Sinopfbd, 2024; 9(1): 26-35 **https://doi.org/10.33484/sinopfbd.1343041**

Sinop Üniversitesi Fen Bilimleri Dergisi
Sinop Uni J Nat Sci
E-ISSN: 2564-7873

https://dergipark.org.tr/tr/pub/sinopfbd

Bounds For Spectral Radius and Energy of PIS Graphs

Esra ÖZTÜRK SÖZEN[1] **and Elif ERYAŞAR**[1]

How to cite: Öztürk Sözen, E., & Eryaşar, E. (2024). Bounds for spectral radius and energy of PIS graphs. *Sinop Üniversitesi Fen Bilimleri Dergisi*, *9*(1), 26-35. https://doi.org/10.33484/sinopfbd.1343041

Research Article

Corresponding Author
Elif ERYAŞAR
eeryasar@sinop.edu.tr

ORCID of the Authors
E.Ö.S.: 0000-0002-2632-2193
E.E.: 0000-0002-9852-6662

Received: 14.08.2023
Accepted: 29.01.2024

Abstract

Once the spectral radius and energy of a graph structure have been defined, many properties have been studied. The spectral radius and energy of a graph are related to the eigenvalues of the adjacency matrix of the graph. In this paper, we define an adjacency matrix for a prime ideal sum (PIS) graph and then extend the concepts of spectral radius and energy to PIS graphs. Some bound theorems on the energy and spectral radius of PIS graph structures are given. A SageMath code for plotting these graphs is also provided.

Keywords: Graph energy, prime ideal sum graph, spectral radius

PIS Grafların Spektral Yarıçapı ve Enerjisi Için Sınırlar

[1]Sinop University,
Department of Mathematics,
57000, Sinop, Türkiye

Öz

Bir graf yapısının spektral yarıçapı ve enerjisi tanımlandıktan sonra birçok özelliği incelenmiştir. Grafların spektral yarıçapı ve enerjisi komşuluk matrisin özdeğerleriyle ilişkilidir. Bu çalışmada bir asal ideal toplam (PIS) graf için bir komşuluk matrisi tanımlanmıştır ve daha sonra spektral yarıçap ve enerji kavramları PIS grafları için genişletilmiştir. PIS graf yapılarının enerjisi ve spektral yarıçapına ilişkin bazı sınır teoremleri verilmiştir. Ayrıca bu grafları çizmek için bir SageMath kodu da sunulmaktadır.

Anahtar Kelimeler: Graf enerjisi, asal ideal toplam graf, spektral yarıçap

Introduction

The graph structure is represented by $G = (V, E)$, where $V = \{v_1, v_2, ..., v_n\}$ is the set of vertices and $E = E(G)$ is the set of edges. Let $i, j \in V$, and we say that two vertices are adjacent if there is at least one edge between i and j. If vertices i and j are adjacent, they are denoted by $i{\sim}j$. The degree of a point i is the number of edges connected to i and it is denoted by d_i [1]. Spectral graph theory has used spectra of certain matrices associated with a graph, such as the adjacency matrix, the Laplace matrix, or other forms of these, to provide information about a graph. It is possible to characterize certain graph structures with a spectrum (with the help of one of these matrices) [2]. The adjacency matrix is an $n \times n$

matrix, denoted by $A(G)$, and is defined as follows [3]:

$$A(G) = \begin{cases} 1, & \text{i}\sim\text{j} \\ 0, & \text{otherwise} \end{cases}$$

The adjacency matrix is a real and symmetric matrix, and all eigenvalues are real. Given different eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of the adjacency matrix, the inequality $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ exists for these eigenvalues. The largest eigenvalue λ_1 is called the spectral radius of the graph [4].
Another concept related to the spectrum of a graph is energy. Let $A(G)$ be the adjacency matrix of a graph G, $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of this matrix, the energy of a graph is denoted by $E(G)$ and $E(G) = \sum_{i=1}^{n} |\lambda_i|$ [5].
Defining zero divisor graphs over commutative rings is the beginning of graph theory in pure algebra [6, 7]. Nowadays, studies on the zero divisor graph of $\mathbb{Z}_n$ is a trending field in spectral and chemical graph theory [8–10]. Studies on these topics in the literature motivated us to study computing different topological descriptors with respect to the prime ideal sum graphs. The prime ideal sum graph of a ring is described as a graph whose vertices are all non-trivials of the ring and whose edges connect two vertices whenever the sum of these vertices is a prime ideal [11].
In this study, motivated by the works mentioned above, we investigate the spectral radius and energy of some commutative rings according to the prime ideal sum graph structures of them. A SageMath code for drawing PIS graph structures is also provided.

Main Results

Spectral Radius and Energy of PIS Graphs

Definition 1. Let $I, J \in \mathbb{Z}_n$ non-trivial two ideals. Then,

$$AG(\mathbb{Z}_n) = \begin{cases} 1, & I + J \text{ prime} \\ 0, & \text{otherwise} \end{cases}$$

where $AG(\mathbb{Z}_n)$ will represent the adjacency matrix of $PIS(\mathbb{Z}_n)$.

Theorem 1. Let p be a prime number. The sum of the eigenvalues of the adjacency matrix of $\mathbb{Z}_{p^\alpha}$ is zero.

Proof. Let us list all ideals of $\mathbb{Z}_{p^\alpha}$. Then we have $\{0\}, p^{\alpha-1}\mathbb{Z}_{p^\alpha}, p^{\alpha-2}\mathbb{Z}\mathbb{Z}_{p^\alpha}, p^2\mathbb{Z}_{p^\alpha}, p\mathbb{Z}_{p^\alpha}, \mathbb{Z}_{p^\alpha}$. Note that $p\mathbb{Z}_{p^\alpha}$ is a prime ideal of $\mathbb{Z}_{p^\alpha}$.
Taking into account the PIS graph over $\mathbb{Z}_{p^\alpha}$, $p\mathbb{Z}_{p^\alpha}$ is adjacent to $p^{\alpha-1}\mathbb{Z}_{p^\alpha}, p^{\alpha-2}\mathbb{Z}_{p^\alpha}$,...,$p^3\mathbb{Z}_{p^\alpha}, p^2\mathbb{Z}_{p^\alpha}$ as $\mathbb{Z}_{p^\alpha}$ is local and so the sum of these ideals is equal to $\mathbb{Z}_{p^\alpha}$. Considering the following graph representation of $\mathbb{Z}_{p^\alpha}$ Figure 1 the adjacency matrix of $\mathbb{Z}_{p^\alpha}$ is as follows.

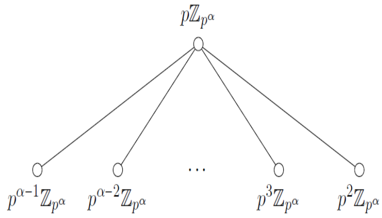

Figure 1. *Prime Ideal Sum Graph of* $\mathbb{Z}_{p^\alpha}$

$$AG(\mathbb{Z}_{p^\alpha}) = \begin{bmatrix} 0 & 1 & 1 & . & . & . & 1 \\ 1 & 0 & 0 & . & . & . & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{nxn}$$

For a matrix of this form, $n-2$ of the n eigenvalues are zero. Since two non-zero eigenvalues have opposite numbers, therefore $\lambda_1 + \lambda_2 + ... + \lambda_n = 0$.

Theorem 2. For two non-zero eigenvalues λ_1, λ_2 of $\mathbb{Z}_{p^\alpha}$ with spectral radius λ_1
$\lambda_1 \lambda_2 \leq \lambda_1$.

Proof. It is clear from thm 1.

Theorem 3. Let $E(G)$ denotes the graph energy of $\mathbb{Z}_{p^\alpha}$. Then $E(G) = |\lambda_1| + |\lambda_2|$. Here, since $\lambda_2 = -\lambda_1,\ E(G) = 2\,|\lambda_1|$.

Proof. The adjacency matrix of $\mathbb{Z}_{p^\alpha}$ has two non-zero eigenvalues and since they have opposite numbers, $E(G) = \sum_{i=1}^{n} |\lambda_i| = |\lambda_1| + |\lambda_2| = |\lambda_1| + |-\lambda_1| = 2\,|\lambda_1|$.

Theorem 4. Let p, q be prime numbers. The spectral radius and energy of $\mathbb{Z}_{pq}$ are zero.

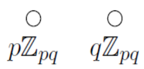

Figure 2. *Prime Ideal Sum Graph of* $\mathbb{Z}_{pq}$

Proof. For the prime ideals $p\mathbb{Z}_{pq}$ and $q\mathbb{Z}_{pq}$, the prime ideal sum graph of $\mathbb{Z}_{pq}$ is as Figure 2. As seen from Figure 2 both $p\mathbb{Z}_{pq}$ and $q\mathbb{Z}_{pq}$ are isolated vertices and so has degree zero. Therefore, its spectral radius and energy are zero.

Theorem 5. Let p, q be prime numbers. For the PIS graph representation of $\mathbb{Z}_{p^2q}$ for the spectral radius λ_1 with $|V(G)| = n = 4$,

$$\sum_{i=1}^{4} \lambda_i \leq \lambda_1 \leq \sum_{u_i \in V(G)} \frac{d(u_i)}{n}.$$

Proof. Let us consider the ideals $u_1 = p\mathbb{Z}_{p^2q}, u_2 = q\mathbb{Z}_{p^2q}, u_3 = pq\mathbb{Z}_{p^2q}, u_4 = p^2\mathbb{Z}_{p^2q}$, where u_1 and u_2 are the prime ones. Then $PIS(\mathbb{Z}_{p^2q})$ is as Figure 3. Thus the adjacency matrix of $\mathbb{Z}_{p^2q}$ is as follows.

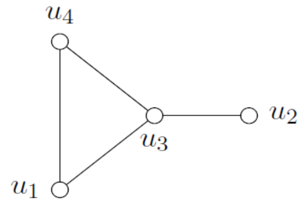

Figure 3. *Prime Ideal Sum Graph of* $\mathbb{Z}_{p^2q}$

$$AG(\mathbb{Z}_{p^2q}) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}_{4x4}$$

Therefore, the eigenvalues are $\lambda_1 = 2.170, \lambda_2 = 0.311, \lambda_3 = -1.481$ and $\lambda_4 = -3$, from which the spectral radius is $\lambda_1 = 2.170$. Also $d(u_1) = 2, d(u_2) = 1, d(u_3) = 3$ and $d(u_4) = 2$, $\sum_{i=1}^{4} \lambda_i \leq \lambda_1 \leq \sum_{u_i \in V(G)} \frac{d(u_i)}{n}$.

Theorem 6. Let $E(G)$ denotes the graph energy of $\mathbb{Z}_{p^2q}$. Then, $E(G) \leq \frac{n\sum_{u_i \in V(G)} d(u_i)}{4}$.

Proof. The graph energy of $\mathbb{Z}_{p^2q}$ is clear from thm 5 that $E(G) = 6.96$. Considering the number of vertices $n = 4$ and the sum of degrees of u_i, we have $E(G) \leq \frac{n\sum_{u_i \in V(G)} d(u_i)}{4}$.

Theorem 7. Let p, q be prime numbers. For the PIS graph representation of $\mathbb{Z}_{p^2q^2}$ for the spectral radius λ_1 with $|V(G)| = n = 7$,

$\sum_{i=1}^{7} \lambda_i \leq \lambda_1 \leq \sum_{u_i \in V(G)} \frac{d(u_i)}{n-1}$.

Proof. Let us consider the ideals $u_1 = p\mathbb{Z}_{p^2q^2}, u_2 = q\mathbb{Z}_{p^2q^2,}, u_3 = pq\mathbb{Z}_{p^2q^2}, u_4 = p^2q\mathbb{Z}_{p^2q^2}, u_5 = pq^2\mathbb{Z}_{p^2q^2}, u_6 = p^2\mathbb{Z}_{p^2q^2}, u_7 = q^2\mathbb{Z}_{p^2q^2}$, where u_1 and u_2 are the prime ones. Then $PIS(\mathbb{Z}_{p^2q^2})$ is as Figure 4. Thus the adjacency matrix of $\mathbb{Z}_{p^2q^2}$ is as follows.

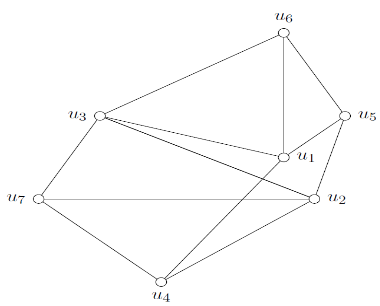

Figure 4. *Prime Ideal Sum Graph of* $\mathbb{Z}_{p^2q^2}$

$$AG(\mathbb{Z}_{p^2q^2}) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}_{7x7}.$$

Therefore, the eigenvalues are $\lambda_1 = 3.48, \lambda_2 = 1.41, \lambda_3 = 0.20, \lambda_4 = 0, \lambda_5 = -1.49, \lambda_6 = -1.41$ and $\lambda_7 = 2.56$ from which the spectral radius is $\lambda_1 = 3.48$. Also $d(u_1) = 4, d(u_2) = 4, d(u_3) = 4, d(u_4) = 3, d(u_5) = 3, d(u_6) = 3$ and $d(u_7) = 3$, $\sum_{i=1}^{7} \lambda_i \leq \lambda_1 \leq \sum_{u_i \in V(G)} \frac{d(u_i)}{n-1}$.

Theorem 8. Let $E(G)$ denotes the graph energy of $\mathbb{Z}_{p^2q^2}$. Then, $E(G) \leq 2\sum_{u_i \in V(G)} \frac{d(u_i)}{n-3}$.

Proof. The graph energy of $\mathbb{Z}_{p^2q^2}$ is clear from thm 7 that $E(G) = 10.56$. Considering the number of vertices $n = 7$ and the sum of degrees of u_i, we have, $E(G) \leq 2\sum_{u_i \in V(G)} \frac{d(u_i)}{n-3}$.

Theorem 9. Let p, q and r be prime numbers. From the PIS graph representation of $\mathbb{Z}_{pqr}$ for the spectral radius λ_1 with $|V(G)| = n = 6$,

$\sum_{i=1}^{6} \lambda_i \leq \lambda_1 \leq \frac{\sum_{u_i \in V(G)} d(u_i)}{2(n-4)}$.

Proof. Let us consider the ideals $u_1 = p\mathbb{Z}_{pqr}, u_2 = q\mathbb{Z}_{pqr}, u_3 = r\mathbb{Z}_{pqr}, u_4 = pq\mathbb{Z}_{pqr}, u_5 = pr\mathbb{Z}_{pqr}, u_6 = qr\mathbb{Z}_{pqr}$, where u_1, u_2 and u_3 prime ones. Then $PIS(\mathbb{Z}_{pqr})$ is as Figure 5. Thus the adjacency matrix of $\mathbb{Z}_{pqr}$ is as follows.

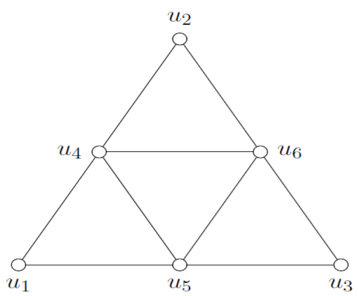

Figure 5. *Prime Ideal Sum Graph of* $\mathbb{Z}_{pqr}$

$$AG(\mathbb{Z}_{pqr}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}_{6x6}.$$

Therefore, the eigenvalues are $\lambda_1 = 3.24, \lambda_{2,3} = 0.62, \lambda_{4,5} = -1.62$ and $\lambda_6 = -1.23$ from which the spectral radius is $\lambda_1 = 3.24$. Also $d(u_1) = 2, d(u_2) = 2, d(u_3) = 2, d(u_4) = 4, d(u_5) = 4$ and $d(u_6) = 4$,
$\sum_{i=1}^{6} \lambda_i \leq \lambda_1 \leq \frac{\sum_{u_i \in V(G)} d(u_i)}{2(n-4)}$.

Theorem 10. Let $E(G)$ denotes the graph energy of $\mathbb{Z}_{pqr}$. Then, $E(G) \leq \frac{\sum_{u_i \in V(G)} d(u_i)}{2}$.

Proof. The graph energy of $\mathbb{Z}_{pqr}$ is clear from thm 9 that $E(G) = 8.95$. Considering the number of vertices $n = 6$ and the sum of degrees of u_i, we have

$$E(G) \leq \frac{\sum_{u_i \in V(G)} d(u_i)}{2}.$$

Theorem 11. Let p, q be prime numbers. For the PIS graph representation of $\mathbb{Z}_{p^3q}$ for the spectral radius λ_1 with $|V(G)| = n = 6$,
$\sum_{i=1}^{6} \lambda_i \leq \lambda_1 \leq \frac{\sum_{u_i \in V(G)} d(u_i)-1}{n-1}$

Proof. Let us consider the ideal $u_1 = p\mathbb{Z}_{p^3q}, u_2 = q\mathbb{Z}_{p^3q}, u_3 = p^2\mathbb{Z}_{p^3q}, u_4 = pq\mathbb{Z}_{p^3q}, u_5 = p^3\mathbb{Z}_{p^3q}, u_6 = p^2q\mathbb{Z}_{p^3q}$, where u_1 and u_2 prime ones. Then $PIS(\mathbb{Z}_{p^3q})$ is as Figure 6. Thus the adjacency matrix of $\mathbb{Z}_{p^3q}$ is as follows.

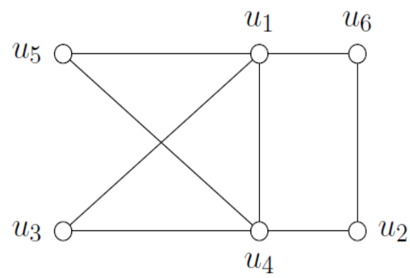

***Figure 6.** Prime Ideal Sum Graph of $\mathbb{Z}_{p^3q}$*

$$AG(\mathbb{Z}_{p^3q}) = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}_{6x6}.$$

Therefore, the eigenvalues are $\lambda_1 = 2.903, \lambda_2 = 0.806, \lambda_{3,4} = 0, \lambda_5 = -1.709$ and $\lambda_6 = -2$ from which the spectral radius is $\lambda_1 = 2.903$. Also $d(u_1) = 4, d(u_2) = 2, d(u_3) = 2, d(u_4) = 4, d(u_5) = 2$ and $d(u_6) = 2$, $\sum_{i=1}^{6} \lambda_i \leq \lambda_1 \leq \frac{\sum_{u_i \in V(G)} d(u_i)-1}{n-1}$.

Theorem 12. Let $E(G)$ denotes the graph energy of $\mathbb{Z}_{p^3q}$. Then, $E(G) \leq \frac{\sum_{u_i \in V(G)} d(u_i)}{2}$.

Proof. The graph energy of $\mathbb{Z}_{p^3q}$ is clear from thm 11 that $E(G) = 7.42$. Considering the number of vertices $n = 6$ and the sum of degrees of u_i, we have $E(G) \leq \frac{\sum_{u_i \in V(G)} d(u_i)}{2}$.

Theorem 13. Let p, q and r be prime numbers . From the PIS graph representation of $\mathbb{Z}_{p^2qr}$ for the spectral radius λ_1 with $V(G) = n = 10$, $\sum_{i=1}^{10} \lambda_i \leq \lambda_1 \leq \frac{\sum_{u_i \in V(G)} d(u_i)}{n-2}$.

Proof. Let us consider the ideal $u_1 = p\mathbb{Z}_{p^2qr}, u_2 = q\mathbb{Z}_{p^2qr}, u_3 = r\mathbb{Z}_{p^2qr}, u_4 = p^2\mathbb{Z}_{p^2qr}, u_5 = pq\mathbb{Z}_{p^2qr}, u_6 = pr\mathbb{Z}_{p^2qr}, u_7 = p^2q\mathbb{Z}_{p^2qr}, u_8 = p^2r\mathbb{Z}_{p^2qr}, u_9 = qr\mathbb{Z}_{p^2qr}, u_{10} = pqr\mathbb{Z}_{p^2qr}$, where u_1, u_2 and u_3 prime ones. Then $PIS(\mathbb{Z}_{p^2qr})$ is as Figure 7. Thus the adjacency matrix of $\mathbb{Z}_{p^2qr}$ is as follows.

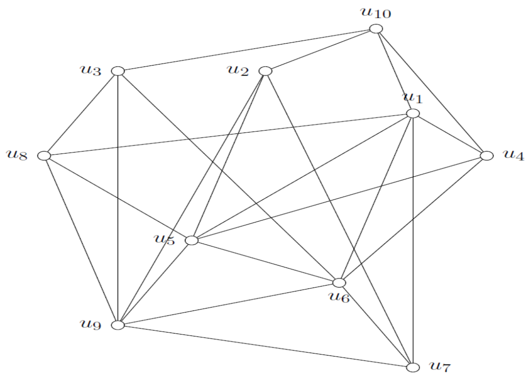

Figure 7. *Prime Ideal Sum Graph of* $\mathbb{Z}_{p^2qr}$

$$AG(\mathbb{Z}_{p^2qr}) = \begin{bmatrix} 0&0&0&1&1&1&1&1&0&1 \\ 0&0&0&0&1&0&1&0&1&1 \\ 0&0&0&0&0&1&0&1&1&1 \\ 1&0&0&0&1&1&0&0&0&1 \\ 1&1&0&1&0&1&0&1&1&0 \\ 1&0&1&1&1&0&1&0&1&0 \\ 1&1&0&0&0&1&0&0&1&0 \\ 1&0&1&0&1&0&0&0&1&0 \\ 0&1&1&0&1&1&1&1&0&0 \\ 1&1&1&1&0&0&0&0&0&0 \end{bmatrix}_{10x10}.$$

Therefore the eigenvalues are $\lambda_1 = 4.98, \lambda_2 = 1.48, \lambda_3 = 1, \lambda_4 = 0.79, \lambda_5 = 0.41, \lambda_6 = -0.09, \lambda_7 = -1.49, \lambda_8 = -2, \lambda_9 = -2.41$ and $\lambda_{10} = -2.67$ from which the spectral radius $\lambda_1 = 4.98$. Also $d(u_1) = 6, d(u_2) = 4, d(u_3) = 4, d(u_4) = 4, d(u_5) = 6, d(u_6) = 6, d(u_7) = 4, d(u_8) = 4, d(u_9) = 6$ and $d(u_{10}) = 4$, $\sum_{i=1}^{10} \lambda_i \leq \lambda_1 \leq \frac{\sum_{u_i \in V(G)} d(u_i)}{n-2}$.

Theorem 14. Let $E(G)$ denotes the graph energy of $\mathbb{Z}_{p^2qr}$. Then, $E(G) \leq \frac{\sum_{u_i \in V(G)} d(u_i)+n}{3}$.

Proof. The graph energy of $\mathbb{Z}_{p^2qr}$ is clear from thm 13 that $E(G) = 17.33$. Considering the number of vertices $n = 10$ and the sum of degrees of u_i, we have $\frac{\sum_{u_i \in V(G)} d(u_i)+n}{3}$.

Theorem 15. Let p, q, r and s be prime numbers . From the PIS graph representation of $\mathbb{Z}_{pqrs}$ for the spectral radius λ_1 with $V(G) = n = 14$, $\sum_{i=1}^{14} \lambda_i \leq \lambda_1 \leq \frac{\sum_{u_i \in V(G)} d(u_i)}{n-2}$.

Proof. Let us consider the ideal $u_1 = p\mathbb{Z}_{pqrs}, u_2 = q\mathbb{Z}_{pqrs}, u_3 = r\mathbb{Z}_{pqrs}, u_4 = pq\mathbb{Z}_{pqrs}, u_5 = s\mathbb{Z}_{pqrs}, u_6 = pr\mathbb{Z}_{pqrs}, u_7 = ps\mathbb{Z}_{pqrs}, u_8 = qr\mathbb{Z}_{pqrs}, u_9 = qs\mathbb{Z}_{pqrs}, u_{10} = pqr\mathbb{Z}_{pqrs}, u_{11} = rs\mathbb{Z}_{pqrs}, u_{12} = pqs\mathbb{Z}_{pqrs}, u_{13} = prs\mathbb{Z}_{pqrs}, u_{14} = qrs\mathbb{Z}_{pqrs}$, where u_1, u_2, u_3 and u_5 prime ones. Then $PIS(\mathbb{Z}_{pqrs})$ is as Figure 8. Thus the adjacency matrix of $\mathbb{Z}_{pqrs}$ is as follows.

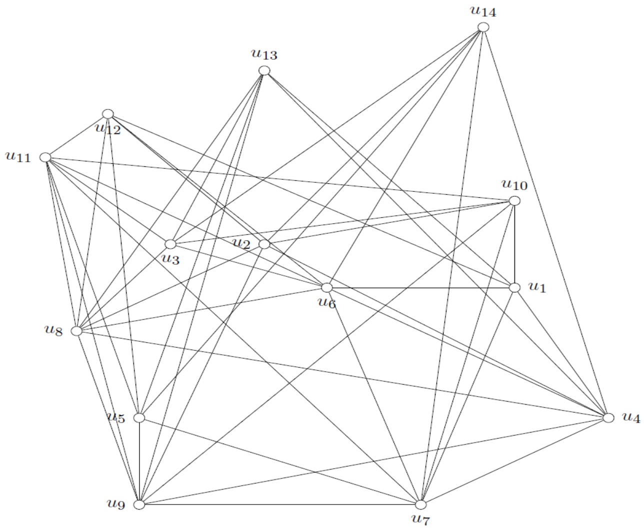

Figure 8. *Prime Ideal Sum Graph of* $\mathbb{Z}_{pqrs}$

$$AG(\mathbb{Z}_{pqrs}) = \begin{bmatrix} 0&0&0&1&0&1&1&0&0&1&0&1&1&0 \\ 0&0&0&1&0&0&0&1&1&1&0&1&0&1 \\ 0&0&0&0&0&1&0&1&0&1&1&0&1&1 \\ 1&1&0&0&0&1&1&1&1&0&0&0&1&1 \\ 0&0&0&0&0&0&1&0&1&0&1&1&1&1 \\ 1&0&1&1&0&0&1&1&0&0&1&1&0&1 \\ 1&0&0&1&1&1&0&0&1&1&1&0&0&1 \\ 0&1&1&1&0&1&0&0&1&0&1&1&1&0 \\ 0&1&0&1&1&0&1&1&0&1&1&0&1&0 \\ 1&1&1&0&0&0&1&0&1&0&1&0&0&0 \\ 0&0&1&0&1&1&1&1&1&1&0&1&0&0 \\ 1&1&0&0&1&1&0&1&0&0&1&0&0&0 \\ 1&0&1&1&1&0&0&1&1&0&0&0&0&0 \\ 0&1&1&1&1&1&1&0&0&0&0&0&0&0 \end{bmatrix}_{14x14}.$$

Therefore the eigenvalues are $\lambda_1 = 7, \lambda_{2,3,4} = 1, \lambda_5 = 0, \lambda_{6,7,8} = -2.56, \lambda_{9,10} = -2, \lambda_{11,12,13} = 1.56$ and $\lambda_{14} = -3$ from which the spectral radius $\lambda_1 = 7$. Also $d(u_1) = 6, d(u_2) = 6, d(u_3) = 6, d(u_4) = 8, d(u_5) = 6, d(u_6) = 8, d(u_7) = 8, d(u_8) = 8, d(u_9) = 8$,$d(u_{10}) = 6, d(u_{11}) = 8$,$d(u_{12}) = 6, d(u_{13}) = 6$ and $d(u_{14}) = 6$, $\sum_{i=1}^{14} \lambda_i \leq \lambda_1 \leq \frac{\sum_{u_i \in V(G)} d(u_i)}{n-2}$.

Theorem 16. Let $E(G)$ denotes the graph energy of $\mathbb{Z}_{pqrs}$. Then, $E(G) \leq \frac{5\sum_{u_i \in V(G)} d(u_i)}{n}$.

Proof. The graph energy of $\mathbb{Z}_{pqrs}$ is clear from thm 15 that $E(G) = 29.36$. Considering the number of vertices $n = 14$ and the sum of degrees of u_i, we have $E(G) \leq \frac{5\sum_{u_i \in V(G)} d(u_i)}{n}$.

Sage code algorithm for drawing $PIS(\mathbb{Z}_n)$

```
n=
V=[ ]
for i in [2..(n-1)]:
if n%i==0:
V.append(i)
E=[ ]
for a in V:
for b in V:
if gcd(a,b).is_prime()==True and a!=b:
E.append((a,b))
G=Graph()
G.add_vertices(V)
G.add_edges(E)
G.plot()
```

Acknowledgments -

Funding/Financial Disclosure The authors have no received any financial support for the research, authorship, or publication of this study.

Ethics Committee Approval and Permissions The work does not require ethics committee approval and any private permission.

Conflict of Interests The authors stated that there are no conflict of interest in this article.

Authors Contribution Authors contributed equally to the study.

References

[1] Bondy, J., & Murty, U. (1982). *Graph theory with applications*. Elsevier Science Publishing.

[2] Hogben, L. (2005). Spectral graph theory and the inverse eigenvalue problem of a graph. *The Electronic Journal of Linear Algebra*, *14*, 12–31. https://doi.org/10.13001/1081-3810.1174

[3] Bapat, R. (2013). On the adjacency matrix of a threshold graph. *Linear Algebra and its Applications*, *439*(10), 3008–3015. https://doi.org/10.1016/j.laa.2013.08.007

[4] Das, K., & Kumar, P. (2004). Some new bounds on the spectral radius of graphs. *Discrete Mathematics*, *281*(1-3), 149–161. https://doi.org/10.1016/j.disc.2003.08.005

[5] Gutman, I. (1978). The energy of a graph. *Ber Math— Statist Sekt Forschungsz Graz*, *103*, 1–22.

[6] Anderson, D., & Livingston, P. (1999). The zero-divisor graph of a commutative ring. *Journal of Algebra*, *217*, 434–447. https://doi.org/10.1006/jabr.1998.7840

[7] Beck, I. (1988). Coloring of commutative rings. *Journal of Algebra*, *116*(1), 208–226. https://doi.org/10.1016/0021-8693(88)90202-5

[8] Banerjee, S. (2022). Laplacian spectrum of comaximal graph of the ring $\mathbb{Z}_n$. *Journal of Algebra*, *10*(1), 285–298. https://doi.org/10.48550/arXiv.2005.02316

[9] Fasfous, W., Rajat, K., & Sharafdini, R. (2020). Various spectra and energies of commuting graphs of finite rings. *Hacettepe Journal of Mathematics and Statistics*, *49*(6), 1915–1925. https://doi.org/10.15672/hujms.540309

[10] Sözen, E., Eryaşar, E., & Abdioğlu, C. (2022). Forgotten topological and wiener indices of prime ideal sum graph of $\mathbb{Z}_n$. *Current Organic Synthesis*. https://doi.org/10.2174/1570179420666230606140448

[11] Saha, M., Çelikel, E., & Abdioğlu, C. (2023). Prime ideal sum graph of a commutative ring. *Hacettepe Journal of Mathematics and Statistics*, *22*(06), 2350121–. https://doi.org/10.1142/S0219498823501219