$$

In this model (1.1), $S-$ sensitive individuals who have not been infected with the disease but are susceptible to the disease; $I-$ infected individuals who have contracted the disease and infect others; and $R-$ individuals who have the disease and but have recovered. $A, d, \gamma, \varepsilon, \lambda$ are positive parameters. $A-$ individuals added to the population by birth, $d-$ natural mortality rate in the population, $\gamma-$ natural recovery

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rate of infected individuals, $\varepsilon-$ disease-related mortality rate, $\lambda-$ infection coefficient. In study [35], local stability and bifurcation analyses were studied by transforming a continuous SIR epidemic model given in study [4] into a discrete-time system by using forward Euler method as follows:

$$\begin{aligned} S_{t+1} &= S_t + \delta\left(A - dS_t - \lambda S_t I_t\right) \\ I_{t+1} &= I_t + \delta\left(\lambda S_t I_t - zI_t - kI_t\right) \end{aligned} \tag{1.2}$$

where $z = d + \gamma + \varepsilon$ is the sum of natural death, recovery, and death from disease, respectively. $T(I) = kI$ is the treatment function such that $0 \leq I \leq I_0$ and k is a positive parameter. In this study, we will consider the following discrete-time model that we developed under the given immigration effect:

$$\begin{aligned} S_{t+1} &= S_t + \delta\left(A - dS_t - \lambda S_t I_t + pS_t\right) \\ I_{t+1} &= I_t + \delta\left(\lambda S_t I_t - zI_t - kI_t + qI_t\right) \end{aligned} \tag{1.3}$$

where pS_t- immigration effect on susceptible individuals and qI_t- immigration effect on infected individuals. In this article, our aim is to examine the dynamics of model (1.3) subject to immigration by briefly recalling the analysis of the model (1.2) without immigration, and then to compare dynamics of these models in order to see the effect of the immigration factor on the system (1.2). We can give references to some studies that are necessary for the basic concepts used in the analyses made throughout the study [36–38].

This article is organized as follows: Section 2 briefly mentions from the analysis of the positive equilibrium point of the model (1.2), which does not include the immigration factor. In Section 3, the equilibrium points of the model (1.3) created by including the immigration factor were obtained; and stability analyses of the obtained equilibrium points are made. Then, the flip bifurcation conditions are obtained for the positive equilibrium point. The resulting chaos was controlled in Section 4. Section 5 presents numerical simulations that validate the criteria obtained. A brief summary of the results is presented in Section 6.

2. Analysis of the SIR Epidemic Model (1.2)

Let us briefly recall the existence of equilibrium points of the system (1.2), stability of the positive equilibrium point, and flip bifurcation condition (see [35]).

Remark 2.1. *The model (1.2) has two equilibrium points such that* $(S_*, I_*) = \left(\frac{A}{d}, 0\right)$ *and* $(S^*, I^*) = \left(\frac{k+z}{\lambda}, \frac{A}{k+z} - \frac{d}{\lambda}\right)$.

Remark 2.2. *Assume that* $\frac{A\lambda}{k+z} > d$. *Regarding the dynamics of the positive equilibrium point* $(S^*, I^*) = \left(\frac{k+z}{\lambda}, \frac{A}{k+z} - \frac{d}{\lambda}\right)$, *the followings are true:*

Proposition 2.3. *If* $\delta < \frac{A\lambda}{(k+z)(A\lambda - d(k+z))} - \sqrt{\frac{4d(k+z)^3 + A\lambda(-4(k+z)^2 + A\lambda)}{(k+z)^2(d(k+z) - A\lambda)^2}}$ *is provided such that* $\frac{A\lambda(4(k+z)^2 - A\lambda)}{4(k+z)^3} < d$, *the* (S^*, I^*) *is locally asymptotically stable.*

Proposition 2.4. *For* $\delta = \frac{4(k+z)}{A\lambda + \sqrt{4d(k+z)^3 + A\lambda(-4(k+z)^2 + A\lambda)}}$, *there can be flip bifurcation such that* $B = \frac{\delta\lambda A}{k+z} \neq 2, 4$.

3. Analysis of the SIR Epidemic Model (1.3)

In this section, the aim is to discretize by adding immigration parameters to the model discussed in [4]; and then to examine the dynamics of the obtained discrete-time model. Thus, the continuous SIR epidemic model, based on different rates of immigration of both susceptible and diseased individuals, is as follows:

$$\begin{aligned} \frac{dS}{dt} &= A - dS - \lambda SI + pS \\ \frac{dI}{dt} &= \lambda SI - (d + \gamma + \varepsilon)I - T(I) + qI \\ \frac{dR}{dt} &= \gamma I - kI - dR. \end{aligned}$$

It is sufficient to consider the following model reduced to $2-$dimensions, since the first two variables are independent of the variable R

$$\begin{aligned} \frac{dS}{dt} &= A - dS - \lambda SI + pS \\ \frac{dI}{dt} &= \lambda SI - (d + \gamma + \varepsilon)I - T(I) + qI. \end{aligned}$$

Now, if we use the forward Euler method in the continuous SIR epidemic model such that, $z = d + \gamma + \varepsilon$ and $T(I) = kI$; we get discretized the system as follows:

$$\begin{aligned} S_{t+1} &= S_t + \delta\left(A - dS_t - \lambda S_t I_t + pS_t\right) \\ I_{t+1} &= I_t + \delta\left(\lambda S_t I_t - zI_t - kI_t + qI_t\right) \end{aligned}$$

with $\frac{dS}{dt} \approx \frac{S_{t+1} - S_t}{\delta}$ and $\frac{dI}{dt} \approx \frac{I_{t+1} - I_t}{\delta}$. The following Lemma is useful for analysis of the positive equilibrium point.

Lemma 3.1. *[9, 25] Let $F(\lambda)=\lambda^2+B\lambda+C$ be a quadratic polynomial with real coefficients. Suppose that this polynomial has roots λ_1, λ_2, and $F(1)>0$. Then the following statements apply:*

(i) $|\lambda_1|<1$ *and* $|\lambda_2|<1 \Leftrightarrow F(-1)>0$, *and* $C<1$. *(In this case, the equilibrium point is stable.)*
(ii) $\lambda_1=-1$ *and* $\lambda_2\neq 1 \Leftrightarrow F(-1)=0$ *and* $B\neq 0,2$. *(In this case, flip bifurcation may occur.)*

Then we can give the following analyses for this model.

3.1. Local stability

We see that the model (1.3) has two equilibrium points $(S_*,I_*)=\left(\frac{A}{d-p},0\right)$ and $(S_*,I_*)=\left(\frac{k+z-q}{\lambda},\frac{A\lambda+(p-d)(k+z-q)}{(k+z-q)\lambda}\right)$. Let us now examine the local asymptotic stability conditions of the positive equilibrium point (S^*,I^*). For this, let us take

$$f(S)=S+\delta\left(A-dS-\lambda SI+pS\right)$$
$$g(I)=I+\delta\left(\lambda SI-zI-kI+qI\right).$$

So we can write the following Jacobian matrix:

$$J(S,I)=\begin{bmatrix} f_S(S,I) & f_I(S,I)\\ g_S(S,I) & g_I(S,I)\end{bmatrix}=\begin{bmatrix} 1+\delta\left(-d-\lambda I+p\right) & -\delta\lambda S\\ \delta\lambda I & 1+\delta\left(\lambda S-z-k+q\right)\end{bmatrix}.$$

The Jacobian matrix evaluated around the positive equilibrium point (S^*,I^*) is given by

$$\begin{aligned} J(S^*,I^*)&=J\left(\frac{k+z-q}{\lambda},\frac{A\lambda+(p-d)(k+z-q)}{(k+z-q)\lambda}\right)\\ &=\begin{bmatrix} 1-\dfrac{\delta\lambda A}{k+z-q} & -\delta(k+z-q)\\ \delta\left(\dfrac{A\lambda+(k+z-q)(p-d)}{k+z-q}\right) & 1\end{bmatrix},\end{aligned}$$

where $det(J)=1-\frac{\delta\lambda A}{k+z-q}+\delta^2\lambda A+\delta^2(p-d)(k+z-q)$ and $Trace(J)=2-\frac{\delta\lambda A}{k+z-q}$. Then the characteristic polynomial corresponding to the Jacobian matrix has the form:

$$F(\mu)=\mu^2-\left(2-\frac{\delta\lambda A}{k+z-q}\right)\mu+1-\frac{\delta\lambda A}{k+z-q}+\delta^2(\lambda A+(p-d)(k+z-q))$$

and the roots of this polynomial are found:

$$\mu_{1,2}=1-\frac{\delta\lambda A}{2(k+z-q)}\pm\frac{\delta}{2(k+z-q)}\sqrt{4(d-p)(k-q+z)^3-4(k-q+z)^2A\lambda+A^2\lambda^2}.$$

For the stability of the equilibrium point, the magnitudes of these two eigenvalues must remain less than 1. To determine the conditions on the parameters, we first make use of the following conditions:

$$|\lambda_1|<1 \quad\text{and}\quad |\lambda_2|<1\Leftrightarrow F(-1)>0, F(1)>0 \quad\text{and}\quad C<1.$$

Proposition 3.2. $F(1)>0\Rightarrow F(1)=\delta^2\left(A\lambda+(p-d)(k+z-q)\right)>0$. *Thus* $\frac{A\lambda}{k+z-q}>d-p>0$ *must be provided such that* $0<q\leq z$.

Proposition 3.3.

$$F(-1)>0\Rightarrow F(-1)=\frac{4(k+z-q)-2\delta\lambda A+\delta^2(\lambda A(k+z-q)+(k+z-q)^2(p-d))}{k+z-q}>0.$$

If this inequality is solved, we have

$$\delta_1<\frac{\lambda A}{(k+z-q)(\lambda A+(p-d)(k-q+z))}-\sqrt{\frac{4(d-p)(k-q+z)^3-4(k-q+z)^2A\lambda+A^2\lambda^2}{(k+z-q)^2((d-p)(k-q+z)-A\lambda)^2}}.$$

or

$$\delta_2>\frac{\lambda A}{(k+z-q)(\lambda A+(p-d)(k-q+z))}+\sqrt{\frac{4(d-p)(k-q+z)^3-4(k-q+z)^2A\lambda+A^2\lambda^2}{(k+z-q)^2((d-p)(k-q+z)-A\lambda)^2}}.$$

such that $\frac{A\lambda(4(k+z-q)^2-A\lambda)}{4(k+z-q)^3}+p<d$.

Proposition 3.4.

$$C<1\Rightarrow C=\frac{(k+z-q)-\delta\lambda A+\delta^2(\lambda A(k+z-q)+(k+z-q)^2(p-d))}{k+z-q}<1.$$

If this inequality is solved, we have

$$\delta<\frac{A\lambda}{(k+z-q)(A\lambda+(p-d)(k+z-q))}.$$

Theorem 3.5. *Assume that* $\frac{A\lambda}{k+z-q} > d-p > 0$. *If* $\frac{A\lambda(4(k+z-q)^2 - A\lambda)}{4(k+z-q)^3} + p < d$ *and*

$$\delta < \frac{\lambda A}{(k+z-q)(\lambda A + (p-d)(k-q+z))} - \sqrt{\frac{4(d-p)(k-q+z)^3 - 4(k-q+z)^2 A\lambda + A^2\lambda^2}{(k+z-q)^2((d-p)(k-q+z) - A\lambda)^2}}$$

are provided, the positive equilibrium point of the model (1.3) is locally asymptotic stable.

3.2. Flip bifurcation

Let's $\frac{A\lambda}{k+z-q} > d-p > 0$. We know that when $B \neq 0, 2$ and $F(-1) = 0$, flip bifurcation can occur. Therefore, we will consider these conditions such that $B = -trace(J)$, $C = det(J)$.

Proposition 3.6. *From the condition* $F(-1) = 0$, *we get the roots*

$$\delta_{1,2} = \frac{4(k-q+z)}{\lambda A \pm \sqrt{4(d-p)(k-q+z)^3 - 4(k-q+z)^2 A\lambda + A^2\lambda^2}}.$$

Proposition 3.7. *From the condition* $B \neq 0$, *we reach*

$$B = -trace(J) = -\left(2 - \frac{\delta\lambda A}{k+z-q}\right) \quad \text{and} \quad -trace(J) \neq 0 \Leftrightarrow \delta \neq \frac{2(k+z-q)}{\lambda A}.$$

Proposition 3.8. *From the condition* $B \neq 2$, *we obtain*

$$B = -trace(J) = -\left(2 - \frac{\delta\lambda A}{k+z-q}\right) \quad \text{and} \quad -trace(J) \neq 2 \Leftrightarrow \delta \neq \frac{4(k+z-q)}{\lambda A}.$$

Theorem 3.9. *If the condition*

$$\delta = \frac{4(k-q+z)}{\lambda A + \sqrt{4(d-p)(k-q+z)^3 - 4(k-q+z)^2 A\lambda + A^2\lambda^2}}$$

are met, the model (1.3) has flip bifurcation such that $\delta \neq \frac{2(k+z-q)}{\lambda A}$ *and* $\delta \neq \frac{4(k+z-q)}{\lambda A}$.

If

$$\delta = \delta_{FB} = \frac{4(k-q+z)}{\lambda A + \sqrt{4(d-p)(k-q+z)^3 - 4(k-q+z)^2 A\lambda + A^2\lambda^2}}$$

then $\lambda_1 = -1$ with

$$|\lambda_2| \neq 1. \tag{3.1}$$

These conditions can be presented by the following set

$$FB_{(S_*,I_*)} = \left\{ A,d,k,z,\lambda,p,q,\delta \in \mathbb{R}^+ : \delta = \delta_{FB} = \frac{4(k-q+z)}{\lambda A + \sqrt{4(d-p)(k-q+z)^3 - 4(k-q+z)^2 A\lambda + A^2\lambda^2}}, |\lambda_2| \neq 1 \right\}.$$

Using the transformation $u = x - \frac{k-q+z}{\lambda}$, $v = y - \frac{A}{k-q+z} - \frac{p-d}{\lambda}$, the fixed point (S_*, I_*) is shifted to the origin. Therefore, we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix} \to J_{(S_*,I_*)} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F_1(u,v) \\ F_2(u,v) \end{pmatrix}$$

where

$$\begin{aligned} F_1(u,v) &= -\delta\lambda uv \\ F_2(u,v) &= \delta\lambda uv \end{aligned}$$

such that $U = (u,v)^T$. From there, the system (1.3) can be written as

$$(U_{n+1}) \to J_{(S_*,I_*)}(U_n) + \frac{1}{2}B(u_n,u_n) + \frac{1}{6}C(u_n,u_n,u_n) + O(\|u_n\|^4),$$

with the multilinear vector functions of $u,v,w \in \mathbb{R}^2$:

$$B(u,v) = \begin{pmatrix} B_1(u,v) \\ B_2(u,v) \end{pmatrix}$$

and

$$C(u,v,w) = \begin{pmatrix} C_1(u,v,w) \\ C_2(u,v,w) \end{pmatrix}.$$

These vectors are expressed by

$$B_1(u,v) = \sum_{j,k=1}^{2} \frac{\partial^2 F_1}{\partial \xi_j \partial \xi_k} |_{\xi=0}\, u_j v_k = -\delta\lambda(u_2 v_1 + u_1 v_2)$$
$$B_2(u,v) = \sum_{j,k=1}^{2} \frac{\partial^2 F_2}{\partial \xi_j \partial \xi_k} |_{\xi=0}\, u_j v_k = \delta\lambda(u_2 v_1 + u_1 v_2)$$

$$C_1(u,v,w) = \sum_{j,k=1}^{2} \frac{\partial^3 F_1}{\partial \xi_j \partial \xi_k \xi_l} |_{\xi=0}\, u_j v_k w_l = 0$$
$$C_2(u,v,w) = \sum_{j,k=1}^{2} \frac{\partial^3 F_2}{\partial \xi_j \partial \xi_k \xi_l} |_{\xi=0}\, u_j v_k w_l = 0$$

and $\delta = \delta_{FB}$. Let $q, p \in \mathbb{R}^2$ be eigenvectors of $J_{(S_*,I_*)}(\delta_{FB})$ and transposed matrix $J^T_{(S_*,I_*)}(\delta_{FB})$ respectively for $\lambda_1(\delta_{FB}) = -1$. Then, we have $J_{(S_*,I_*)}(\delta_{FB})q = -q$ and $J^T_{(S_*,I_*)}(\delta_{FB})p = -p$. We use standard scalar product $< p, q >= p_1q_1 + p_2q_2$ in $\mathbb{R}^2$ in order to normalize p with respect to q, such that $< p, q >= 1$. To determine the direction of the flip bifurcation, we need to get the sign of the coefficient $c(\delta_{FB})$ as follows:

$$c(\delta_{FB}) = \frac{1}{6} < p, C(q,q,q) > -\frac{1}{2} < p, B(q,(J-I)^{-1}B(q,q) > .$$

The following theorem gives the result on flip bifurcation regarding the coefficient of the critical normal form.

Theorem 3.10. *If (3.1) becomes valid, $c(\delta_{FB}) \neq 0$, and the parameter a changes its value around δ_{FB}, then the system (1.3) undergoes a flip bifurcation at positive coexistence fixed point (S_*, I_*). Furthermore, if $c(\delta_{FB}) > 0$ ($c(\delta_{FB}) < 0$), then the period 2 orbits that bifurcate from (S_*, I_*) are stable (unstable).*

4. Chaos Control

In this section, we will use the chaos control method to control the chaos that occurs in systems (1.2) and (1.3). Chaos theory, a method of qualitative and quantitative analysis for investigating the behavior of dynamic systems, explains how a small change in one state of a nonlinear system can lead to large differences in a later state. In some cases, long-term prediction of the behavior of a chaotic system may become impossible, especially due to sensitive dependence on initial conditions, and even the deterministic nature of the system does not make them predictable. Due to the infinite number of unstable periodic orbits, system behavior becomes unpredictable. Control of chaos is the stabilization of one of the selected unstable periodic orbits through small system perturbations. The aim is to make the chaotic behavior more stable and predictable by directing the trajectories towards the desired position by adding an appropriate control parameter to the system. A state feedback control method [16, 18, 36] is used to stabilize chaotic orbit at an unstable fixed point of the system (1.2) and (1.3).

4.1. Chaos control analysis for the (1.2) model

The controlled form of the system (1.2) is obtained by incorporating a feedback control parameter as the control force into system (1.2). So we define the controller of the system (1.2) as follows:

$$S_{t+1} = S_t + \delta(A - dS_t - \lambda S_t I_t) + U_t$$
$$I_{t+1} = I_t + \delta(\lambda S_t I_t - zI_t - kI_t)$$

where U_t is a control force such that $U_t = -p_1(S_t - S^*) - p_2(I_t - I^*)$. The Jacobian matrix at the positive equilibrium point of this system is

$$J(S^*, I^*) = J\left(\frac{k+z}{\lambda}, \frac{A\lambda - (k+z)d}{(k+z)\lambda}\right) = \begin{bmatrix} 1 - \dfrac{\delta A\lambda}{k+z} - p_1 & -\delta(k+z) - p_2 \\ \delta\left(\dfrac{A\lambda - (k+z)d}{k+z}\right) & 1 \end{bmatrix}$$

and the characteristic equation obtained through the Jacobian matrix is written as;

$$F(\mu) = \mu^2 - \left(2 - \frac{\delta A\lambda}{k+z} - p_1\right)\mu + 1 - \frac{\delta A\lambda}{k+z} - p_1 + \delta^2(A\lambda - d(k+z)) + \delta p_2\left(\frac{A\lambda - d(k+z)}{k+z}\right)$$

and $\mu_{1,2}$ be the eigenvalues of this characteristic equation. Then we have

$$\mu_1 + \mu_2 = 2 - \frac{\delta A\lambda}{k+z} - p_1$$

and

$$\mu_1\mu_2 = 1 - \frac{\delta A\lambda}{k+z} - p_1 + \delta^2(A\lambda - d(k+z)) + \delta p_2\left(\frac{A\lambda - d(k+z)}{k+z}\right).$$

We must solve the equations $\mu_1\mu_2 = 1, \mu_1 = 1$ and $\mu_1 = -1$. So, we get the marginal line I_1, I_2 and I_3 as follows:

$$\begin{aligned}
\mu_1\mu_2 = det(J) = 1 &\Rightarrow I_1 = -\frac{\delta A\lambda}{k+z} - p_1 + \delta^2(A\lambda - d(k+z)) + \delta p_2\left(\frac{A\lambda - d(k+z)}{k+z}\right)\\
\mu_1 = 1 &\Rightarrow I_2 = \delta^2(A\lambda - d(k+z)) + \delta p_2\left(\frac{A\lambda - d(k+z)}{k+z}\right)\\
\mu_1 = -1 &\Rightarrow I_3 = 4 - \frac{2\delta A\lambda}{k+z} - 2p_1 + \delta^2(A\lambda - d(k+z)) + \delta p_2\left(\frac{A\lambda - d(k+z)}{k+z}\right)
\end{aligned}$$

The region bounded by I_1, I_2 and I_3 gives stable eigenvalues of magnitude less than 1.

4.2. Chaos control analysis for the (1.3) model

We define the controller of the system (1.3) as follows:

$$\begin{aligned}
S_{t+1} &= S_t + \delta\left(A - dS_t - \lambda S_t I_t + pS_t\right) + U_t\\
I_{t+1} &= I_t + \delta\left(\lambda S_t I_t - zI_t - kI_t + qI_t\right)
\end{aligned}$$

where U_t is a control force such that $U_t = -p_1\left(S_t - S^*\right) - p_2\left(I_t - I^*\right)$. The Jacobian matrix at the positive equilibrium point of this system is

$$\begin{aligned}
J\left(S^*, I^*\right) &= J\left(\frac{k+z-q}{\lambda}, \frac{A\lambda - (p-d)(k+z-q)}{(k+z-q)\lambda}\right)\\
&= \begin{bmatrix} 1 - \frac{\delta A\lambda}{k+z-q} - p_1 & -\delta(k+z-q) - p_2\\ \delta\left(\frac{A\lambda - (p-d)(k+z-q)}{k+z-q}\right) & 1 \end{bmatrix}.
\end{aligned}$$

The characteristic equation we get by means of the Jacobian matrix is

$$F(\mu) = \mu^2 - trace(J) + det(J),$$

where,

$$trace(J) = 2 - \frac{\delta A\lambda}{k+z-q} - p_1$$

and

$$det(J) = 1 - \frac{\delta A\lambda}{k+z-q} - p_1 + \delta^2(A\lambda - (p-d)(k+z-q)) + \delta p_2\left(\frac{A\lambda - (p-d)(k+z-q)}{k+z-q}\right).$$

The eigenvalues of the characteristic equation $F(\mu)$ are μ_1 and μ_2. By providing the conditions $\mu_1\mu_2 = 1$, $\mu_1 = 1$ and $\mu_1 = -1$, we have the marginal line I_1, I_2 and I_3 as follows:

$$\begin{aligned}
I_1 &= -\frac{\delta A\lambda}{k+z-q} - p_1 + \delta^2(A\lambda - (p-d)(k+z-q)) + \delta p_2\left(\frac{A\lambda - (p-d)(k+z-q)}{k+z-q}\right)\\
I_2 &= \delta^2(A\lambda - (p-d)(k+z-q)) + \delta p_2\left(\frac{A\lambda - (p-d)(k+z-q)}{k+z-q}\right)\\
I_3 &= 4 - \frac{2\delta A\lambda}{k+z-q} - 2p_1 + \delta^2(A\lambda - (p-d)(k+z-q)) + \delta p_2\left(\frac{A\lambda - (p-d)(k+z-q)}{k+z-q}\right).
\end{aligned}$$

The region bounded by I_1, I_2 and I_3 gives stable eigenvalues of magnitude less than 1.

5. Numerical Simulations

We give the following examples to verify our theoretical results. Time series, phase and bifurcation graphs are presented by using Matlab program (see also the Mathematical Software program [39,40]).

Example 5.1. *We can write system (1.2) as*

$$\begin{aligned}
S_{t+1} &= S_t + \delta\left(3 - 0.1S_t - S_t I_t\right)\\
I_{t+1} &= I_t + \delta I_t\left(S_t - 0.3 - 0.2\right)
\end{aligned} \tag{5.1}$$

with parameter values $A = 3, \lambda = 1, k = 0.2, z = 0.3, d = 0.1$. The presented graphs show the dynamic behavior of the system (5.1) with the initial condition $(S_0, I_0) = (2.1, 0.9)$.

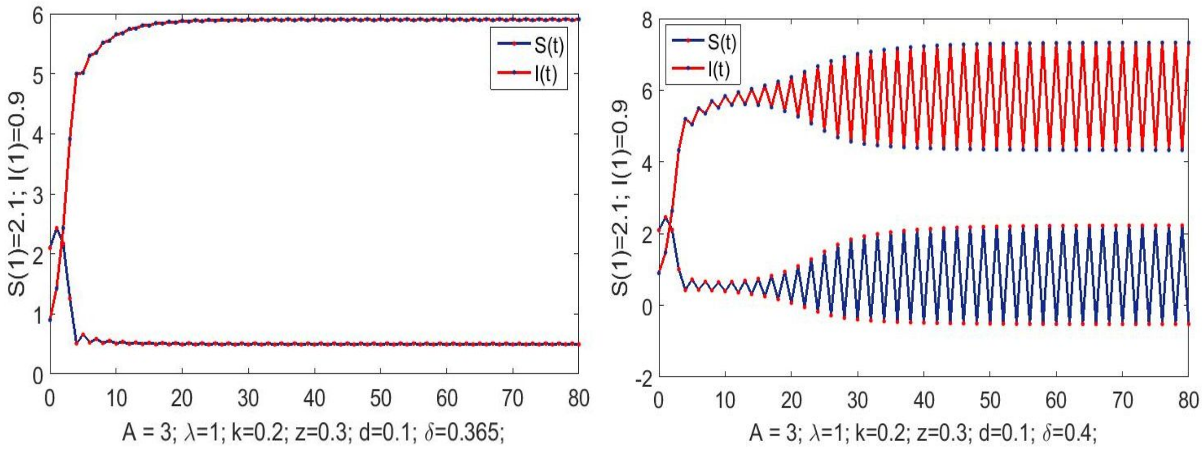

Figure 5.1: Time Series Graph of System (5.1) when (a) $\delta = 0.365$ (b) $\delta = 0.4$

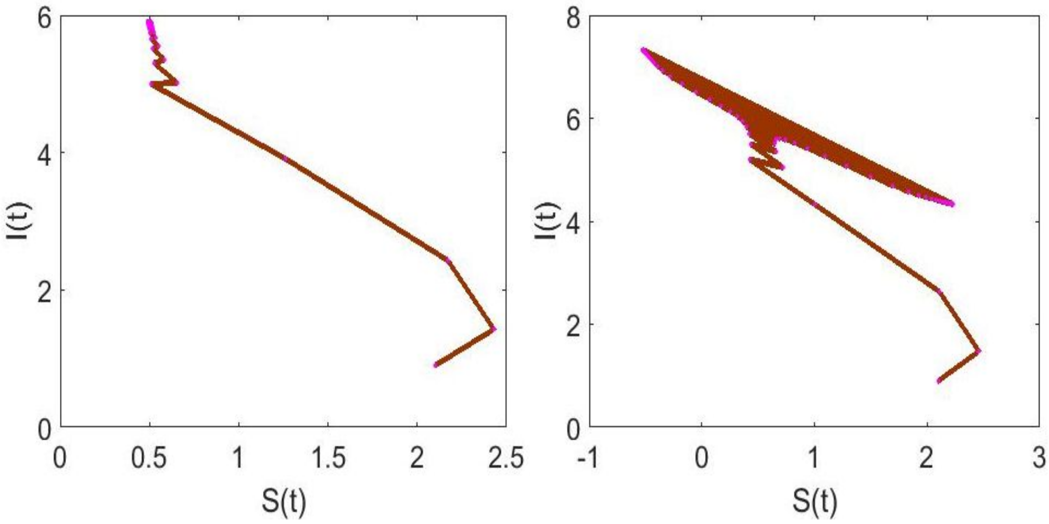

Figure 5.2: Phase Graph of System (5.1)

While we observe that the system (5.1) is locally asymptotic stable ($\delta = 0.365 < 0.366$) with the appropriate parameter values in Figure 5.1-(a), we see that the system (5.1) is unstable ($\delta = 0.4 > 0.366$) when the value δ is increased in Figure 5.1-(b). Figure 5.2, corresponding to Figure 5.1 with the same parameter values, is the phase portraits of the system (5.1).

Also, in Figure 5.3, we present the flip bifurcation graph of the system (5.1) for the parameter values $A = 3, \lambda = 1, k = 0.2, z = 0.3, d = 0.1$ and $0.3 < \delta < 0.5$. Here, we can see that flip bifurcation occurs at (S_*, I_*) when the parameter changes in a small neighborhood of $\delta_{FB} = 0.366322$. The computation yields $(S_*, I_*) = (0.5, 5.9)$. The Jacobian matrix is $J = \begin{bmatrix} -1.19793 & -0.183161 \\ 2.1613 & 1 \end{bmatrix}$. The eigenvalues are $\lambda_1 = -1$, and $\lambda_2 = 0.802067$ such that $|\lambda_2| \neq 1$. This defines that the fixed point (S_*, I_*) is stable for $\delta < 0.366322$, and there exists a period doubling phenomena for $\delta > 0.366322$. By direct calculations, we can write

$$
\begin{aligned}
F_1(u,v) &= -0.366322uv \\
F_2(u,v) &= 0.366322uv \\
B_1(u,v) &= -0.366322(u_2v_1 + u_1v_2) \\
B_2(u,v) &= 0.366322(u_2v_1 + u_1v_2) \\
C_1(u,v,w) &= 0 \\
C_2(u,v,w) &= 0
\end{aligned}
$$

$$
\begin{aligned}
B(q,q) &= \begin{pmatrix} 0.365223 \\ -0.365223 \end{pmatrix} \\
C(q,q,q) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

and $p \sim (-0.733965, -0.679187)^T$, $q \sim (-0.679187, 0.733965)^T$. Here, $p \sim (-4.40731.10^{15}, -4.07838.10^{15})^T$ is obtained as normalized vector according to q, such that $< p, q >= 1$. Upon the necessary calculations, we obtain $c(\delta_{FB}) = 0.000285917 > 0$. The period-2 orbits that bifurcate from (S_*, I_*) are stable.

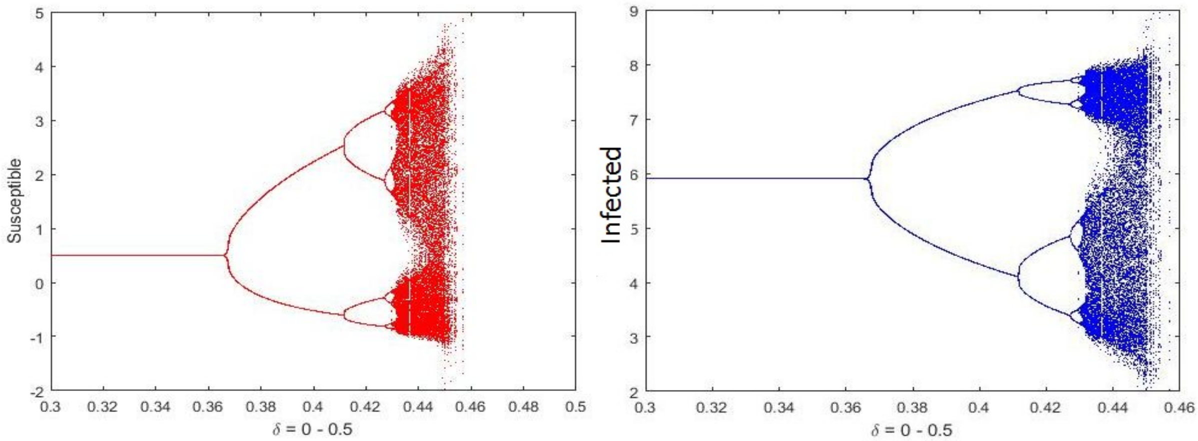

Figure 5.3: Flip Bifurcation Graph of System (5.1).

Example 5.2. *We can write system (1.3) as*

$$\begin{aligned} S_{t+1} &= S_t + \delta\,(3 - 0.1S_t - S_tI_t + 0.01S_t) \\ I_{t+1} &= I_t + \delta I_t\,(S_t - 0.3 - 0.2 + 0.1) \end{aligned} \tag{5.2}$$

with parameter values $A = 3, \lambda = 1, k = 0.2, z = 0.3, d = 0.1, q = 0.1, p = 0.01$. The following graphs are obtained for the dynamic behavior of the system (5.2) with $(S_0, I_0) = (2.1, 0.9)$. The time series and phase diagram graphs are displayed in Figures 5.4 & 5.5. In Figure 5.8, we present the flip bifurcation graph of the system (5.2) for the parameter values $A = 3, \lambda = 1, k = 0.2, z = 0.3, d = 0.1, p = 0.01, q = 0.1$ and $0.25 < \delta < 0.35$. Here, it can seen that flip bifurcation occurs at (S_*, I_*) when the parameter changes in a small neighborhood of $\delta_{FB} = 0.282428$. The computation yields $(S_*, I_*) = (0.4, 7.41)$. The Jacobian matrix is $J = \begin{bmatrix} -1.1181 & -0.112971 \\ 2.08279 & 1 \end{bmatrix}$. The eigenvalues are $\lambda_1 = -1$, and $\lambda_2 = 0.881788$ such that $|\lambda_2| \neq 1$. So, the fixed point (S_*, I_*) is stable for $\delta < 0.282428$ and there exists a period doubling phenomena for $\delta > 0.282428$. By direct calculations, we obtain

$$\begin{aligned} F_1(u,v) &= -0.282428uv \\ F_2(u,v) &= 0.282428uv \end{aligned}$$

$$\begin{aligned} B_1(u,v) &= -0.282428(u_2v_1 + u_1v_2) \\ B_2(u,v) &= 0.282428(u_2v_1 + u_1v_2) \\ C_1(u,v,w) &= 0 \\ C_2(u,v,w) &= 0 \\ B(q,q) &= \begin{pmatrix} 0.0318047 \\ -0.0318047 \end{pmatrix} \\ C(q,q,q) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned}$$

and $p \sim (-0.998408, -0.0563957)^T$, $q \sim (0.0563957, -0.998408)^T$. Here, $p \sim (-7.19429.10^{16}, -4.06374.10^{15})^T$ is obtained as normalized vector according to q, such that $< p, q >= 1$. Upon the necessary calculations, we obtain $c\,(\delta_{FB}) = -0.00204406 < 0$. The period-2 orbits that bifurcate from (S_*, I_*) are unstable.

Example 5.3. *We can write system (1.3) as*

$$\begin{aligned} S_{t+1} &= S_t + \delta\,(3 - 0.1S_t - S_tI_t + 0.2S_t) \\ I_{t+1} &= I_t + \delta I_t\,(S_t - 0.3 - 0.2 + 0.1) \end{aligned} \tag{5.3}$$

with parameter values $A = 3, \lambda = 1, k = 0.2, z = 0.3, d = 0.1, q = 0.1, p = 0.2$. The following graphs are obtained for the dynamic behavior of the system (5.2) with $(S_0, I_0) = (2.1, 0.9)$. The time series and phase diagram graphs are displayed in Figures 5.6 & 5.7.

The flip bifurcation graph of the system (5.3) for the parameter values $A = 3, \lambda = 1, k = 0.2, z = 0.3, d = 0.1, p = 0.2, q = 0.1$ and $0.25 < \delta < 0.32$ exhibited in Figure 5.9. The flip bifurcation emerges at (S_*, I_*) when the parameter changes in a small neighborhood of $\delta_{FB} = 0.282885$. The computation yields $(S_*, I_*) = (0.4, 7.6)$. The Jacobian matrix is $J = \begin{bmatrix} -1.12164 & -0.113154 \\ 2.14992 & 1 \end{bmatrix}$. The eigenvalues are $\lambda_1 = -1$, and $\lambda_2 = 0.878364$ such that $|\lambda_2| \neq 1$. This defines that the fixed point (S_*, I_*) is stable for $\delta < 0.282885$, and there exists a period

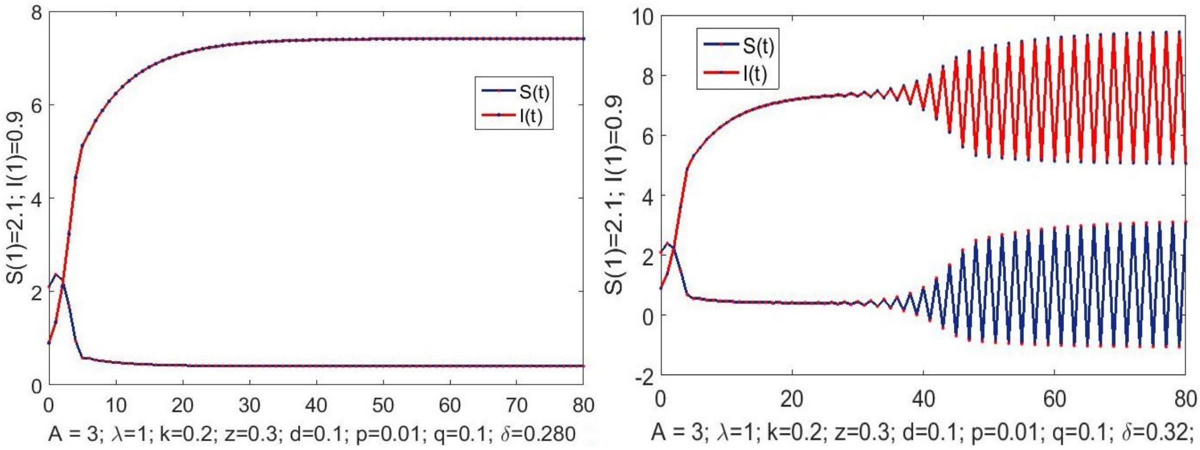

Figure 5.4: Time Series Graph of System (5.2) when (a) $\delta = 0.28$ (b) $\delta = 0.32$

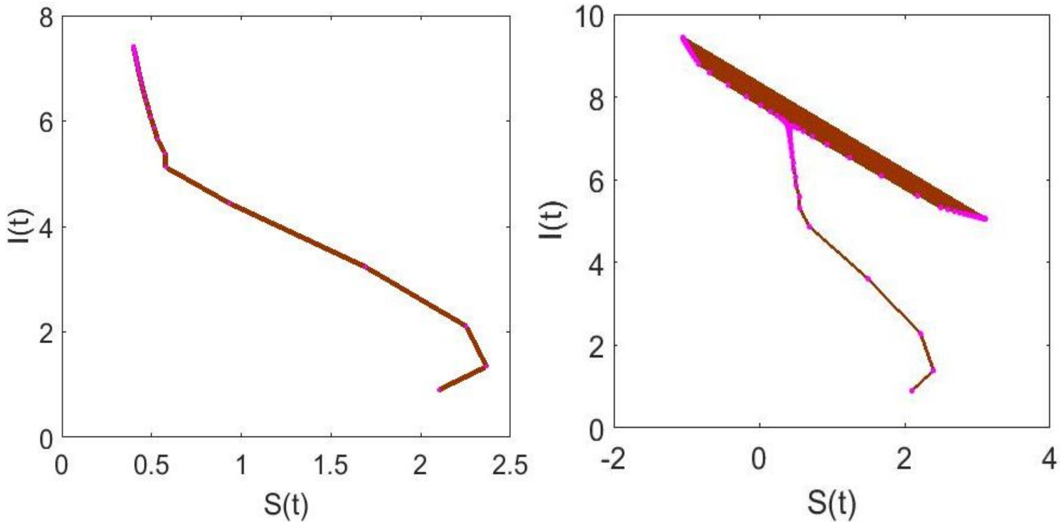

Figure 5.5: Phase Graph of System (5.2)

doubling phenomena for $\delta > 0.282885$. By direct calculations, we can write

$$
\begin{aligned}
F_1(u,v) &= -0.282885uv \\
F_2(u,v) &= 0.282885uv \\
B_1(u,v) &= -0.282885(u_2v_1+u_1v_2) \\
B_2(u,v) &= 0.282885(u_2v_1+u_1v_2) \\
C_1(u,v,w) &= 0 \\
C_2(u,v,w) &= 0 \\
B(q,q) &= \begin{pmatrix} 0.0319074 \\ -0.0319074 \end{pmatrix} \\
C(q,q,q) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

and $p \sim (-0.998403, -0.0564866)^T$, $q \sim (0.0564866, -0.998403)^T$. Here, $p \sim (1.43885.10^{17}, 8.14058.10^{15})^T$ is obtained as normalized vector according to q, such that $< p,q >= 1$. Upon the necessary calculations, we obtain $c(\delta_{FB}) = -0.00904053 < 0$. The period-2 orbits that bifurcate from (S_*, I_*) are unstable.

Example 5.4. *For controlled system (5.1) with parameter values* $A = 3; \lambda = 1; z = 0.3; d = 0.1; k = 0.2$ *and* $\delta = 0.44$*, we get the marginal lines are*

$$
\begin{aligned}
&I_1 = -2.06888 - p_1 + 2.596p_2 \\
&I_2 = 0.57112 + 2.596p_2 \\
&I_3 = -0.70888 - 2p_1 + 2.596p_2.
\end{aligned}
$$

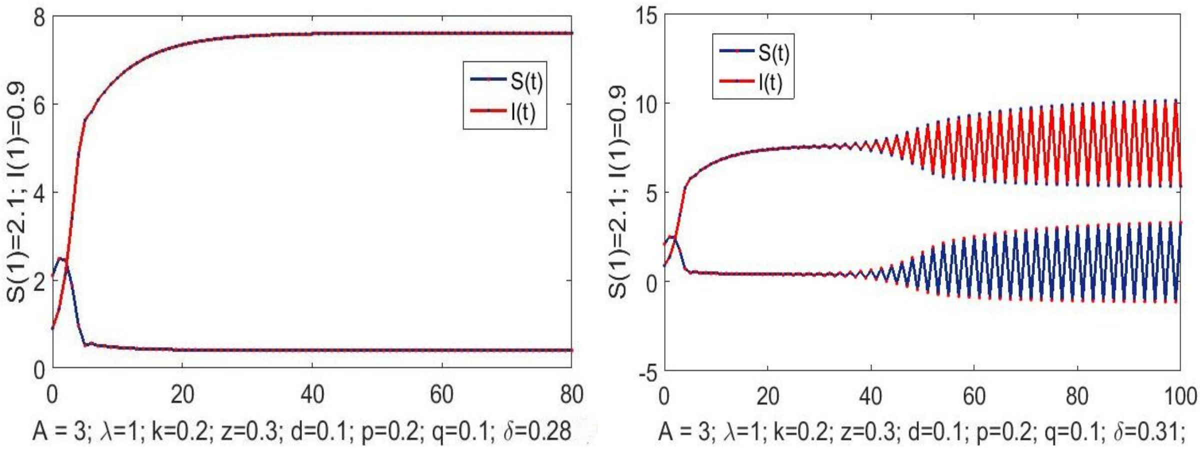

Figure 5.6: Time Series Graph of System (5.3) when (a) $\delta = 0.287$ (b) $\delta = 0.31$

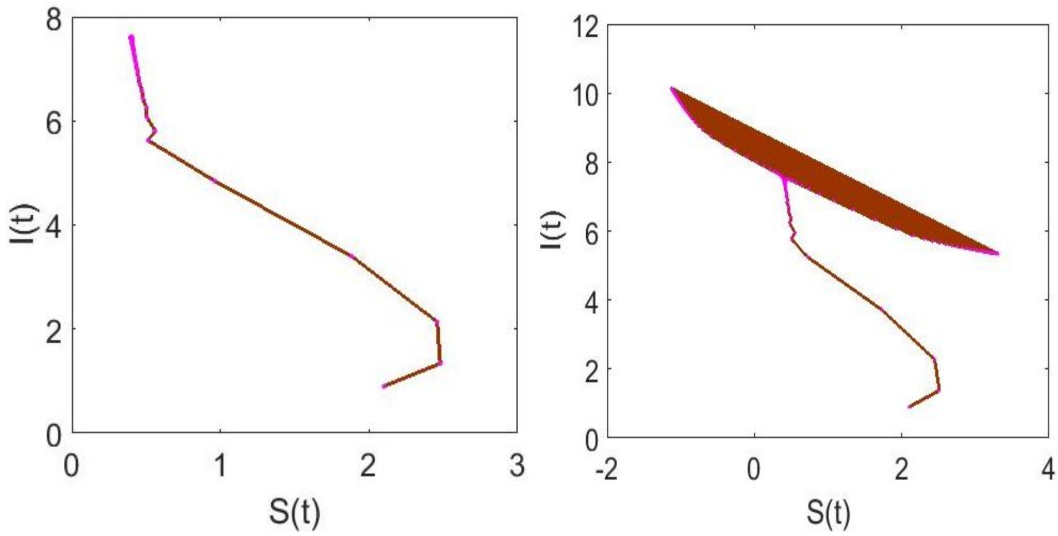

Figure 5.7: Phase Graph of System (5.3)

Example 5.5. *For controlled system (5.2) with parameter values* $A = 3; \lambda = 1; k = 0.2; z = 0.3; d = 0.1; p = 0.01; q = 0.1$ *and* $\delta = 0.34$*, we get the marginal lines are*

$$
\begin{aligned}
I_1 &= -2.71223 - p_1 + 3.3396p_2 \\
I_2 &= 0.5877 + 3.3396p_2 \\
I_3 &= -2.01223 + 2p_1 + 3.3396p_2.
\end{aligned}
$$

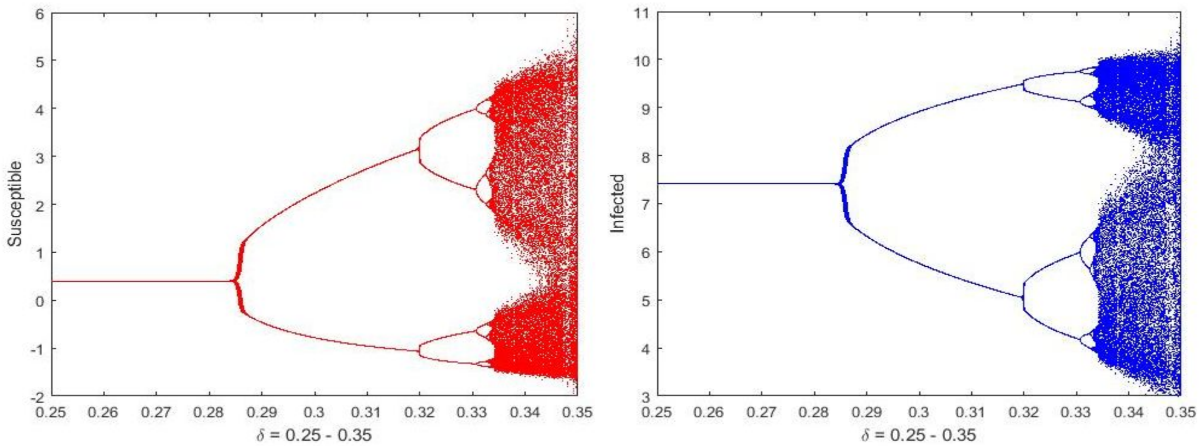

Figure 5.8: Flip Bifurcation Graph of System (5.2) for $0.25 < \delta < 0.35$.

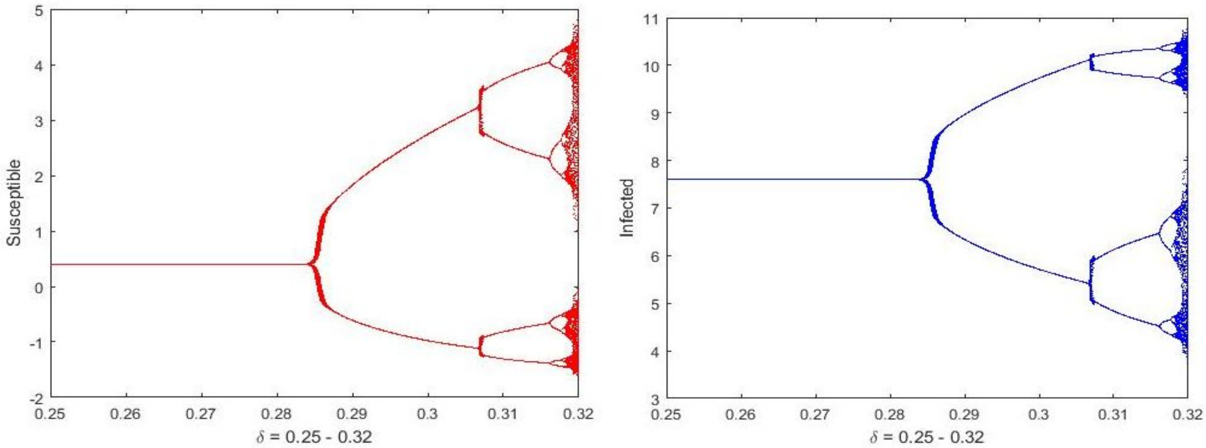

Figure 5.9: Flip Bifurcation Graph of System (5.3) for $0.25 < \delta < 0.32$.

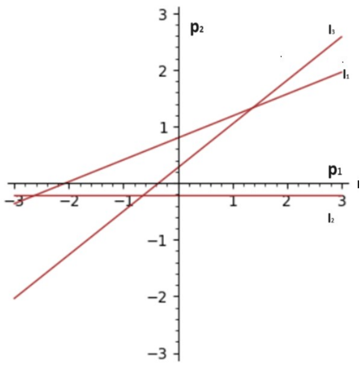

Figure 5.10: Chaos control lines of the system (5.1).

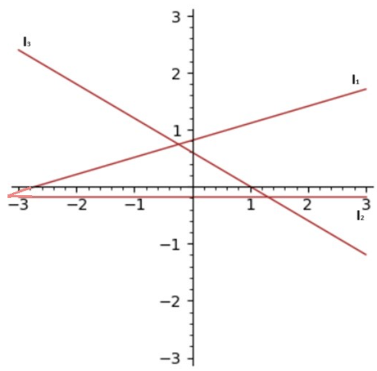

Figure 5.11: Chaos control lines of the system (5.2).

6. Conclusion

In this study, first of all, the existence of the equilibrium points of the discrete-time system (1.2), the local stability of the equilibrium points, the conditions of flip bifurcation are summarized analytically. The model (1.3) is created by adding the immigration effect to the model (1.2); and the dynamics of model (1.3) are examined. A comparison is presented for the dynamic behavior of model (1.2) and model (1.3). Finally, numerical simulations are included to support the theoretical results obtained.
Figure 5.4 and Figure 5.6 are time series graphs with immigration parameters added at different rates to susceptible individuals. Note that and bifurcation values are calculated $\delta = 0.366322$ and $\delta = 0.282428$ for without immigration and with immigration, respectively (see Figures 5.3 and 5.8). Considering Figures 5.3 and 5.8, we see that immigration parameters lead the system to faster flip bifurcation.
Finally, Figure 5.8 and Figure 5.9 show that flip bifurcation will be delayed as the number of immigration added to susceptible individuals increases. Also, Example 5.4 and Example 5.5 give chaos control lines of the system (5.1) and (5.2), respectively. The stable triangular region is determined by these marginal lines.

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