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Approximate Solutions of Nonlinear Boundary Value Problems by Collocation Methods Compared to Newer Methods

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Abstract

A large variety of new methods are being developed for fast and efficient solutions of nonlinear boundary value problems. Some of these methods are, Adomian decomposition (ADM), differential transform (DTM), least squares vector machines (LSSVMM), and multiple variational iteration (MVIM). A natural question arises as to how efficient and simple to use these newer methods are compared to classical methods. One of the simplest and widely applicable classical methods is the collocation method. The overall performance of collocation method and the newer methods are compared on a number of problems, which were previously used to benchmark the newer methods. It is concluded that, at least for the problems considered, the collocation method performs as successfully as the newer methods.

Keywords: Collocation method, nonlinear differential equations, boundary value problems, approximate solution.

1. INTRODUCTION

Nonlinear ordinary differential equations are encountered in many problems, including modelling of the spread of diseases, modelling some economical and virtually all technical phenomena.

There are quite a few methods for solving linear differential equations, but very few and usually severely restricted ones for nonlinear problems. The main reason for this is the non-applicability of the method of superposition in nonlinear problems, whereas, linear problems are usually solved by superposing simple solutions, called the superposition principle. The case of linear equations with constant coefficients is especially well understood and developed.

Two of the most common methods for solving equations with variable coefficients are, employing a transformation to change the problem into one with constant coefficients,

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and the other is the method of series expansions. In some problems, both methods are insufficient and in that case, finding the solution at a discrete set of points by numerical approximations is preferred. Methods specific to equations with variable coefficients sometimes are tried for nonlinear equations. However, difficulty of determining a suitable transformation, and the complicated structure of the recurrence relations needed in series solution method are the main obstacles in applying these methods to nonlinear problems. Clearly, efficient numerical methods suitable for nonlinear equations should be developed and used.

The Adomian decomposition method (ADM) developed for nonlinear equations, is a method used for both linear and nonlinear differential equations and boundary value problems in various disciplines [1]. Wazwaz [2] noted that this method is simpler and more effective than the Taylor series method. Sanchez [3] applied the ADM to nonlinear initial value problems.

Another method of interest for solving differential equations of higher order encountered in engineering and mathematics is the Differential transform method (DTM). This method was used by Kaya [4] for investigating the vibrations of a rotating Timoshenko beam. Gökdoğan et al. [5] solved some specific first and second order equations of interest by using the same method.

Comparing the approximate solutions of nonlinear differential equations obtained by relatively new methods, to solutions obtained by traditional or conventional methods is an important aspect in the development of progressively stronger and reliable toolbox of methods for attacking various physical or other phenomena modelled by differential equations. Comparison of ADM and Variational iteration method (VIM) by He [6] and comparison of DTM, VIM and Homotopy perturbation method (HPM) for solving nonlinear vibration problems by Ghafoori [7] can be shown as examples to this approach.

There is a large and growing literature on the solution of nonlinear boundary value problems. Classical methods can be classified as purely numerical methods that approximate the problem with some type of finite difference method and result in a nonlinear difference equation, and another group of methods generally called the weighted residual methods. In weighted residual methods, the approximate solution is expressed as some combination of suitably chosen basis functions and the combination coefficients are found so as to minimize the residual (error) in some way, when the approximate solution is substituted in the equation.

Variational methods can also be considered a member of weighted residual methods. The simplest of these is the collocation method which simply makes the residual zero at a chosen set of collocation points, resulting in a system of algebraic equations for the coefficients. The expectation is that the error at points other than the collocation points should not be much larger than zero. In fact, there are convergence proofs for both classes of methods.

The collocation method used in this study is an efficient and reliable numerical method that can be applied to the engineering problems listed below and defined by nonlinear equations.

1-) Such types of systems arise in the mathematical modeling of viscoelastic and inelastic flows, deformation of beams and plate deflection theory [8].

2-) In physics and fluid mechanics, in the solution of the Blasius boundary layer, which describes the constant two-dimensional boundary layer formed on a semi-infinite plate held parallel to a constant unidirectional flow [9].

3-) It can be used in rocket motion, thin film flow, heat transfer, dynamic programming, flow problems on a rigid body [10], which have been studied in the past years.

As a result, second and third-order boundary value problems have been of enormous attention over the last three decades, and so many theoretical and numerical studies dealing with such equations have appeared in the literature.

2. DESCRIPTION OF THE METHOD

Method of weighted residuals (MWR) is one of the most general methods in approximately solving differential and other equations. This method involves approximating the analytical solution by linear combinations of suitably chosen test or shape functions, the coefficients of the linear combination being the main unknowns to be solved [11]. In this manner, the solution of the differential equation is reduced to the solution of an algebraic system of equations for the coefficients. There is a variety of MWR developed all of which give different approximate solutions. Rather than finding a transformation to solve the equation, using some arbitrary points within the solution domain to form an algebraic system is much easier, even though some of the MWR involve evaluating some simple integrals.

This means that MWR methods do not require long and complex processing steps like analytical solutions or other approximate solutions. They need a simple algorithm to be programmed. The systems of equations that will obtain in the studied non-linear problems can be solved by various existing numerical methods.

The main difference between weighted residual methods and finite element methods is the choosing of the trial functions or shape functions [12]. In addition, weighted residual methods use trial functions defined over the whole domain, while finite element methods use shape functions defined on an element with elements aggregated to cover the whole domain. The method is asymptotically stable for first order differential equations [13].

Amin et al. [14] developed Haar wavelet collocation technique for solving non-linear delayed integro-differential equations defined for wireless sensor network and industrial internet of things.

Originally developed for aerospace and astrodynamic applications, direct collocation methods have become very popular and widely used in the context of trajectory optimization and model predictive control. These methods have proven to be powerful tools for solving optimal control problems in robotics [15].

Four representative methods and four (DTM-ADM-LSSVM-MVIM methods have been considered) example problems will be considered in the following [16-20]. The logic and application of these methods can be found in the related publications. In the next section, method of weighted residuals and collocation method will be summarized for nonlinear boundary value problems for ordinary differential equations to set the stage. And in the following sections, collocation method will be compared with each of the five methods.

2.1. Weighted residuals and Collocation Method

Denoting the unknown function as $y(x)$, a general form of a second order nonlinear problem can be written as

$$M(y) = 0 \quad , \quad a < x < b \tag{1}$$

with the boundary conditions

$$B(y) = 0 \quad , \quad x = a, b \tag{2}$$

Here the solution domain is $a \leq x \leq b$. N and B are some nonlinear ordinary differential operators.

The solution is expressed as a finite linear combination of chosen (known) base functions $f_n(x)$

$$y(x) = \sum_{n=1}^{N} C_n f_n(x) \tag{3}$$

The coefficients C_n are free (unknown). When substituted in the equation and the boundary conditions, the result will be the residual in the equation

$$R(x, C_1, C_2, \dots, C_N) = M\left(\sum_{n=1}^{N} C_n f_n(x)\right) \neq 0 \quad , \quad a < x < b \tag{4}$$

and the residual in the boundary conditions

$$G(x, C_1, C_2, \dots, C_N) = B\left(\sum_{n=1}^{N} C_n f_n(x)\right) \neq 0 \quad , \quad x = a, b \tag{5}$$

There are several possibilities for determining the coefficients so that the residuals will be minimum in some sense. A natural choice is to make the square error minimum, i.e. define total square error as

$$E(C_1, C_2, \dots, C_N) = \int_a^b R^2 dx + G^2]_{x=a} + G^2]_{x=b} \tag{6}$$

and minimize it with respect to the free coefficients

$$\frac{\partial E}{\partial C_n} = 0 \quad , \quad n = 1,2, \dots, N \tag{7}$$

This results in a nonlinear system of algebraic equations for C_n and involves a fair amount of error-prone intermediary steps.

The collocation method uses suitably chosen collocation points, including the boundaries

$$a = x_1 < x_2 < \cdots < x_{N-1} < x_N = b$$

and equates the residulas to zero at these points

$$R(x_n, C_1, C_2, \dots, C_N) = 0 \ , \ n = 2,3, \dots, N-1 \tag{8.a}$$

$$G(x_1, C_1, \dots, C_N) = G(x_N, C_1, \dots, C_N) = 0 \tag{8.b}$$

This gives a system of nonlinear algebraic equations for C_n. When applied carefully, the collocation method gives quick and correct results.

2.2. Application of collocation method to nonlinear differential equations

In this section, the solution of several problems by collocation method, which were previously solved by others using other methods, will be given. The comparison of collocation method with these newer methods will be given altogether in the next section.

The following problem was solved in [16] by the DTM.

$$\begin{aligned} & y''(x) - \frac{1}{2} y^3(x) = 0 \\ & y(1) = \frac{2}{3} \quad , \quad y(3) = \frac{2}{5} \end{aligned} \tag{9}$$

The exact solution is $2/(2+\mathrm{x})$. For collocation solution, basis functions are chosen as simple polynomials, $N = 3$

$$y(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 \tag{10}$$

Applying the boundary conditions

$$\begin{aligned} & y(1) = \frac{2}{3} \quad \Rightarrow \quad C_0 + C_1 + C_2 + C_3 = \frac{2}{3} \\ & y(3) = \frac{2}{5} \quad \Rightarrow \quad C_0 + 3C_1 + 9C_2 + 27C_3 = \frac{2}{5} \end{aligned} \tag{11}$$

Solving C_0 and C_1 from (11) and substituting in (10),

$$\begin{aligned} y(x) = & \left(\frac{4}{5} + 3C_2 + 12C_3\right) + \left(-\frac{2}{15} - 4C_2 - 13C_3\right)x \\ & + C_2 x^2 + C_3 x^3 \end{aligned} \tag{12}$$

Now that the boundary conditions are satisfied, the residual of the equation

$$R(x) = 2C_2 + 6xC_3 - \frac{1}{2}\left(\frac{4}{5} + 3C_2 + x^2C_2 + x\left(-\frac{2}{15} - 4C_2 - 13C_3\right) + 12C_3 + x^3C_3\right)^3 \quad (13)$$

Are to be made zero at two points, which we choose as $x_1=5/3$, $x_2=7/3$. This gives two equations

$$R(\frac{5}{3}) = 0$$
$$\Rightarrow 2C_2 + 10C_3 - 0.5\begin{pmatrix} 0.8 + 5.778C_2 \\ +1.667(-0.133 - 4C_2 - 13C_3) \\ +16.629C_3 \end{pmatrix}^3 = 0 \quad (14.a)$$

$$R(\frac{7}{3}) = 0$$
$$\Rightarrow 2C_2 + 14C_3 - 0.5\begin{pmatrix} 0.8 + 8.445C_2 \\ +2.334(-0.1333 - 4C_2 - 13C_3) \\ +24.703C_3 \end{pmatrix}^3 = 0 \quad (14.b)$$

Since this is a nonlinear system of algebraic equations, it may have real as well as complex solutions. Of these solutions, the real ones with the smallest absolute values will be taken; this choice will be applied for the other examples below also. This gives the coefficients as C_2=0.080961, C_3= -0.008039, and the approximate solution for two collocation point is

$$y_2(x) = -0.008x^3 + 0.0809x^2 - 0.352x + 0.946 \quad (15)$$

When polynomials of degree 4, 5 and 6 are taken, going through the similar procedure, the approximate solutions are found as

$$y_3(x) = 0.002x^4 - 0.024x^3 + 0.129x^2 - 0.414x + 0.973 \quad (16)$$

$$y_4(x) = -0.0005x^5 + 0.0072x^4 - 0.0450x^3 + 0.1692x^2 - 0.4514x + 0.9872 \quad (17)$$

$$y_5(x) = 0.0001x^6 - 0.0020x^5 + 0.015x^4 - 0.0653x^3 + 0.1982x^2 - 0.4731x + 0.9938 \quad (18)$$

The next example considered is a non-homogeneous nonlinear problem

$$y''(x) + y^2(x) = x^4 + 2 \quad , \quad 0 < x < 1$$
$$y(0) = 0 \quad , \quad y(1) = 1 \quad (19)$$

This problem was solved using ADM by Jafari [17] and using DTM by Ertürk [18]. The exact solution is

$$y(x) = x^2 \quad (20)$$

Taking the approximate solution as a third order polynomial

$$y(x) = C_0 + C_1 x + C_2x^2 + C_3x^3 \quad (21)$$

the boundary condition $y(0) = 0$ gives $C_0 = 0$, resulting in

$$y(x) = C_1 x + C_2x^2 + C_3x^3 \quad (22)$$

Applying the boundary condition $y(1) = 1$ to (22) gives

$$C_1 + C_2 + C_3 = 1 \quad (23)$$

Substituting (22) into (19) the residual can be found as

$$R(x) = C_1^2x^2 + C_2^2x^4 + 6xC_3 + C_3^2x^6 + C_1\left(2x^3C_2 + 2x^4C_3\right) + C_2\left(2 + 2x^5C_3\right) - 2 - x^4 \quad (24)$$

Residual $R(x)$ now involves three unknown coefficients; one equation is (23), and we need two more equations. These two equations are found by equating the residual to zero at two arbitrarily chosen collocation points. Taking the collocation points as 1/3 and 2/3, the equations $R\left(\frac{1}{3}\right) = 0$ and $R\left(\frac{2}{3}\right) = 0$ become

$$\begin{aligned}&-2.0123 + 0.11C_1^2 + 0.012C_2^2\\ &+C_2\left(2 + 0.008C_3\right) + C_1\left(0.07C_2 + 0.024C_3\right)\\ &+2C_3 + 0.001C_3^2 = 0\end{aligned} \tag{25}$$

$$\begin{aligned}&-2.197 + 0.444C_1^2 + 0.197C_2^2\\ &+C_2\left(2 + 0.26C_3\right) + C_1\left(0.59C_2 + 0.395C_3\right)\\ &+4C_3 + 0.087C_3^2 = 0\end{aligned} \tag{26}$$

Solving the system (23), (25), (26) $C_1 = 0$, $C_2 = 1$ and $C_3 = 0$, and the approximate solution is

$$y(x) = x^2 \tag{27}$$

which is the same as the exact solution, but this is just a coincidence due to the exact solution x^2 being chosen as one of the base functions.

As another example consider the following problem

$$\begin{aligned}&y''(x) - \frac{y^3 - 2y^2}{2x^2} = 0\\ &y(1) = 1 \quad , \quad y(2) = \frac{4}{3}\end{aligned} \tag{28}$$

which was also studied by Yanfei [19] using the Least squares support vector machines method. The exact solution is $2x/(x+1)$. Taking the approximate solution as

$$y(x) = C_0 + C_1x + C_2x^2 + C_3x^3 \tag{29}$$

Applying the boundary conditions gives

$$y(1) = 1 \;\Rightarrow\; C_0 + C_1 + C_2 + C_3 = 1 \tag{30.a}$$

$$y(2) = \frac{4}{3} \;\Rightarrow\; C_0 + 2C_1 + 4C_2 + 8C_3 = \frac{4}{3} \tag{30.b}$$

which allows solving C_0, C_1 in terms of C_2 and C_3

$$\begin{aligned}&C_0 = \frac{2}{3} + 2C_2 + 3C_3\\ &C_1 = \frac{1}{3} - 3C_2 - 7C_3\end{aligned} \tag{31}$$

Now the approximate solution becomes

$$\begin{aligned}y(x) = &\frac{2}{3} + \frac{x}{3} + \left(2 - 3x + x^2\right)C_2\\ &+\left(6 - 7x + x^3\right)C_3\end{aligned} \tag{32}$$

Evaluating the residual expression and taking the collocation points as 4/3 and 5/3

$$R(x_1) = R\left(\frac{4}{3}\right) = 0, R(x_2) = R\left(\frac{5}{3}\right) = 0 \tag{33}$$

and solving the resulting algebraic system for C_2 and C_3 , the coefficients are

$$\begin{aligned}&C_0 = 0.24695 \quad && C_2 = -0.366843\\ &C_1 = 1.06757 \quad && C_3 = 0.0523282\end{aligned} \tag{34}$$

Thus the approximate solution of (28) is, substituting (34) in (29),

$$\begin{aligned}y(x) = &0.24695 + 1.06757x\\ &-0.366843x^2 + 0.0523282x^3\end{aligned} \tag{35}$$

Utilizing three collocation points 5/4, 6/4 and 7/4 gives the approximate solution

$$\begin{aligned}y(x) = &0.14688 + 1.34485x - 0.64964x^2\\ &+0.17890x^3 - 0.02099x^4\end{aligned} \tag{36}$$

As the last example, we consider the following nonlinear problem with Robin boundary conditions.

$$\begin{aligned}&y''(x) = \frac{1}{2}(1 + x + y(x))^3 \;, \qquad 0 \le x \le 1\\ &y'(0) - y(0) = -\frac{1}{2} \quad , \quad y'(1) + y(1) = 1\end{aligned} \tag{37}$$

Ghorbani [20] used MVIM to find the solution. The exact solution is $\frac{2}{2-x} - x - 1$. Taking the approximate solution as

$$y(x)=\sum_{n=0}^{9}C_n x^n \tag{38}$$

involving ten collocation points (two of them being the boundary points), the approximate solution becomes

$$\begin{aligned} y(x) = &\, 0.04x^9 - 0.12x^8 + 0.19x^7 \\ & - 0.15x^6 + 0.12x^5 + 0.03x^4 \\ & + 0.13x^3 + 0.24x^2 - 0.49x + 0.00003 \end{aligned} \tag{39}$$

3. RESULTS

The approximate and exact solutions of the first problem (9) are compared in Table 1. Where N denotes the number of collocation points. Absolute errors for DTM are also given in the table.

The nonlinear problems naturally lead to a nonlinear system of algebraic equations which are considerably harder to solve than linear systems. It was observed that, of possible multiple solutions of the nonlinear system, the ones with the smallest absolute values give very good approximate solutions.

Table 1 DTM and Collocation solutions of (9)

					Absolute Errors in Collocation Solutions, Eq. (9)			
x	**Exact Solution**	**DTM**	**Absolute Error in DTM**	**Collocation Solution (N=2)**	**N=2**	**N=3**	**N=4**	**N=5**
1.0	0.6666	0.6667	$1.0\text{x}10^{-4}$	0.6666	$1\text{x}10^{-16}$	$1\text{x}10^{-16}$	0.000	$1\text{x}10^{-16}$
1.1	0.6451	0.6479	$2.7\text{x}10^{-3}$	0.6452	$5\text{x}10^{-4}$	$1\text{x}10^{-4}$	$2\text{x}10^{-5}$	$5\text{x}10^{-6}$
1.2	0.6250	0.6305	$5.5\text{x}10^{-3}$	0.6251	$9\text{x}10^{-4}$	$1\text{x}10^{-4}$	$3\text{x}10^{-5}$	$6\text{x}10^{-6}$
1.3	0.6060	0.6143	$8.2\text{x}10^{-3}$	0.6061	$1\text{x}10^{-3}$	$1\text{x}10^{-4}$	$3\text{x}10^{-5}$	$5\text{x}10^{-6}$
1.4	0.5882	0.5992	$1.1\text{x}10^{-2}$	0.5883	$1\text{x}10^{-3}$	$1\text{x}10^{-4}$	$3\text{x}10^{-5}$	$4\text{x}10^{-6}$
1.5	0.5714	0.5850	$1.5\text{x}10^{-2}$	0.5715	$1\text{x}10^{-3}$	$1\text{x}10^{-4}$	$2\text{x}10^{-5}$	$3\text{x}10^{-6}$
1.6	0.5556	0.5717	$1.6\text{x}10^{-2}$	0.5556	$9\text{x}10^{-4}$	$1\text{x}10^{-4}$	$2\text{x}10^{-5}$	$3\text{x}10^{-6}$
1.7	0.5405	0.5591	$1.9\text{x}10^{-2}$	0.5406	$8\text{x}10^{-4}$	$7\text{x}10^{-5}$	$2\text{x}10^{-5}$	$2\text{x}10^{-6}$
1.8	0.5263	0.5471	$2.1\text{x}10^{-2}$	0.5263	$7\text{x}10^{-4}$	$6\text{x}10^{-5}$	$2\text{x}10^{-5}$	$2\text{x}10^{-6}$
1.9	0.5128	0.5355	$2.3\text{x}10^{-2}$	0.5128	$6\text{x}10^{-4}$	$4\text{x}10^{-5}$	$2\text{x}10^{-5}$	$1\text{x}10^{-6}$
2.0	0.5000	0.5242	$2.4\text{x}10^{-2}$	0.5000	$6\text{x}10^{-4}$	$4\text{x}10^{-5}$	$2\text{x}10^{-5}$	$1\text{x}10^{-6}$
2.1	0.4878	0.5131	$2.5\text{x}10^{-2}$	0.4878	$5\text{x}10^{-4}$	$3\text{x}10^{-5}$	$2\text{x}10^{-5}$	$7\text{x}10^{-7}$
2.2	0.4761	0.5020	$2.6\text{x}10^{-2}$	0.4761	$6\text{x}10^{-4}$	$2\text{x}10^{-5}$	$2\text{x}10^{-5}$	$2\text{x}10^{-7}$
2.3	0.4651	0.4909	$2.6\text{x}10^{-2}$	0.4650	$6\text{x}10^{-4}$	$4\text{x}10^{-6}$	$2\text{x}10^{-5}$	$2\text{x}10^{-7}$
2.4	0.4545	0.4796	$2.5\text{x}10^{-2}$	0.4543	$6\text{x}10^{-4}$	$1\text{x}10^{-5}$	$2\text{x}10^{-5}$	$6\text{x}10^{-7}$
2.5	0.4444	0.4679	$2.4\text{x}10^{-2}$	0.4442	$6\text{x}10^{-4}$	$1\text{x}10^{-5}$	$2\text{x}10^{-5}$	$1\text{x}10^{-6}$
2.6	0.4347	0.4558	$2.1\text{x}10^{-2}$	0.4344	$6\text{x}10^{-4}$	$5\text{x}10^{-5}$	$2\text{x}10^{-5}$	$1\text{x}10^{-6}$
2.7	0.4255	0.4430	$1.8\text{x}10^{-2}$	0.4252	$6\text{x}10^{-4}$	$7\text{x}10^{-5}$	$2\text{x}10^{-5}$	$2\text{x}10^{-6}$
2.8	0.4166	0.4296	$1.3\text{x}10^{-2}$	0.4164	$5\text{x}10^{-4}$	$7\text{x}10^{-5}$	$2\text{x}10^{-5}$	$3\text{x}10^{-6}$
2.9	0.4081	0.4152	$7.0\text{x}10^{-3}$	0.4080	$3\text{x}10^{-4}$	$5\text{x}10^{-5}$	$1\text{x}10^{-5}$	$2\text{x}10^{-6}$
3.0	0.4000	0.3999	$1.0\text{x}10^{-4}$	0.4002	$2\text{x}10^{-16}$	$2\text{x}10^{-16}$	$3\text{x}10^{-16}$	$3\text{x}10^{-16}$

These results show that the collocation method is a powerful and simple to apply method which can easily be adopted to other types of equations such as fractional differential equations.

The approximate and exact solutions of the first problem (9) are compared in Table 1. Where N denotes the number of collocation points. Absolute errors for DTM are also given in the table.

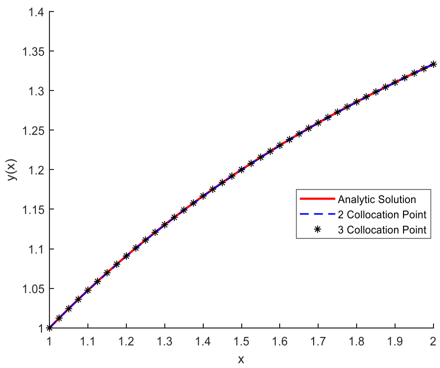

Figure 1 Comparison between the collocation and analytic solutions for the third problem

Table 2 Absolute Errors for various approximate solutions of (28)

	Absolute Errors in Collocation Solutions, Eq. (28)	
x	**N=2**	**N=3**
1.0	2.22045×10^{-16}	2.22045×10^{-16}
1.1	5.78945×10^{-4}	8.53546×10^{-5}
1.2	7.12085×10^{-4}	7.94841×10^{-5}
1.3	6.50105×10^{-4}	5.30591×10^{-5}
1.4	5.49578×10^{-4}	3.3029×10^{-5}
1.5	4.91793×10^{-4}	2.14438×10^{-5}
1.6	4.97229×10^{-4}	9.93213×10^{-6}
1.7	5.36816×10^{-4}	9.03769×10^{-6}
1.8	5.40787×10^{-4}	3.33098×10^{-5}
1.9	4.05688×10^{-4}	4.40236×10^{-5}
2.0	6.66134×10^{-16}	6.66134×10^{-16}

The last (fourth) example considered in this study is (37). The collation solution of the last problem is compared with other methods in Table 2.

Table 3 Errors for various approximate solutions of (37)

x	OVIM	SVIM	PRESENT STUDY	MVIM
0.0	6.66×10^{-1}	2.45×10^{-2}	3.12×10^{-5}	6.47×10^{-6}
0.2	8.42×10^{-1}	3.03×10^{-2}	2.91×10^{-5}	7.98×10^{-6}
0.4	1.14×10^{0}	3.83×10^{-2}	2.78×10^{-5}	9.99×10^{-6}
0.6	1.66×10^{0}	4.91×10^{-2}	2.92×10^{-5}	1.26×10^{-6}
0.8	2.54×10^{0}	6.07×10^{-2}	3.43×10^{-5}	1.60×10^{-6}
1.0	4.03×10^{0}	6.21×10^{-2}	4.25×10^{-5}	1.88×10^{-6}

The abbreviations OVIM, SVIM, and MVIM in Table 3 are different versions of the variational iteration method.

For the example differential equations considered here, all approximate solutions have been taken as polynomial expressions. The coefficients in these approximate solutions are the solutions of the algebraic equations formed by the use of the boundary conditions and the differential equation. Looking at Table 1, it is seen that the two-point collocation solution has a better absolute error than the one in DTM [16].

Figure 1 and Table 2 show the decrease in absolute error as the order of the approximate solution polynomial is increased, which means more unknown coefficients have to be solved. The number of collocation points also have to be increased; thus approximate solution converges to the exact solution with the increase of the number of collocation points.

Looking at Table 3, it is seen that the collocation solution is a better approximation to the exact solution than both OVIM and SVIM; and is comparable to MVIM.

If the boundary value problem is linear, this gives a linear system of algebraic equations for the unknown coefficients. For nonlinear problems, the resulting nonlinear system of algebraic equations may result in multiple and/or complex solutions. In this case the smallest real solutions need to be chosen. The other solutions will give unacceptable errors.

The collocation method is not restricted to polynomials as base functions. Depending on the type of differential equation under consideration, trigonometric, exponential and many other different types of functions can be chosen. The collocation points have to be chosen within the solution domain, but they do not need to be equidistant from each other. Choosing equidistant points may be preferred since it is simpler and easier to control.

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The Declaration of Ethics Committee Approval

This study does not require ethics committee permission or any special permission.

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The authors of the paper declare that they comply with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that they do not make any falsification on the data collected. In addition, they declare that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

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