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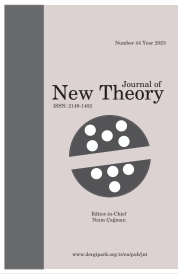

A New Form of Smooth Cubic Surfaces with 9 Lines

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Abstract — A smooth cubic surface has at most 27 lines, with equality if and only if the underlying field is algebraically closed. Only a few cases are possible regarding the number of lines over fields that are not algebraically closed. The next two cases of interest are smooth cubic surfaces with 15 or 9 lines. The author has recently settled the case of 15 lines. In this paper, we address the case of smooth cubic surfaces with 9 lines. We describe a way to create some cubic surfaces with 9 or more lines based on a set of six field elements. Conditions on the six parameters are given under which the surface has exactly 9, 15, or 27 lines. However, the problem of generating all cubic surfaces with 9 lines remains open.

Keywords *Cubic surface, parametrization, non-algebraically closed fields*

Mathematics Subject Classification (2020) 14G27, 68W30

1. Introduction

It is well known that a smooth cubic surface over an algebraically closed field has exactly 27 lines [1]. However, the number of lines over a non-algebraically closed field varies. Naturally, the following question arises: How many lines can a smooth cubic surface have over a non-algebraically closed field? The problem of determining these numbers over the fields of $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{F}_q$ where q is odd, and $\mathbb{F}_2$ has been considered by several authors [2–5]. In [6], the author gives the possible number of lines of smooth cubic surfaces over $\mathbb{F}_q$ where q is even. The number of lines of a smooth cubic surface is one of 27, 15, 9, 7, 5, 3, 2, 1, or 0 [3]. The results on cubic surfaces with 15 or 27 lines over a given field are found in [7–11], as well as using alternative methods in papers [12–14]. All the classification results agree with an enumerative formula recently found by Das [15].

In this paper, we focus on smooth surfaces with at least 9 lines over various fields, characteristic 0 or p. In [2], Schlafli described 27 lines of the cubic surface, explaining the line intersection properties. Each line intersects ten others and skews to 16. He defined the term "double-six", which has 12 lines with some special properties, and another 15 can be produced by these 12. To give the intersection properties of the lines for 9, we use the same idea of the double-six but for double-three. Smooth cubic surfaces may only have less than 27 lines if the field is nonalgebraically closed. However, over the algebraic closure of that field, the surface will have 27 lines. Therefore, we can use Schlafli labeling to notate 9 lines. We will then prove that the line intersection graph of the smooth cubic surfaces with 9 lines is unique.

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We go back to the original proof by Cayley and Salmon that the surface has 27 lines over the algebraically closed fields as we did in [11]. We see that there is a discriminant condition certain polynomial of degree 5, which can have irreducible factors of degree 3 over the field $\mathbb{F}$, which is not algebraically closed. When this happens, we end up with cubic surfaces with 9 lines. Considering the rational lines over a given field $\mathbb{F}$, we formulate the conditions that the surface has 9 lines over the given field. We describe smooth cubic surfaces with at least 9 lines using six parameters. Our approach is experimental. We study some examples of smooth cubic surfaces with 9 lines that we obtain using the computer algebra system Orbiter [16]. Once we observe the pattern, we make the computer free proof. The proof is based on the symmetry of the projective group. We use the computer algebra system Maple for the symbolic computations. When we extend the field over the algebraic closure $\overline{\mathbb{F}}$, the surface is complete to 27 lines since these surfaces are smooth. Using our model, we will show examples of cubic surfaces over a field of characteristic zero and p. These examples would have 9, 15, or 27 lines depending on whether the special polynomial is irreducible or reducible into two irreducible polynomials or splits completely over the base field.

We give the rational parameterization of points of our new form. To do this, we study the birational map between cubic surfaces and a plane, [9, 17]. There is an exceptional locus, and the birational map is defined outside the exceptional locus bijective. The exceptional locus of the map on the plane is two conics and a line. The exceptional locus of the map on the surface is two skew lines and one transversal line. We give them explicitly.

The smooth cubic surface has q^2+tq+1 points where t is between -2 and 7, but 6 is never possible [18]. Studying the birational map helps us to prove that the smooth cubic surfaces with 9 lines have q^2+4q+1 points. If the cubic surface has a double-six, then the surface is smooth and has exactly 27 lines. However, if the surface has a double-three, it does not necessarily have exactly 9 lines. It can have more lines, in which case the surface is singular. Hence, a necessary condition for the smoothness of our new form is needed. We give this condition using the rational parameterization of our new form.

In section 2, we will provide some basic theory about the cubic surfaces with 27 lines since the structure of smooth cubic surfaces with 9 lines is the sub-configuration of the structure of cubic surfaces with 27 lines. In section 3, we show the uniqueness of 9 lines and investigate the configuration. In this section, we also give our new form for smooth cubic surfaces with at least 9 lines using six parameters, and we provide the conditions when the surface has exactly 9, 15, or 27 over the given field. In section 4, we provide examples of various fields, including $\mathbb{Q}$ and some finite fields. In section 5, we give the rational parameterization of our new form and the condition when our form is smooth. In section 6, we discuss future work.

2. Preliminaries

In this section, we provide some background material on cubic surfaces and projective geometry over finite fields. For a deeper treatment, we refer to [11, 19, 20].

A finite field is a field with only a finite number of elements. $\mathbb{F}_q$ is a finite field of order $q = p^k$ where p is a prime number. The characteristic of the field is the smallest n such that $\underbrace{1+1+1+\cdots+1}_{n\ times} = 0$. The characteristic of $\mathbb{F}_q$ is p. $\overline{\mathbb{F}}_p = \mathbb{F}_p$ adjoints all the roots of polynomials over $\mathbb{F}_p$. $\overline{\mathbb{F}}_p$ is an algebraically closed field of characteristic p. $\overline{\mathbb{F}}_p$ contains every $\mathbb{F}_{p^e}$, for all $e \geq 1$. Each $\mathbb{F}_p$ has a unique $\overline{\mathbb{F}}_p$.

Let $\mathbb{F}$ be a field. A projective space $\mathrm{PG}(n, \mathbb{F})$ is a partially ordered set of subspaces of a vector space $\mathbf{v}(n+1, \mathbb{F})$. It is a lattice with respect to "join" and "meet". Join is the span of two subspaces. Meet is the intersection of two subspaces. In $\mathrm{PG}(3,\mathbb{F})$, a point is denoted by $P = \mathbf{P}(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$. A line through the points $\mathbf{P}(\beta_0, \beta_1, \beta_2, \beta_3)$ and $\mathbf{P}(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ is denoted by

$$\ell = \mathbf{L}\begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$$

The plane consists of the non-collinear points $\mathbf{P}(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$, $\mathbf{P}(\beta_0, \beta_1, \beta_2, \beta_3)$, and $\mathbf{P}(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ and is denoted by

$$\pi = \mathbf{v}(c_0x_0 + c_1x_1 + c_2x_2 + c_3x_3) = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$$

where c_0, c_1, c_2, and c_3 are elements of the field $\mathbb{F}$.

A conic is a curve of degree 2 in $\mathrm{PG}(2,\mathbb{F})$. It is either an irreducible conic, two distinct lines, or a double line. The space of quadratic polynomials in three variables has dimension 6. To determine conic in the associated projective space, 5 linearly independent conditions are required. A cubic curve is a curve of degree three in $\mathrm{PG}(2,\mathbb{F})$. It is one of the following: an irreducible cubic, a conic, and a line, 3 different lines, or 2 different lines such that one of which is a double or a triple line. To determine a conic in the associated projective space, 9 linearly independent conditions are required. A k-arc in $\mathrm{PG}(2,\mathbb{F})$ is a set of k points where no three are collinear.

Let π be a plane in $\mathrm{PG}(3,\mathbb{F})$, and Q be a point on π. Let ℓ_1 and ℓ_2 be two skew lines in $\mathrm{PG}(3,\mathbb{F}) \setminus \pi$. Then, there is a unique transversal line of ℓ_1 and ℓ_2 through Q.

The automorphism group $\mathrm{P\Gamma L}(n+1,\mathbb{F})$ of $\mathrm{PG}(n,\mathbb{F})$ is the group of bijective mappings that preserve collinearity. The collineation group contains $\mathrm{PGL}(n+1,\mathbb{F})$ as subgroup which is the group of projectivities of $\mathrm{PG}(n,\mathbb{F})$. $\mathrm{PGL}(4,\mathbb{F})$ is transitive on the points, lines, and planes of $\mathrm{PG}(3,\mathbb{F})$. In $\mathrm{PG}(n,\mathbb{F})$, any $(n+2)$-arc can be mapped to any other $(n+2)$-arc. The pointwise stabilizer of a hyperplane π in the $\mathrm{PGL}(4,\mathbb{F})$ is transitive on the set of two skew lines of $\mathrm{PG}(3,\mathbb{F})$ not in π which meet the fixed plane π in two points.

Let f be a homogeneous cubic equation in 4 variables over the field $\mathbb{F}$. A *cubic surface* $\mathcal{F}$ in $\mathrm{PG}(3,\mathbb{F})$ is the zero set of f. For instance,

$$\mathcal{F} = \mathbf{v}(f) = \mathbf{v}(x_0^3 + x_1^3 + x_2^3 + x_3^3 - (x_0 + x_1 + x_2 + x_3)^3)$$

The cubic surface is smooth if the following system of equations has no solution:

$$\begin{cases} f(x_0, x_1, x_2, x_3) = 0 \\ \frac{\partial f(x_0,x_1,x_2,x_3)}{\partial x_0} = 0 \\ \frac{\partial f(x_0,x_1,x_2,x_3)}{\partial x_1} = 0 \\ \frac{\partial f(x_0,x_1,x_2,x_3)}{\partial x_2} = 0 \\ \frac{\partial f(x_0,x_1,x_2,x_3)}{\partial x_3} = 0 \end{cases}$$

To define a cubic surface $\mathcal{F}$ in $\mathrm{PG}(3,\mathbb{F})$, it is sufficient to specify 19 linearly independent points on it. A line in $\mathrm{PG}(3,\mathbb{F})$ either intersects cubic surfaces in three points, or it is the line of $\mathcal{F}$. Therefore, if the 4 points of the line are on the cubic surface, then the line lies on it. A cubic surface intersects a plane in a cubic curve. If the surface is smooth, then that cubic curve is one of the following: an irreducible cubic, a line, an irreducible conic, or 3 different lines. If the cubic surface intersects a plane in 3 different lines, then that plane is called a tritangent plane.

A "double-six" in $\mathrm{PG}(3,\mathbb{F})$ is the set of 12 lines

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{matrix}$$

such that a_i intersects b_j if and only if $i \neq j$, a_i are pairwise skew, and b_i are pairwise skew.

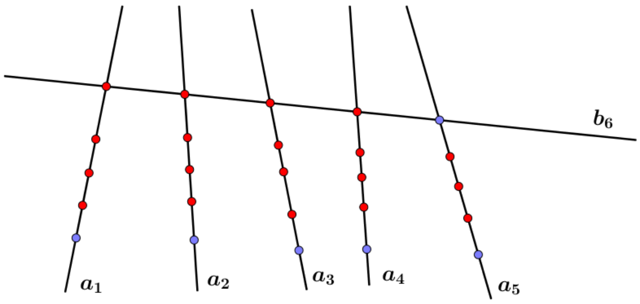

Figure 1. 19 independent points of cubic surfaces with 27 lines

A double-six determines a unique cubic surface with 27 lines. 15 further lines c_{ij} are given by $< a_i, b_j > \cap < a_j, b_i >$ [2]. The red points in Figure 1 represent the 19 independent points that determine the cubic surface with 27 lines.

When three lines of the cubic surface are concurrent at a point, then this point is called an Eckardt point. From line intersection properties, only two cases are possible: either a_i, b_j, and c_{ij} are concurrent where $i \neq j$ or c_{ij}, c_{kl}, and c_{mn} are concurrent where i, j, k, l, m, and n are all different. In the first case, we notate the Eckardt point as E_{ij}, and for the second case, $E_{ij,kl,mn}$.

There is a map between the cubic surface in $\mathrm{PG}(3,\mathbb{F})$ and a plane. This map is called Clebsch map in [9]. Here, we refer to [9,11,17,20] and repeat the description of the map. Let $\mathcal{F}$ be a cubic surface and π be a plane in $\mathrm{PG}(3,\mathbb{F})$. Let ℓ_1 and ℓ_2 be two skew lines of $\mathcal{F}$ not lying on the plane π, and $P = \mathbf{P}(X)$ be a point of $\mathcal{F}$ which is neither on ℓ_1 nor on ℓ_2. There is a unique line ℓ through P which is the transversal to ℓ_1 and ℓ_2. The line ℓ meets π in a unique point $Q = \mathbf{P}(Y)$. Let Q be the image of P. Therefore,

$$\begin{aligned} \Phi : \mathcal{F} &\to \pi \\ P &\mapsto Q \end{aligned}$$

There is a unique line through Q that is transversal to ℓ_1 and ℓ_2. This line intersect $\mathcal{F}$ in 3 points. Two of them are on ℓ_1 and ℓ_2, let the third one to be P. The inverse Φ^{-1} of this map moves Q to P.

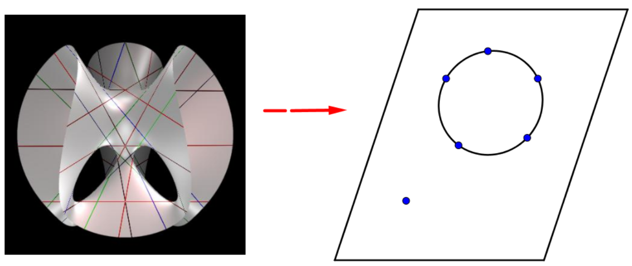

Figure 2. Birational map from cubic surfaces with 27 lines to 6-arc not on a conic

Some properties of this map are as follows: Consider that the surface $\mathcal{F}$ has 27 lines. Each line of the half double-six of $\mathcal{F}$ maps to a single point in π under Φ. These six points form a 6-arc not on a conic in the plane, see Figure 2. Outside these six lines, the map is bijective.

The following lemma is elementary. For a proof of the lemma, see Proposition 7.3 in [21]. For the sake of completeness, we include the reference of the previous paper of the author [11].

Lemma 2.1. [11,21] Let $\mathcal{F}$ be a smooth cubic surface with at least one line. The number of tritangent planes through a line of $\mathcal{F}$ is one of 0, 1, 2, 3, and 5 but never 4.

Lemma 2.2. [1] If two lines of a smooth cubic surface intersect, then they span a tritangent plane.

We introduce the following notation. Two tritangent planes are called disjoint if their line of intersection does not belong to $\mathcal{F}$.

Lemma 2.3. [22] Any two disjoint tritangent planes of $\mathcal{F}$ determine a third.

The three tritangent planes in Lemma 2.3 give rise to 9 lines of $\mathcal{F}$. These 9 lines give rise to three further tritangent planes [11,20].

A trihedral pair consists of two sets of three tritangent planes, which pairwise intersect in 9 lines of $\mathcal{F}$ [22].

Lemma 2.4. Let $\mathcal{F}$ be a smooth cubic surface with at least 9 lines over the field $\mathbb{F}$. There exist at least four tritangent planes of $\mathcal{F}$.

Proof.
As 9 lines of $\mathcal{F}$ cannot be pairwise skew, there exist two lines ℓ_1 and ℓ_2 which intersect. From Lemma 2.2, there is a third line ℓ_3 such that ℓ_1, ℓ_2, and ℓ_3 form a tritangent plane. Any line not contained in a hyperplane intersects the hyperplane at a point. In the case of a tritangent plane and a line of the surface, this point of intersection must be on one of the three lines of the tritangent plane. Hence, each m_i such that $i \in \{1, \dots, 6\}$ intersect one of the ℓ_j such that $j \in \{1, 2, 3\}$. This gives rise to 6 pairs (ℓ_j, m_i) of intersecting lines. By Lemma 2.2, these 6 pairs create at least 3 tritangent planes different from the tritangent plane through ℓ_1, ℓ_2, and ℓ_3. $\square$

Lemma 2.5. Let $\mathcal{F}$ be a smooth cubic surface with at least 9 lines over the field $\mathbb{F}$. Lemma 2.4 guarantees that there are at least 4 tritangent planes. If these 4 tritangent planes of $\mathcal{F}$ intersect pairwise in a line of the surface, then they all intersect in the same line of $\mathcal{F}$.

Proof.
Let π_1, π_2, π_3, and π_4 be the tritangent planes arising from Lemma 2.4. Let π_1 and π_2 intersect in the line a_1 of $\mathcal{F}$. Without loss of generality, we may assume that π_1 is spanned by a_1, b_2, and c_{12}, and π_2 is spanned by a_1, b_3, and c_{13}. The lines b_2 and c_{12} are skew to the lines b_3 and c_{13}. Therefore, if the third tritangent plane π_3 intersects π_1 and π_2 in lines of $\mathcal{F}$, this line must be a_1. The same holds true for π_4. $\square$

Lemma 2.6. Let $\mathcal{F}$ be a smooth cubic surface with at least 9 lines over the field $\mathbb{F}$. Then, there is at least one pair of disjoint tritangent planes.

Proof.
From Lemma 2.4, there exist 4 tritangent planes of $\mathcal{F}$. To show that a disjoint pair of tritangent planes exist, we assume the opposite. From Lemma 2.5, these planes are all through the same line. By Lemma 2.1, there exists a fifth tritangent plane in this pencil. This configuration gives rise to 11 lines of the surface. Because of Lemma 2.2, any further line would create another tritangent plane not passing through a_1, contradicting Lemma 2.5, and it is not possible that a smooth cubic surface to have 11 lines. $\square$

3. Construction of a Smooth Cubic Surface with 9 Lines

This section proves the uniqueness of the line intersection graph of a smooth cubic surface with 9 lines and defines double-three. Besides, is provides a new form of smooth cubic surfaces with at least 9 lines involving 6 parameters.

The result of Cayley (and Salmon) in [1] is strengthened for smooth cubic surfaces with 15 lines in [11]. In this section, we will strength this result for cubic surfaces with 9 lines.

Theorem 3.1. The line intersection graph of a smooth cubic surface with exactly 9 lines over $\mathbb{F}$ is unique.

PROOF.
Let $\mathcal{F}$ be a smooth cubic surface with 9 lines over $\mathbb{F}$. By Lemma 2.6, there is a pair of disjoint tritangent planes. Because of Lemma 2.3, there is a unique third tritangent plane which is also disjoint to others. The 9 lines obtained in this way give rise to 3 more tritangent planes. Hence, there exists a trihedral pair of $\mathcal{F}$. In addition, there are 2 tritangent planes through each line of $\mathcal{F}$. Moreover, each line in the trihedral pair intersects 4 others and is skew to 4. Therefore, it is unique. $\square$

Let ℓ be a line of $\mathcal{F}$ and $\pi(\mu)$ be the plane through ℓ. $\pi(\mu)$ intersects $\mathcal{F}$ in a conic $C(\mu)$ and the line ℓ. Let $\mathcal{Q}(\mu)$ be a quadratic polynomial which represents $C(\mu)$. We define $\Delta(\mu)$ as the discriminant of the quadratic polynomial $\mathcal{Q}(\mu)$ as in the proof of Lemma 2.1.

Theorem 3.2. Let $\mathbb{F}$ be a non-algebraically closed field, and $\mathcal{F}$ be a smooth cubic surface with at least one line. The smooth cubic surface $\mathcal{F}$ has exactly 9 lines over $\mathbb{F}$ if and only if $\Delta(\mu)$ consists of an irreducible polynomial of degree 3 and 2 linear factors over $\mathbb{F}$.

PROOF.
If $\Delta(\mu)$ has two linear factors over the field $\mathbb{F}$, then there exists 2 tritangent planes through ℓ. With a similar argument in the proof of Theorem 11 in [11], we start with a fixed tritangent plane. There is one more tritangent plane through each of the three lines of the fixed plane. This gives 3 tritangent planes different from the one we started with. Each tritangent plane gives rise to 2 new lines. Counting all lines gives $1 + 2 + 3 \cdot 1 \cdot 2 = 9$ lines. If $\mathcal{F}$ has exactly 9 lines, there are 2 tritangent planes through each line of $\mathcal{F}$ from Theorem 3.1. Therefore, there are exactly 2 distinct solutions for $\mu \in \mathbb{F}$ in the $\Delta(\mu)$. Hence, $\Delta(\mu)$ has exactly two distinct linear factors. $\square$

To give the intersection properties of the lines for smooth cubic surfaces with exactly 9 lines, we used the same idea of double-six but for double-three.

Definition 3.3. Let $\mathbb{F}$ be a field. A *double-three* in $\mathrm{PG}(3, \mathbb{F})$ is the set of 6 lines

$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}$$

such that each line is skew to the ones in the same column or row and meets others. One row from the array is called half double-three.

Once a smooth cubic surface include a double-three, then it follows with 3 more lines which arise from the intersection of the following planes:

$$c_{12} = \langle a_1, b_2 \rangle \cap \langle a_2, b_1 \rangle$$

$$c_{13} = \langle a_1, b_3 \rangle \cap \langle a_3, b_1 \rangle$$

and

$$c_{23} = \langle a_2, b_3 \rangle \cap \langle a_3, b_2 \rangle$$

The intersection table of 9 lines can be seen in Table 1. We insert 1 in the table if the lines intersect; otherwise, 0.

Table 1. Pairwise intersection table of the 9 lines

	a_1	a_2	a_3	b_1	b_2	b_3	c_{12}	c_{13}	c_{23}
a_1	–	0	0	0	1	1	1	1	0
a_2	0	–	0	1	0	1	1	0	1
a_3	0	0	–	1	1	0	0	1	1
b_1	0	1	1	–	0	0	1	1	0
b_2	1	0	1	0	–	0	1	0	1
b_3	1	1	0	0	0	–	0	1	1
c_{12}	1	1	0	1	1	0	–	0	0
c_{13}	1	0	1	1	0	1	0	–	0
c_{23}	0	1	1	0	1	1	0	0	–

The following theorem presents a new form of a smooth cubic surfaces with 9 lines.

Theorem 3.4. Let $\mathbb{F}$ be a field with at least 4 elements. Let $a, c \in \mathbb{F} \setminus \{0, 1\}$, $b, d \in \mathbb{F} \setminus \{0, -1\}$, and $f, g \in \mathbb{F} \setminus \{0\}$ such that $b \neq d$ and $f \neq g$. Let $\mathcal{F}_{a,b,c,d,f,g} = \mathbf{v}(\mathfrak{f}_{a,b,c,d,f,g})$ be the variety over $\mathbb{F}$ given by the equation $\mathfrak{f}_{a,b,c,d,f,g}$

$$\begin{aligned} & d_{002}x_0^2x_2 + d_{012}x_0x_1x_2 + d_{013}x_0x_1x_3 + d_{022}x_0x_2^2 + d_{023}x_0x_2x_3 \\ & +d_{112}x_1^2x_2 + d_{113}x_1^2x_3 + d_{122}x_1x_2^2 + d_{123}x_1x_2x_3 + d_{133}x_1x_3^2 = 0 \end{aligned} \tag{1}$$

where

$$\begin{aligned} d_{002} &= bg\kappa_3 & d_{113} &= d_{112} & \kappa_1 &= c + d + 1 \\ d_{012} &= g(a+b)\kappa_2 - f(bf+a)\kappa_1 & d_{023} &= -abg\kappa_3 & \kappa_2 &= dg + c + g \\ d_{013} &= f\kappa_1\kappa_4 & d_{122} &= -cf\kappa_4 & \kappa_3 &= cf + df - dg - c + f - g \\ d_{022} &= ag\kappa_3 & d_{133} &= cdf\kappa_4 & \kappa_4 &= ag - bf + bg - a \\ d_{112} &= -(1+d)f\kappa_4 & d_{123} &= (d-1)cf\kappa_4 & \kappa_5 &= abd + acd + bcd + ab \\ & & & & \kappa_6 &= abc + abd + bcd + ab \end{aligned}$$

$$\kappa_3 \neq 0, \ \kappa_4 \neq 0, \ a\kappa_1 + bc \neq 0, \quad \text{and} \quad a\kappa_2 + bcf \neq 0$$

Let $\mathbf{g}_{a,b,c,d,f,g}$ be the polynomial

$$\mathbf{g}_{a,b,c,d,f,g} = A_3\mu^3 + A_2\mu^2 + A_1\mu + A_0 \tag{2}$$

in μ such that

$$A_0 = -bcdf(ac + ad + cd + a)(ag - bf + bg - a)$$

$$A_1 = g^2 A_{12} + g(c^2 A_{112} + cA_{111} + A_{110}) + A_{10}$$

$$A_2 = a^2(fA_{221} + A_{220}) - abc(f^2 A_{212} + fA_{211} + A_{210}) + A_{20}$$

$$A_3 = (f - g)(ac + ad + bc + a)(adg + bcf + ac + ag)$$

and

$$A_{12} = -a(d + 1)b(abd + acd + bcd + ab) \qquad A_{220} = g(dg + c + g)(bc + 2bd + cd + 2b)$$

$$A_{112} = -(a + b)(bd^2 f + abd - abf - adf - 2bdf) \qquad A_{212} = (c + d + 1)(b + d - 1)$$

$$A_{111} = a(d + 1)(ab^2 f - ab^2 + abf + adf + b^2 f + bdf) \qquad A_{211} = -2cdg + 2cd + 2cg + dg - c + g$$

$$A_{110} = a^2 b^2 f(d + 1)^2 \qquad A_{210} = -g(dg + c + g)(b + d)$$

$$A_{221} = -(c + d + 1)(2bdg + bc + 2bg + cd + cg - c) \qquad A_{20} = -b^2 c^2 f(f - g)(2d - 1)$$

$$A_{10} = cf(bf + a)(abcd + abd^2 + bcd^2 - abc - acd - ad^2 - 2bcd - ab - ad)$$

Assume that the surface $\mathcal{F}_{a,b,c,d,f,g}$ is smooth over $\mathbb{F}$. The surface $\mathcal{F}_{a,b,c,d,f,g}$ has at least 9 lines, six of which form a double-three. The conditions on the exact number of lines of $\mathcal{F}_{a,b,c,d,f,g}$ depends on the polynomial $\mathbf{g}_{a,b,c,d,f,g}$ of degree three.

i. If the polynomial $\mathbf{g}_{a,b,c,d,f,g}$ is irreducible over the field $\mathbb{F}$, then the surface $\mathcal{F}_{a,b,c,d,f,g}$ has exactly 9 lines.

ii. If $\mathbf{g}_{a,b,c,d,f,g}$ is reducible into one irreducible quadratic polynomial and one linear over the field $\mathbb{F}$, then the surface $\mathcal{F}_{a,b,c,d,f,g}$ has exactly 15 lines.

iii. If $\mathbf{g}_{a,b,c,d,f,g}$ splits completely to 3 linear factors over the field $\mathbb{F}$, then the surface $\mathcal{F}_{a,b,c,d,f,g}$ has exactly 27 lines.

In Table 2, the parametrization of the 9 lines of $\mathcal{F}_{a,b,c,d,f,g}$ can be observed.

Table 2. Lines of $\mathcal{F}_{a,b,c,d,f,g}$

$a_1 = \mathbf{L}\begin{bmatrix} a(b+1) & 0 & -b & b \\ a+b & b & 0 & 0 \end{bmatrix}$	$a_2 = \begin{cases} \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & d_{133} & 0 & -d_{112} \end{bmatrix}, & \kappa_1 = 0 \\ \mathbf{L}\begin{bmatrix} -cd & 0 & 0 & \kappa_1 \\ 1+d & \kappa_1 & 0 & 0 \end{bmatrix}, & \text{otherwise} \end{cases}$	$a_3 = \mathbf{L}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
$b_1 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$b_2 = \mathbf{L}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$	$b_3 = \mathbf{L}\begin{bmatrix} 1 & 1 & 1 & 0 \\ c(b-d)a & 0 & \kappa_6 & a\kappa_1 + bc \end{bmatrix}$
$c_{12} = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$c_{13} = \mathbf{L}\begin{bmatrix} a & 0 & 0 & 1 \\ 0 & 0 & b & 1 \end{bmatrix}$	$c_{23} = \mathbf{L}\begin{bmatrix} 0 & 0 & d & 1 \\ 0 & c & -1 & 1 \end{bmatrix}$

PROOF.
The group $\mathrm{PGL}(4, \mathbb{F})$ is transitive on the planes. Therefore, we can start from any hyperplane. Hence, we may pick $\pi = \mathbf{v}(x_3)$. Consider that we want to construct a cubic surface $\mathcal{F}$ with 9 lines, including

the double-three. Considering the sub-configuration of Schlafli configuration and the labeling for 9 lines, we may choose the labels that a_1, a_2, a_3, b_1, b_2, b_3, c_{12}, c_{13}, and c_{23}. To determine this surface $\mathcal{F}$, we have to specify 19 independent conditions. Nine of them should be on the plane π since the cubic surface intersects π in a cubic curve. We can assume that the coordinates of 4 of those 9 points are $P_1 = \mathbf{P}(1,0,0,0)$, $P_2 = \mathbf{P}(1,0,0,0)$, $P_3 = \mathbf{P}(1,0,0,0)$, and $P_4 = \mathbf{P}(1,0,0,0)$ since the projective group of the plane is transitive on the quadrangles. Since $\mathrm{PGL}(4,\mathbb{F})$ is transitive on the lines, we can pick the first line of the surface $\mathcal{F}$ without any constraints. Let assume that the first line of $\mathcal{F}$ is

$$c_{12} = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Since c_{12} lies on the surface $\mathcal{F}$, there are 2 more independent points on the intersection $c_{12} \cap \mathcal{F}$. Assume that $P_7 = \mathbf{P}(1,1,0,0)$ and $P_8 = \mathbf{P}(-1,1,0,0)$ are the points of $\mathcal{F}$ as well as c_{12} is transitive. It is known that the pointwise stabilizer of the hyperplane π in the $\mathrm{PGL}(4,\mathbb{F})$ is transitive on the set of two skew lines of $\mathrm{PG}(3,\mathbb{F})$ not in π which meet the fixed plane π in two points P_1 and P_2. Therefore, we are free to choose two skew lines not on the plane π through P_1 and P_2. Let

$$b_1 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b_2 = \mathbf{L}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

are the two skew lines. b_1 are through P_1 and $P_5 = \mathbf{P}(0,0,0,1)$ and b_2 are through P_2 and $P_6 = \mathbf{P}(0,0,-1,1)$. It is known that there is a unique line through P_2 and transversal to b_1 and b_2. We call this line is a line of $\mathcal{F}$. Because of the configuration of 9 lines, we can label that line as a_3 and it is the line through P_5 and P_6,

$$a_3 = \mathbf{L}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

From line intersection properties, we know that there exists a line c_{23} which meets b_2 and a_3 and skew to c_{12}, i.e., not in π. When a_3, b_2, and c_{23} are not concurrent which means the Eckardt point E_{32} does not exist, we may set the line c_{23} through $P_9 = \mathbf{P}(0,c,-1,1)$ and $P_{10} = \mathbf{P}(0,0,d,1)$. Since P_2, P_6, and P_9 are three distinct points from our assumption, we have $0 \neq c \neq 1$.

From line intersection properties, we know that there exists a line c_{13} which meets b_1 and a_3 and skew to c_{12}, i.e., not in π. When a_3, b_1, and c_{13} are not concurrent which means the Eckardt point E_{31} does not exist, we may set the line c_{13} through $P_{11} = \mathbf{P}(a,0,0,1)$ and $p' = \mathbf{P}(0,0,b,1)$. Since P_1, P_5, and P_{11} are three distinct points from our assumption, we have $0 \neq a \neq 1$. Therefore, two lines of $\mathcal{F}$ are

$$c_{13} = \mathbf{L}\begin{bmatrix} a & 0 & 0 & 1 \\ 0 & 0 & b & 1 \end{bmatrix} \quad \text{and} \quad c_{23} = \mathbf{L}\begin{bmatrix} 0 & c & -1 & 1 \\ 0 & 0 & d & 1 \end{bmatrix}$$

Since P_3, P_5, P_6, P_{10}, and p' are five distinct points on the line a_3, we have $b, d \notin \{0,-1\}$ and $b \neq d$. Since there are already 4 independent points on a_3, we do not count the point p'. The plane π intersects the line c_{13} at the point $P_{13} = \mathbf{P}(-a,0,b,0)$ and intersects the line c_{23} at the point $P_{12} = \mathbf{P}(0,-c,d+1,0)$. From line intersection properties, we know that there exists a line b_3 which meets c_{13} and c_{23} and skew to a_3, b_1, b_2, and c_{12}. The line b_3 cannot intersect the line through P_1 and P_3 and the line through P_2 and P_3 since those lines already have 3 points of $\mathcal{F}$ each. The new line b_3 is either P_4 or $P_{19} = \mathbf{P}(f,g,1,0)$. For this new form, we consider the b_3 is through P_4. We know that the line c_{13} and c_{23} are not in π, and they are skew. Hence, the transversal line b_3 to c_{13} and c_{23} through P_4 is uniquely determined. It intersects c_{13} at the point $P_{14} = \mathbf{P}(ac(b-d),0,\kappa_6,a\kappa_1+bc)$ and intersects c_{23} at the point $P_{15} = \mathbf{P}(0,-ac(b-d),\kappa_5,a\kappa_1+bc)$ where $\kappa_1 = c+d+1$, $\kappa_5 = abd+acd+bcd+ab$, and $\kappa_6 = abc + abd + bcd + ab$. Thus,

$$b_3 = \mathbf{L}\begin{bmatrix} 1 & 1 & 1 & 0 \\ c(b-d)a & 0 & \kappa_6 & a\kappa_1 + bc \end{bmatrix}$$

The line c_{13} already has 4 independent points of $\mathcal{F}$ which are p', P_{11}, P_{13}, and P_{14}. Hence, it is a line of $\mathcal{F}$. The line c_{23} already has 4 independent points of $\mathcal{F}$ which are P_{12}, P_{10}, P_9, and P_{15}. Thus, it is a line of $\mathcal{F}$. The lines b_1, b_2, and b_3 have 3 points of $\mathcal{F}$ each. Therefore, we force the points $P_{16} = \mathbf{P}(1,0,0,1)$, $P_{17} = \mathbf{P}(0,1,-1,1)$, and $P_{18} = a \cdot P_{15} + P_{14}$ to be on the surface $\mathcal{F}$. The following 18 points are chosen to force the lines

$$b_1, b_2, b_3, a_3, c_{12}, c_{13}, \quad \text{and} \quad c_{23}$$

to be on the cubic surface. More conditions arise from the fact that the cubic surface intersects the plane $\mathbf{v}(x_3)$ in a cubic curve which consists of the line c_{12} and an irreducible conic. Besides, we have considered the 8 points on π

$$P_1,\ P_2,\ P_3,\ P_4,\ P_7,\ P_8,\ P_{12}, \quad \text{and} \quad P_{13}$$

We need to force one further point of the surface to lie on this plane. We may pick the point $P_{19} = \mathbf{P}(f, g, 1, 0)$. The conic through P_3, P_4, P_{12}, P_{13}, and P_{19} is irreducible. This gives the restriction on the parameters a, b, c, d, f, and g such that $a, c, f, g \neq 0$, $f \neq g$, $\kappa_3 \neq 0$, $\kappa_4 \neq 0$, $a\kappa_1 + bc \neq 0$, and $a\kappa_2 + bcf \neq 0$.

The points $P_1, \ldots, P_{19}$ define the cubic surface uniquely since they are linearly independent. Using Maple, we compute the equation $\mathfrak{f}_{a,b,c,d,f,g}$ as in Equation 1 which is the unique equation of a cubic surface $\mathcal{F}_{a,b,c,d,f,g}$ defined by these 19 points. This surface also involves a double-three. By Theorem 3.1, we know that there is only one way to complete the configuration of the lines. By Definition 3.3, further lines will be defined as a_1 and a_2. These further lines a_1 and a_2 can be computed using Maple. All these 9 lines can be seen in Table 2.

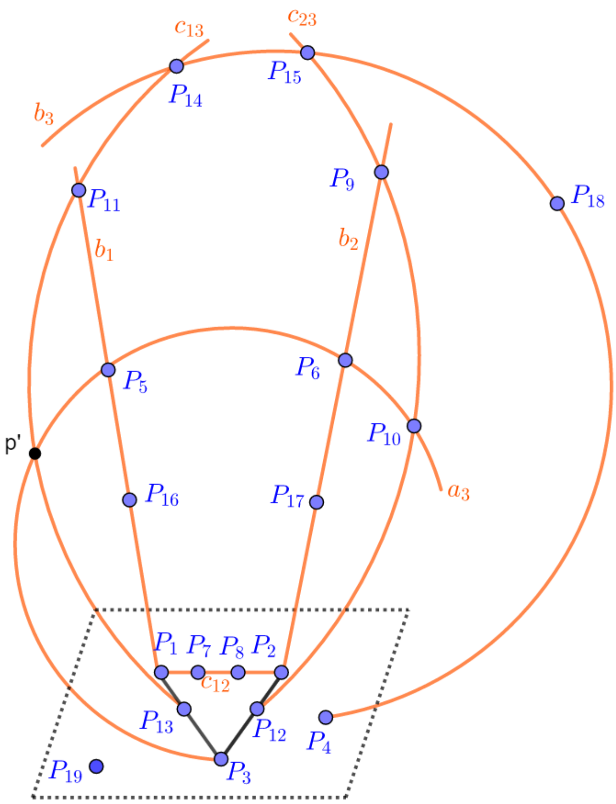

Figure 3. Configuration of 19 points

We check whether the cubic surface $\mathcal{F}_{a,b,c,d,f,g}$ has any further lines. The only possible way would be the surface having 15 or 27 lines. This depends on the discriminant condition. By Theorem 3.2 and Theorem 11 in [11], we know that it depends on the factors of the discirimant. Therefore, we need to compute it. As in the proof of Theorem 3.2, we start with a fixed tritangent plane $\pi = \mathbf{v}(x_3)$. Consider the line

$$c_{12} = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and the planes

$$\mathbf{v}(x_2 - \mu x_3)$$

through this line c_{12}. Each of these planes intersects $\mathcal{F}_{a,b,c,d,f,g}$ in c_{12} and in a conic $C(\mu)$. Substituting $x_2 = \mu x_3$ into the equation of Equation 1, we find the conic equation $C(\mu)$ in x_0, x_1, and x_3. Hence, we calculate the discriminant $\Delta(\mu)$ of $C(\mu)$. It has a factor of the polynomial $\mathbf{g}_{a,b,c,d,f,g}$ of degree 3 as in the Equation 2. Similarly, considering the planes through a_1 and b_2, two further polynomials $\mathbf{f}_{a,b,c,d,f,g}$ and $\mathbf{h}_{a,b,c,d,f,g}$ of degree three arise. These polynomials are not stated due to space restrictions. For a given field $\mathbb{F}$, if the polynomial $\mathbf{g}_{a,b,c,d,f,g}$ is irreducible, then $\mathbf{f}_{a,b,c,d,f,g}$ and $\mathbf{h}_{a,b,c,d,f,g}$ are irreducible, and vice versa.

There are 2 tritangent planes of $\mathcal{F}_{a,b,c,d,f,g}$ through each created 9 lines. Hence, there are either one or 3 more tritangent planes through each line. Then, these further lines would arise from the new tritangent planes.

If $\mathbf{g}_{a,b,c,d,f,g}$ is reducible into one linear factor and one irreducible quadratic, then the surface $\mathcal{F}_{a,b,c,d,f,g}$ has 15 lines over $\mathbb{F}$ from Part 2 of Theorem 11 in [11]. The further 6 lines arise as following: Each of c_{12}, a_1, and b_2 lies on one further tritangent planes, and each of which gives two lines to the surface. This gives $3 \cdot 1 \cdot 2 = 6$ further lines of $\mathcal{F}$.

If $\mathbf{g}_{a,b,c,d,f,g}$ is reducible into three linear factors, then the surface $\mathcal{F}_{a,b,c,d,f,g}$ has 27 lines over $\mathbb{F}$ from Part 1 of Theorem 11 in [11]. The further 18 lines arise as following: Each of c_{12}, a_1, and b_2 lies on three further tritangent planes, each of which gives two lines to the surface. This gives $3 \cdot 3 \cdot 2 = 18$ further lines of $\mathcal{F}$.

If the polynomial $\mathbf{g}_{a,b,c,d,f,g}$ is irreducible over $\mathbb{F}$, then the surface $\mathcal{F}_{a,b,c,d,f,g}$ has exactly 9 lines over this field by Theorem 3.2. $\square$

Remark 3.5. Figure 3 summarises the content of Theorem 3.4 by giving the configuration of 19 points on the cubic surface and forcing the surface including the double-three.

Remark 3.6. The cubic surface $\mathcal{F}_{a,b,c,d,f,g}$ cannot have 2 specific Eckardt points. Therefore, we cannot create all possible smooth cubic surfaces with 9 lines using this form.

Remark 3.7. The line b_3 of $\mathcal{F}_{a,b,c,d,f,g}$ could be through P_{19} instead P_4. Considering this way would give another form of such surfaces.

4. Illustrative Examples

This section exemplifies the aforesaid form over finite fields and fields of characteristic zero.

Example 4.1. The surface $\mathcal{F}_{-1,1,-1,2,-1,1}$ given by the equation

$$-2x_0^2x_2 - 2x_0x_1x_2 - 2x_0x_1x_3 + 2x_0x_2^2 - 2x_0x_2x_3 + 3x_1^2x_2 + 3x_1^2x_3 - x_1x_2^2 + x_1x_2x_3 + 2x_1x_3^2 = 0$$

is smooth over $\mathbb{C}$. The polynomial

$$\mathbf{g}_{-1,1,-1,2,-1,1} = -6\mu^3 + 10\mu^2 - 16\mu + 16$$

is irreducible over $\mathbb{Q}$. Hence, the surface $\mathcal{F}_{-1,1,-1,2,-1,1}$ has exactly 9 lines over $\mathbb{Q}$. The polynomial $\mathbf{g}_{-1,1,-1,2,-1,1}$ has one real and two complex roots. Therefore, by Theorem 3.4, it has exactly 15 lines over $\mathbb{R}$ and 27 over $\mathbb{C}$.

Example 4.2. Over $\mathbb{F}_5$, the cubic surface $\mathcal{F}_{2,1,4,2,4,1}$ given by the equation

$$3x_0^2x_2 + 3x_1^2x_2 + 3x_1^2x_3 + x_0x_2^2 + 4x_1x_2^2 + 2x_1x_3^2 + 4x_0x_1x_2 + 3x_0x_1x_3 + 4x_0x_2x_3 + x_1x_2x_3 = 0$$

is smooth. Since the polynomial

$$\mathbf{g}_{2,1,4,2,4,1} = 2\mu^3 + 3\mu^2 + 4\mu + 3$$

is reducible into $(\mu + 1)$ and $(2\mu^2 + \mu + 3)$ over $\mathbb{F}_5$, the surface $\mathcal{F}_{2,1,4,2,4,1}$ has exactly 15 lines over this field by Theorem 3.4.

Let $\tau \in \mathbb{F}_{25}$ satisfy the equation $\tau^2 + \tau + 2 = 0$. The polynomial

$$\begin{aligned}\mathbf{g}_{2,1,4,2,4,1} &= 2\mu^3 + 3\mu^2 + 4\mu + 3 \\ &= (\mu + 1)(2\mu^2 + \mu + 3) \\ &= (\mu + 1)(\mu + 4\tau + 1)(\mu + \tau + 2)\end{aligned}$$

is reducible into three linear factors over $\mathbb{F}_{25}$. Therefore, by Theorem 3.4, the surface $\mathcal{F}_{2,1,4,2,4,1}$ has exactly 27 lines over this field.

Example 4.3. Over $\mathbb{F}_5$, the cubic surface $\mathcal{F}_{4,1,4,2,4,1}$ given by the equation

$$3x_0^2x_2 + 3x_1^2x_2 + 3x_1^2x_3 + 2x_0x_2^2 + 4x_1x_2^2 + 2x_1x_3^2 + 3x_0x_1x_2 + 3x_0x_1x_3 + 3x_0x_2x_3 + x_1x_2x_3 = 0$$

is smooth. Since the polynomial

$$\mathbf{g}_{4,1,4,2,4,1} = 3\mu^3 + 3\mu + 2$$

is irreducible over $\mathbb{F}_5$, the surface $\mathcal{F}_{4,1,4,2,4,1}$ has exactly 9 lines over this field by Theorem 3.4. Over $\mathbb{F}_{25}$, this polynomial is also irreducible. Therefore, the cubic surface $\mathcal{F}_{4,1,4,2,4,1}$ has exactly 9 lines over this field.

Let $\psi \in \mathbb{F}_{125}$ satisfy the equation $\psi^3 + \psi^2 + 2 = 0$. The polynomial

$$\begin{aligned}\mathbf{g}_{4,1,4,2,4,1} &= 3\mu^3 + 3\mu + 2 \\ &= (\mu + 4\psi^2 + 3\psi + 3)(\mu + 4\psi^2 + 2\psi + 1)(\mu + 2\psi^2 + 1)\end{aligned}$$

is reducible into three linear factors over $\mathbb{F}_{125}$. Therefore, the surface $\mathcal{F}_{4,1,4,2,4,1}$ has exactly 27 lines over this field by Theorem 3.4.

5. Rational Parametrization of the New Form

This section provides a parametrization of the rational points at lines of the cubic surface $\mathcal{F}_{a,b,c,d,f,g}$ given in Section 3. The proof of the following theorem is based on the birational map between the cubic surface and a plane. In Subsection 4.2 of [11], rational parametrization of a smooth cubic surface with 15 lines is given explicitly. The proof here and the remarks are similar to the proof of Theorem 8 and its following remarks in [11] but for the cubic surface with at least 9 lines $\mathcal{F}_{a,b,c,d,f,g}$.

Theorem 5.1. Let $\mathbb{F}$ be a field with at least 4 elements. Let $\mathcal{F}_{a,b,c,d,f,g}$ be the variety given in Theorem 3.4. Assume that $\mathcal{F}_{a,b,c,d,f,g}$ is smooth. Let $P = \mathbf{P}(x_0, x_1, x_2, x_3)$ be a point on $\mathcal{F}_{a,b,c,d,f,g}$ and $Q = \mathbf{P}(y_0, y_1, y_2)$ be a point in a plane so that P and Q are the images of each other under a birational map

$$\begin{aligned} \Phi_{a,b,c,d,f,g} : \quad \mathcal{F}_{a,b,c,d,f,g} &\to \mathrm{PG}(2,\mathbb{F}) \\ \mathbf{P}(x_0,x_1,x_2,x_3) &\mapsto \mathbf{P}(y_0,y_1,y_2) \end{aligned}$$

Then, (x_0, x_1, x_2, x_3) can be expressed as

$$\begin{cases} x_0 = -g(a+b)\kappa_3 y_0^2 y_1 + ga(b+1)\kappa_3 y_0^2 y_2 - (1+d)f\kappa_4 y_0 y_1^2 - fc(1+d)\kappa_4 y_0 y_1 y_2 \\ x_1 = abg\kappa_3 y_0 y_1 y_2 - cdf\kappa_4 y_1^2 y_2 - bg\kappa_3 y_0^2 y_1 - f\kappa_1\kappa_4 y_0 y_1^2 \\ x_2 = abg\kappa_3 y_0 y_2^2 - cdf\kappa_4 y_1 y_2^2 - bg\kappa_3 y_0^2 y_2 - f\kappa_1\kappa_4 y_0 y_1 y_2 \\ x_3 = gb\kappa_3 y_0^2 y_2 + ag\kappa_3 y_2^2 y_0 - cf\kappa_4 y_2^2 y_1 + d_{012} y_0 y_1 y_2 - (1+d)f\kappa_4 y_1^2 y_2 \end{cases} \tag{3}$$

and (y_0, y_1, y_2) can be expressed as

$$\begin{cases} y_0 = x_0 x_2 \\ y_1 = x_1 x_2 + x_1 x_3 \\ y_2 = x_2^2 + x_2 x_3 \end{cases} \tag{4}$$

up to a nonzero scalar multiple.

Proof.
No generality is lost by picking the plane as $\mathbf{v}(x_3)$, and the two skew lines b_1 and b_2 of $\mathcal{F}_{a,b,c,d,f,g}$

$$b_1 = \mathbf{L}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b_2 = \mathbf{L}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Let $P = \mathbf{P}(x_0, x_1, x_2, x_3)$ be a point on $\mathcal{F}_{a,b,c,d,f,g}$ but neither on b_1 nor on b_2. To find the image Q of P under $\Phi_{a,b,c,d,f,g}$, first we compute the transversal line ℓ of b_1 and b_2 through P. Then, we find the unique point Q where ℓ meets $\mathbf{v}(x_3)$ as follows:

$$\ell = \mathbf{L}\begin{bmatrix} x_0 & 0 & 0 & x_2 + x_3 \\ x_0 x_2 & x_1 x_2 + x_1 x_3 & x_2^2 + x_2 x_3 & 0 \end{bmatrix}$$

and

$$\Phi_{a,b,c,d,f,g}(\mathbf{P}(x_0, x_1, x_2, x_3)) = \mathbf{P}(y_0, y_1, y_2) = \mathbf{P}(x_0 x_2, x_1 x_2 + x_1 x_3, x_2^2 + x_2 x_3)$$

whose coordinates are given in System 4.

Conversely, let $Q = \mathbf{P}(y_0, y_1, y_2, 0)$ be a point in the plane $\mathbf{v}(x_3)$. To find the image Q of P under $\Phi^{-1}_{a,b,c,d,f,g}$, first we compute the transversal line ℓ' of b_1 and b_2 through Q as follows:

$$\ell' = \mathbf{L}\begin{bmatrix} y_0 & 0 & 0 & y_2 \\ y_0 & y_1 & y_2 & 0 \end{bmatrix}$$

Then, we find the three intersection points where ℓ' meets $\mathcal{F}_{a,b,c,d,f,g}$ by substituting the point

$$\mathbf{P}(y_0, 0, 0, y_2) + t \cdot \mathbf{P}(y_0, y_1, y_2, 0)$$

of ℓ' into the equation of $\mathcal{F}_{a,b,c,d,f,g}$. This gives a polynomial of degree three in t, for some $t \in \mathbb{F}$. Solving this polynomial in t gives 3 solutions as follows:

$$t_1 = 0$$

$$t_2 = -1$$

and

$$t_3 = \frac{abg\kappa_3 y_0 y_2 - cdf\kappa_4 y_1 y_2 - bg\kappa_3 y_0^2 - f\kappa_1\kappa_4 y_0 y_1}{gb\kappa_3 y_0^2 + ag\kappa_3 y_2 y_0 - cf\kappa_4 y_2 y_1 + d_{012} y_0 y_1 - (1+d)f\kappa_4 y_1^2}$$

The point $\mathbf{P}(y_0, 0, 0, y_2)$ arising from t_1 is the point where ℓ' meets the line b_1. The point $\mathbf{P}(0, -y_1, -y_2, y_2)$ arising from t_2 is the point where ℓ' meets the line b_2. Let $P = \mathbf{P}(x_0, x_1, x_2, x_3)$ be the third point of intersection of ℓ' with $\mathcal{F}_{a,b,c,d,f,g}$. Hence,

$$x_0 = y_0 + t_3 \cdot y_0$$

$$x_1 = t_3 \cdot y_1$$

$$x_2 = t_3 \cdot y_2$$

and

$$x_3 = y_2$$

as in System 3. Then,

$$\Phi^{-1}_{a,b,c,d,f,g}(\mathbf{P}(y_0, y_1, y_2)) = \mathbf{P}(x_0, x_1 x_2, x_3)$$

□

Remark 5.2. Let $\mathcal{F}_{a,b,c,d,f,g}$ be the smooth cubic surface over the field $\mathbb{F}$ as described in Theorem 3.4, and $\Phi_{a,b,c,d,f,g}$ be the birational map from $\mathcal{F}_{a,b,c,d,f,g}$ to the plane embedded in $\mathrm{PG}(3, \mathbb{F})$ as the hyperplane $\mathbf{v}(x_3)$, described in Theorem 5.1. Consider the lines a_1, a_2, and a_3 of $\mathcal{F}_{a,b,c,d,f,g}$. The map $\Phi_{a,b,c,d,f,g}$ sends the line a_1 to the point $\mathbf{P}(1, 0, 0)$, the line a_2 to the point $\mathbf{P}(0, 1, 0)$, and the line a_3 to the point $\mathbf{P}(0, 0, 1)$.

Remark 5.3. The exceptional locus of $\Phi_{a,b,c,d,f,g}$ on the surface consists of 3 lines of $\mathcal{F}_{a,b,c,d,f,g}$ which are b_1, b_2, and c_{12}. The exceptional locus on the plane can be found explicitly by applying $\Phi^{-1}_{a,b,c,d,f,g}$ to $\mathbf{P}(x_0, x_1, x_2, x_3)$ and then by applying $\Phi_{a,b,c,d,f,g}$ to $\mathbf{P}(y_0, y_1, y_2)$. It consists of one line and two conics in the plane, which are the line L' through $\mathbf{P}(1, 0, 0)$ and $\mathbf{P}(0, 1, 0)$ and two conics

$$D_1 = abg\kappa_3 y_0 y_2 - cdf\kappa_4 y_1 y_2 - bg\kappa_3 y_0^2 - f\kappa_1\kappa_4 y_0 y_1$$

and

$$D_2 = -f(1+d)\kappa_4 y_1^2 + ag(b+1)\kappa_3 y_0 y_2 - cf(1+d)\kappa_4 y_1 y_2 + (-f\kappa_1\kappa_4 + d_{012}) y_0 y_1$$

The points of the lines b_1 and b_2 are mapped to the points of the conics D_1 and D_2 and the points of c_{12} are mapped to the points of L' under $\Phi_{a,b,c,d,f,g}$.

Theorem 5.4. The cubic surface $\mathcal{F}_{a,b,c,d,f,g} = \mathbf{v}(\mathfrak{f}_{a,b,c,d,f,g})$ as described in Theorem 3.4 is non-singular over $\mathbb{F}$ if and only if the certain four sextic curves in $(2, \mathbb{F})$ never intersect at a point of $\mathcal{F}_{a,b,c,d,f,g}$.

Proof.
The idea of the proof is similar to the proof of Theorem 9 in [11] but for $\mathcal{F}_{a,b,c,d,f,g}$. The partial derivatives of $\mathfrak{f}_{a,b,c,d,f,g}$ as follows:

$$\frac{\partial \mathfrak{f}_{a,b,c,d,f,g}}{\partial x_0} = 2d_{002}x_0x_2 + d_{012}x_1x_2 + d_{013}x_1x_3 + d_{022}x_2^2 + d_{023}x_2x_3$$

$$\frac{\partial \mathfrak{f}_{a,b,c,d,f,g}}{\partial x_1} = d_{012}x_0x_2 + d_{013}x_0x_3 + 2d_{112}x_1x_2 + 2d_{113}x_1x_3 + d_{122}x_2^2 + d_{123}x_2x_3 + d_{133}x_3^2$$

$$\frac{\partial \mathfrak{f}_{a,b,c,d,f,g}}{\partial x_2} = d_{002}x_0^2 + d_{012}x_0x_1 + 2d_{022}x_0x_2 + d_{023}x_0x_3 + d_{112}x_1^2 + 2d_{122}x_1x_2 + d_{123}x_1x_3$$

and

$$\frac{\partial \mathfrak{f}_{a,b,c,d,f,g}}{\partial x_3} = d_{013}x_0x_1 + d_{023}x_0x_2 + d_{113}x_1^2 + d_{123}x_1x_2 + 2d_{133}x_1x_3$$

Let $P = \mathbf{P}(x_0, x_1, x_2, x_3)$ be a point of $\mathcal{F}_{a,b,c,d,f,g}$. By substituting the coordinates of the point $\mathbf{P}(y_0, y_1, y_2)$ as given in Theorem 5.1 into the four partial derivatives of $\mathfrak{f}_{a,b,c,d,f,g}$, we get four polynomials S_1, S_2, S_3, and S_4 of degree 6 in three variables y_0, y_1, and y_2. We did not present these polynomials explicitly here so that they do not take up much space. The zeros of these polynomials form curves $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$, and $\mathcal{S}_4$ of degree six in $\mathrm{PG}(2, \mathbb{F})$. The point $P = \mathbf{P}(x_0, x_1, x_2, x_3)$ is a singular point if and only if it appears on all the partial derivatives of $\mathfrak{f}_{a,b,c,d,f,g}$ if and only if the curves $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$, and $\mathcal{S}_4$ intersect at P. Therefore, the surface is non singular if and only if the curves $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$, and $\mathcal{S}_4$ never intersect in a point of $\mathcal{F}_{a,b,c,d,f,g}$. $\square$

6. Conclusion

In this paper, the theorems are proved with a computer-free proof. However, at the beginning of the work, computers are used to produce data. Then, the data is analyzed to make a conjecture. To follow this experimental approach, one needs to have good data. To have good data, one must have the right computer software for computations. We use Orbiter [16], which is an open source. The idea behind that paper has been found this way. This paper shows the uniqueness of the line intersection graph of smooth cubic surfaces with 9 lines. The properties of line intersections are defined. A new form of smooth cubic surfaces $\mathcal{F}_{a,b,c,d,f,g}$ with at least 9 lines is created. The conditions specify when the surface has 9, when 15, and when 27 lines. This form is exemplified over several fields. Besides, the rational parametrization of points and lines of $\mathcal{F}_{a,b,c,d,f,g}$.

The Remarks 3.6 and 3.7 will be considered as a continuation of this work. All possible smooth cubic surfaces with 9 lines can be created by considering Remark 3.6 and 3.7. Studying the sub-configuration of Eckardt points of smooth cubic surfaces with 9 lines and investigating which cases are not covered by the form in this paper are worth reaching that aim. Moreover, the solution can be found for the other cases to generalize. Regarding Remark 3.7, the points P_{14}, P_{15}, and P_{18} need to be reset. Maple can be used to construct another form, covering all possible cases. Once these two remarks are considered, the classification problem of smooth cubic surfaces with 9 lines over the small finite fields can be considered. Therefore, the result herein can be verified by the enumerative formula of Das [15] using the Orbit Stabilizer Theorem. Besides, one may generalize certain cases and find a family over $\mathbb{Q}$.

Even though it was a popular topic in the 19th century, it still maintains its current and interesting nature. For recent related studies, see [23, 24].

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

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