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Research Article

Harmonic maps on the tangent bundle according to the ciconia metric

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Abstract

The focus of this paper revolves around investigating the harmonicity aspects of various mappings. Firstly, we explore the harmonicity of the canonical projection $\pi : (TM, \tilde{g}) \to (M_{2n}, J, g)$, where (M_{2n}, J, g) represents an anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent bundle with the ciconia metric. Additionally, we study the harmonicity of a vector field ξ, treated as mappings from M to TM . In this context, we consider the harmonicity relations between the ciconia metric $\tilde{g}$ and the Sasaki metric ${}^{S}g$, examining their mutual interactions. Furthermore, we investigate the Schoutan-Van Kampen connection and the Vrãnceanu connection, both associated with the Levi-Civita connection of the ciconia metric. Our analysis also includes the computation of the mean connections for the Schoutan-Van Kampen and Vrãnceanu connections, thereby providing insights into their properties. Finally, our exploration extends to the second fundamental form of the identity mapping from $(TM, \tilde{g})$ to $\left(TM, \overline{\nabla}^{m}\right)$ and $\left(TM, \widetilde{\nabla}^{*m}\right)$. Here $\overline{\nabla}^{m}$ and $\widetilde{\nabla}^{*m}$ denote the mean connections associated with the Schoutan-Van Kampen and Vrãnceanu connections, respectively.

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1. Introduction

Let us start with a $2n-$dimensional Riemannian manifold M with a Riemannian metric g. In mathematical terms, a paracomplex manifold on a Riemannian manifold is an almost product manifold (M_{2n}, J), $J^2 = id$, such that the two eigenbundles $T^{+}M$ and $T^{-}M$ linked to the two eigenvalues $+1$ and -1 of J are of the same rank. The fact that the Nijenhuis tensor specified by

$$N_J(A, B) = [JA, JB] - J[JA, B] - J[A, JB] + [A, B]$$

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is zero means that an almost paracomplex structure is integrable. If an almost paracomplex structure is integrable, then it becomes a paracomplex structure. In the context of M_{2n}, an anti-paraHermitian metric is a Riemannian metric satisfying the following expression

$$g\left(JA, JB\right) = g\left(A, B\right)$$

or equivalent to this equation

$$g\left(JA, B\right) = g\left(A, JB\right) \qquad \text{(purity condition)}$$

for any vector fields A, B. The manifold M_{2n} equipped with an almost paracomplex structure and an anti-paraHermitian metric g is called an almost anti-paraHermitian manifold. It is also called anti-paraKähler if the paracomplex structure J is parallel with regard to the Levi-Civita connection (${}^{g}\nabla J = 0$). Recall that the condition ${}^{g}\nabla J = 0$ is equivalent to the paraholomorphicity of the Riemannian metric g, that is, $\check{T}_J g = 0$, where $\check{T}_J$ is the Tachibana operator applied to the Riemannian metric g [26, 31].

The upcoming discussion will center around a crucial source, referred to as [2], which serves as a primary inspiration for the current article. This paper introduces an innovative category of invariant metrics applicable to the tangent bundle of any given almost Hermitian manifold. Termed the "ciconia metric" in the work, this metric broadens its scope by encompassing both Sasaki and Yano metrics with weights. It is worth noting that each instance of the ciconia metric maintains its own status as an almost Hermitian metric.

The frame of our present work is based on harmonicity, which we briefly recall here. The concept of harmonicity is used in a wide range of fields such as differential geometry, partial differential equations, analysis, theoretical physics and so on, which initiated with harmonic functions and then generalized to harmonic maps between (semi-)Riemannian on one side, and to harmonic exterior forms on the other side. The harmonic maps, defined as critical points of the Dirichlet energy functional. This crucial result on harmonic maps was obtained by Eells-Sampson in [16]. After that, harmonicity extended in a wide range of directions, such as harmonic morphisms, (see [5]), harmonic (semi-) Riemannian metrics (see [13]), harmonic sections [12, 14, 24, 25], harmonic endomorphisms and $(1, 1)$-tensor fields (see [8, 9, 18]), harmonic connections [8, 17], quasi-harmonic maps and morphisms (see [4, 7]), etc.

Harmonic mapping is a technique used in complex analysis, mathematical physics, and harmonic analysis. It is employed to analyze and model electromagnetic fields, fluid behavior, heat transfer, and various other phenomena. By decomposing electromagnetic fields into their harmonic components, harmonic mapping enables a better understanding of their behavior. In fluid mechanics, it helps analyze fluid motion and interactions. In heat transfer, it models temperature distribution and thermal radiation. Harmonic mapping also has applications in vibration analysis, audio signal analysis and synthesis, image compression, and recognition. Overall, it is a versatile tool for mathematical modeling, system analysis, and signal processing.

Harmonic vector fields, viewed as mappings from the base manifold (M, g) to its tangent bundle TM endowed with a (semi-)Riemannian metric g, hold a significant position in the theory of harmonic maps. The initial investigations in this area were conducted by Nouhaud [21], Ishihara [20] and Piu [25], wherein they considered the complete lift of g or the Sasaki metric ${}^{S}g$ on TM (see also [23]). After that, the harmonicity of vector fields and unit vector fields was studied for the tangent bundle endowed with metrics of natural lift type (see [6]). Let (M_{2k}, φ, g) be an anti-paraKähler manifold and (TM, g_{BS}) be its tangent bundle with a Berger type deformed Sasaki metric g_{BS}. Altunbas, Simsek and Gezer explored the harmonicity of the canonical projection $\pi : (TM, g_{BS}) \to (M_{2k}, \varphi, g)$ and a vector field ξ treated as a mapping $\xi : (M_{2k}, \varphi, g) \to (TM, g_{BS})$ (see [3]). This paper focuses on addressing issues related to the harmonicity of vector fields and the canonical projection, specifically within the context of the ciconia metric on TM.

Throughout the entirety of this paper, we consistently assume that all manifolds, tensor fields, and connections are differentiable of class C^∞. Additionally, we represent the set of all tensor fields of type (p,q) on M as $\Im_q^p(M)$. The convention of Einstein summation is employed, with the indices i,j,s ranging over $\{1,2,...,n\}$.

1.1. The ciconia metric on tangent bundle

Consider an n-dimensional Riemannian manifold M_n having a Riemannian metric g and TM its tangent bundle. This article employs the C^∞-category to provide a comprehensive explanation, focusing on connected manifolds. The natural projection $\pi : TM \to M_n$ is considered, with particular attention to systems of local coordinates. When a system of local coordinates (U, x^i) is established in M_n, it induces a corresponding system of local coordinates $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = u^i\right)$ in TM, where $\bar{i} = n+i = n+1,...,2n$. Here, (u^i) represent the cartesian coordinates in each tangent space T_PM of $\forall p \in U$. Also, p is an arbitrary point on U.

The Levi-Civita connection of the Riemannian metric g is denoted as ∇. Within the horizontal distribution defined by ∇ and the vertical distribution defined by $\ker \pi_*$, the local frame is provided as follows:

$$E_i = \frac{\partial}{\partial x^i} - u^s \Gamma_{is}^h \frac{\partial}{\partial u^h};\ i = 1,...,n,$$

and

$$E_{\bar{i}} = \frac{\partial}{\partial u^i};\ \bar{i} = n+1,...,2n,$$

where Γ_{is}^h denote the Christoffel's symbols of g. The local frame $\{E_\beta\} = \left(E_i, E_{\bar{i}}\right)$ is referred to as the adapted frame. Consider a vector field $X = X^i \frac{\partial}{\partial x^i}$. The horizontal and vertical lifts of X are determined with respect to the adapted frame as follows: [32]

$$\begin{aligned} {}^H X &= X^i E_i \\ {}^V X &= X^i E_{\bar{i}}. \end{aligned}$$

In TM, the local 1−form system $\left(dx^i, \delta u^i\right)$ serves as the dual frame to the adapted frame $\{E_\beta\}$, where

$$\delta u^i = {}^H\left(dx^i\right) = du^i + u^s \Gamma_{hs}^i dx^h.$$

Definition 1.1. Let (M_{2n}, J, g) be an almost anti-paraHermitian manifold equipped with an almost paracomplex structure J and a Riemannian metric g. Consider TM as its tangent bundle. The ciconia metric $\tilde{g}$ on the tangent bundle TM is defined as follows:

$$\begin{aligned} i)\ \tilde{g}\left({}^V A, {}^V B\right) &= {}^V(bg(A,B)), \\ ii)\ \tilde{g}\left({}^V A, {}^H B\right) &= {}^V(G(A,B)) = {}^V(g(JA,B)), \\ iii)\ \tilde{g}\left({}^H A, {}^H B\right) &= {}^V(ag(A,B)) \end{aligned}$$

for all vector fields A, B on M_{2n}, where $G(A,B) = g(JA,B)$ represents the twin metric and a, b are positive constants [19].

Within the context of the manifold (M,g) and its corresponding tangent bundle TM, various Riemannian or pseudo-Riemannian metrics have been formulated by leveraging the natural lifts of the underlying Riemannian metric g. When used in this manner, these metrics are referred to as g−natural metrics. In [1], the authors have meticulously established the family of all Riemannian g-natural metrics, which depend on six arbitrary functions related to the norm of a vector $u \in TM$. Interestingly, the ciconia metric, which plays a pivotal role in this context, also derived through the lifts of the Riemannian metric and the twin metric defined on the base manifold. From this perspective, the ciconia metric can be considered a natural metric. However, it is important to note that the ciconia metric does not fall within the category of g-natural metrics presented in [1]. This distinction underscores the uniqueness of the ciconia metric and its divergence from the established g-natural metric class. Consequently, this brings to light a novel category within the realm of the Riemannian geometry of tangent bundles, showcasing the richness and diversity of possible metric structures. In the specially chosen adapted frame $\{E_\beta\}$, the explicit expressions for both the ciconia metric and its inverse can be stated as follows:

$$(\tilde{g}_{\gamma\beta}) = \begin{pmatrix} a g_{ji} & G_{ji} \\ G_{ji} & b g_{ji} \end{pmatrix}$$

and

$$(\tilde{g}^{\gamma\varepsilon}) = \begin{pmatrix} \frac{b}{\alpha} g^{jk} & \frac{-1}{\alpha} G^{jk} \\ \frac{-1}{\alpha} G^{jk} & \frac{a}{\alpha} g^{jk} \end{pmatrix},$$

where $\alpha = a.b - 1 \neq 0$, the twin metric $G(A, B) = g(JA, B)$ is locally expressed as $G_{ji} = g_{jh} J_i^h$.

We proceed by determining the Levi-Civita connection $\tilde{\nabla}$ associated with the ciconia metric $\tilde{g}$. The coefficients of this connection ([29]) can be computed using the following procedure:

$$\tilde{\Gamma}^\alpha_{\gamma\beta} = \frac{1}{2}\tilde{g}^{\alpha\varepsilon}(E_\gamma \tilde{g}_{\varepsilon\beta} + E_\beta \tilde{g}_{\gamma\varepsilon} - E_\varepsilon \tilde{g}_{\gamma\beta}) + \frac{1}{2}({\Omega_{\gamma\beta}}^{\alpha} + \Omega^\alpha_{\gamma\beta} + \Omega^\alpha_{\beta\gamma}),$$

where

$$\begin{cases} \Omega^\alpha_{\gamma\beta} = \tilde{g}^{\alpha\varepsilon}\tilde{g}_{\delta\beta}{\Omega_{\varepsilon\gamma}}^{\delta}, \\ {\Omega_{ji}}^{\overline{h}} = -{\Omega_{ij}}^{\overline{h}} = -{\mathcal{R}_{jis}}^{h} y^s, \\ {\Omega_{j\overline{i}}}^{\overline{h}} = -{\Omega_{\overline{i}j}}^{\overline{h}} = \Gamma^h_{ji} \end{cases}$$

and it will be used as $\gamma = j, \overline{j}$ $\quad \beta = i, \overline{i}$ $\quad \alpha = h, \overline{h}$ $\quad \varepsilon = k, \overline{k}$ $\quad \delta = m, \overline{m}$.

Regarding the Levi-Civita connection $\tilde{\nabla}$ associated with the ciconia metric $\tilde{g}$, we present the following proposition:

Proposition 1.2. *Consider an anti-paraKähler manifold (M_{2n}, J, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the ciconia metric. The local expression for the Levi-Civita connection $\tilde{\nabla}$ of the ciconia metric $\tilde{g}$ on TM is as follows:*

$$\begin{aligned}
\tilde{\nabla}_{E_j} E_i &= \left(\Gamma^h_{ji} - \frac{b}{2\alpha} u^s J^t_j \left({R_{tsi}}^{h} + {R_{ist}}^{h}\right)\right) E_h \\
&\quad + \left(\frac{1}{2\alpha} u^s {R_{jsi}}^{h} + \frac{1}{2\alpha} u^s {R_{isj}}^{h} - \frac{1}{2} u^s {R_{jis}}^{h}\right) E_{\overline{h}}, \\
\tilde{\nabla}_{E_j} E_{\overline{i}} &= \left(\frac{b^2}{2\alpha} u^s {R_{sij}}^{h}\right) E_h + \left(\Gamma^h_{ji} + \frac{b}{2\alpha} u^s J^t_j {R_{ist}}^{h}\right) E_{\overline{h}}, \\
\tilde{\nabla}_{E_{\overline{j}}} E_i &= \left(\frac{b^2}{2\alpha} u^s {R_{sji}}^{h}\right) E_h + \left(\frac{b}{2\alpha} u^s J^t_j {R_{tsi}}^{h}\right) E_{\overline{h}}, \\
\tilde{\nabla}_{E_{\overline{j}}} E_{\overline{i}} &= 0,
\end{aligned}$$

where R is the Riemannian curvature tensor of g.

2. Main results

Consider two Riemannian manifolds, denoted as (M, g) and $(\widetilde{M}, \tilde{g})$, where M has dimension n and $\widetilde{M}$ has dimension m. Let us focus on a smooth map $f : M \to \widetilde{M}$. In this context, we utilize the Christoffel symbols $\widetilde{\Gamma}^\alpha_{\beta\gamma}$ associated with the metric $\tilde{g}$ and by $\widetilde{\nabla}$ represents the Levi-Civita connection corresponding to $\tilde{g}$. With respect to the local charts

(U, x^i), $i = 1, ..., n$, around a point $p \in M$ and (V, u^α), $\alpha = 1, ..., m$, around a point $f(p)$. The second fundamental form of f at p, denoted by $\beta(f)_p$, can be expressed locally as follows:

$$\beta(f)_p \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \left(f_{ij}^\alpha - \Gamma_{ij}^k f_k^\alpha + \widetilde{\Gamma}_{\beta\gamma}^\alpha f_i^\beta f_j^\gamma \right) \frac{\partial}{\partial u^\alpha}, \tag{2.1}$$

where $f_k^\alpha = \frac{\partial f^\alpha}{\partial x^k}$ and $f_{ij}^\alpha = \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j}$. The form $\beta(f)$ is $C^\infty(M)$ bilinear and symmetric. It is important to note that the map f is considered totally geodesic if and only if $\beta(f) = 0$.

Moving forward, we define the tension field $\tau(f)$ associated with the map f as:

$$\tau(f) = tr_g \ \beta(f) = g^{ij} \beta(f) \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right). \tag{2.2}$$

A notable case arises when the tension field $\tau(f)$ equals zero, in which case we label the map f as a harmonic map (as described in [15]). Furthermore, if we have a vector field $X \in \chi(M)$ that is $f-$related with $\widetilde{X} \in \chi\left(\widetilde{M}\right)$ (meaning $f_{*,p} X = \widetilde{X}(f(p)), \forall p \in M$) and a similar relationship for vector field $Y \in \chi(M)$, then we can simplify the expression for the second fundamental form as follows:

$$\beta(f)_p (X, Y) = \left(\widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} \right)_{f(p)} - f_{*,p} (\nabla_X Y). \tag{2.3}$$

2.1. Harmonicity of the canonical projection $\pi : TM \to M$

We will now explore the previously discussed extensions within the context of the map $\pi : (TM, \tilde{g}) \to (M_{2n}, J, g)$, where (M_{2n}, J, g) represents an anti-paraKähler manifold and $(TM, \tilde{g})$ denotes its tangent bundle equipped with the ciconia metric. Our calculations will be conducted with respect to the adapted frame. By combining the expression (2.1) and Proposition 1.2, we arrive at the following result:

$$\begin{aligned}
\beta(\pi)(E_i, E_j) &= \left(E_i E_j \pi^\gamma - \widetilde{\Gamma}_{ij}^k E_k \pi^\gamma - \widetilde{\Gamma}_{ij}^{\bar{k}} E_{\bar{k}} \pi^\gamma + \Gamma_{\alpha\beta}^\gamma E_i \pi^\alpha E_j \pi^\beta \right) \frac{\partial}{\partial x^\gamma} \\
&= \left(\underbrace{E_i \overbrace{E_j \pi^\gamma}^{\delta_j^\gamma}}_{0} - \left(\Gamma_{ij}^k + \frac{b}{2\alpha} y^s J_i^t (R_{stj}{}^k + R_{sjt}{}^k) \right) E_k \pi^\gamma + \Gamma_{\alpha\beta}^\gamma E_i \pi^\alpha E_j \pi^\beta \right) \frac{\partial}{\partial x^\gamma} \\
&= \left(-\Gamma_{ij}^\gamma - \frac{b}{2\alpha} y^s J_i^t (R_{stj}{}^\gamma + R_{sjt}{}^\gamma) + \Gamma_{ij}^\gamma \right) \frac{\partial}{\partial x^\gamma} \\
&= \frac{b}{2\alpha} y^s J_i^t (R_{tsj}{}^\gamma + R_{jst}{}^\gamma) \frac{\partial}{\partial x^\gamma},
\end{aligned}$$

$$\begin{aligned}
\beta(\pi)\left(E_{\bar{i}}, E_{\bar{j}}\right) &= \left(E_{\bar{i}} E_{\bar{j}} \pi^\gamma - \widetilde{\Gamma}_{\bar{i}\bar{j}}^k E_k \pi^\gamma - \widetilde{\Gamma}_{\bar{i}\bar{j}}^{\bar{k}} E_{\bar{k}} \pi^\gamma + \Gamma_{\alpha\beta}^\gamma E_{\bar{i}} \pi^\alpha E_{\bar{j}} \pi^\beta \right) \frac{\partial}{\partial x^\gamma} \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\beta(\pi)(E_{\bar{i}}, E_j) &= \left(E_{\bar{i}} E_j \pi^\gamma - \widetilde{\Gamma}_{\bar{i}j}^k E_k \pi^\gamma - \widetilde{\Gamma}_{\bar{i}j}^{\bar{k}} E_{\bar{k}} \pi^\gamma + \Gamma_{\alpha\beta}^\gamma E_{\bar{i}} \pi^\alpha E_j \pi^\beta \right) \frac{\partial}{\partial x^\gamma} \\
&= -\frac{b^2}{2\alpha} y^s R_{sij}{}^k \delta_k^\gamma \frac{\partial}{\partial x^\gamma} \\
&= \frac{b^2}{2\alpha} y^s R_{isj}{}^\gamma \frac{\partial}{\partial x^\gamma}.
\end{aligned}$$

As a result, the following conclusion can be drawn.

Theorem 2.1. *In the context of an anti-paraKähler manifold (M_{2n}, J, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the ciconia metric, the Riemannian submersion $\pi : (TM, \tilde{g}) \to (M_{2n}, J, g)$ is totally geodesic if and only if the anti-paraKähler manifold (M_{2n}, J, g) is locally flat. Moreover, the mapping π is a harmonic map.*

Consider another anti-paraHermitian metric h on the manifold M_{2n} with respect to an almost paracomplex structure J_1. Let us examine the projection $\pi : (TM, \tilde{g}) \to (M_{2n}, J_1, h)$. Then we arrive at the following result.

$$\beta(\pi)\left(E_{\bar{i}}, E_{\bar{j}}\right) = \left(E_{\bar{i}}E_{\bar{j}}\pi^{\gamma} - \widetilde{\Gamma}^{k}_{\bar{i}\bar{j}}E_k\pi^{\gamma} + {}^{h}\Gamma^{\gamma}_{\alpha\beta}E_{\bar{i}}\pi^{\alpha}E_{\bar{j}}\pi^{\beta}\right)\frac{\partial}{\partial x^{\gamma}} = 0,$$

$$\begin{aligned}
\beta(\pi)\left(E_{\bar{i}}, E_j\right) &= \left(E_{\bar{i}}E_j\pi^{\gamma} - \widetilde{\Gamma}^{k}_{\bar{i}j}E_k\pi^{\gamma} + {}^{h}\Gamma^{\gamma}_{\alpha\beta}E_{\bar{i}}\pi^{\alpha}E_j\pi^{\beta}\right)\frac{\partial}{\partial x^{\gamma}} \\
&= \frac{b^2}{2\alpha}y^s R_{isj}{}^{\gamma}\frac{\partial}{\partial x^{\gamma}},
\end{aligned}$$

$$\begin{aligned}
\beta(\pi)\left(E_i, E_j\right) &= \left(E_iE_j\pi^{\gamma} - \widetilde{\Gamma}^{k}_{ij}E_k\pi^{\gamma} + {}^{h}\Gamma^{\gamma}_{\alpha\beta}E_i\pi^{\alpha}E_j\pi^{\beta}\right)\frac{\partial}{\partial x^{\gamma}} \\
&= \left({}^{h}\Gamma^{\gamma}_{ij} - \Gamma^{\gamma}_{ij} + \frac{b}{2\alpha}y^s J^t_i(R_{jst}{}^{\gamma} + R_{tsj}{}^{\gamma})\right)\frac{\partial}{\partial x^{\gamma}},
\end{aligned}$$

where ${}^{h}\Gamma^{\gamma}_{ij}$ are the Christoffel symbols with respect to the metric h. Hence we get proposition below.

Proposition 2.2. *Consider an anti-paraKähler manifold (M_{2n}, J, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the ciconia metric. The map $\pi : (TM, \tilde{g}) \to (M_{2n}, J_1, h)$ is totally geodesic if and only if the anti-paraKähler manifold (M_{2n}, J, g) is locally flat and the map $I : (M_{2n}, J, g) \to (M_{2n}, J_1, h)$ is totally geodesic.*

It is worth noting that if the map $\pi : (TM, \tilde{g}) \to (M_{2n}, J_1, h)$ is totally geodesic, it implies that both (M_{2n}, J, g) and (M_{2n}, J_1, h) are locally flat.

Let g and h represent two anti-paraHermitian metrics on M_{2n}. The metric h is considered to be harmonic with respect to g if the equation $g^{ij}\left({}^{h}\Gamma^{k}_{ij} - \Gamma^{k}_{ij}\right) = 0$ holds ([13]). By employing $\tau(\pi) = trace(\beta(\pi))$, we arrive at the following

$$\begin{aligned}
\tau(\pi) &= \tilde{g}^{\alpha\beta}\beta(\pi)(E_{\alpha}, E_{\beta}) \\
&= \tilde{g}^{ij}\beta(\pi)(E_i, E_j) + \tilde{g}^{i\bar{j}}\beta(\pi)\left(E_i, E_{\bar{j}}\right) + \tilde{g}^{\bar{i}j}\beta(\pi)\left(E_{\bar{i}}, E_j\right) \\
&\quad + \tilde{g}^{\bar{i}\bar{j}}\beta(\pi)\left(E_{\bar{i}}, E_{\bar{j}}\right) \\
&= \frac{b}{\alpha}g^{ij}\beta(\pi)(E_i, E_j) - \frac{2}{\alpha}G^{ij}\beta(\pi)\left(E_i, E_{\bar{j}}\right) \\
&= \left[\frac{b}{\alpha}g^{ij}\left({}^{h}\Gamma^{\gamma}_{ij} - \Gamma^{\gamma}_{ij} + \frac{b}{2\alpha}y^s J^t_i(R_{jst}{}^{\gamma} + R_{tsj}{}^{\gamma})\right) - \frac{b^2}{\alpha^2}G^{ij}y^s R_{isj}{}^{\gamma}\right]\frac{\partial}{\partial x^{\gamma}} \\
&= \left[\frac{b}{\alpha}g^{ij}({}^{h}\Gamma^{\gamma}_{ij} - \Gamma^{\gamma}_{ij}) + \frac{b^2}{\alpha^2}y^s\left(g^{ij}J^{\gamma}_l R_{jsi}{}^{l} - G^{ij}R_{isj}{}^{\gamma}\right)\right]\frac{\partial}{\partial x^{\gamma}} \\
&= \left[\frac{b}{\alpha}g^{ij}({}^{h}\Gamma^{\gamma}_{ij} - \Gamma^{\gamma}_{ij})\right]\frac{\partial}{\partial x^{\gamma}},
\end{aligned}$$

where $\alpha = i, \bar{i}$ $\beta = j, \bar{j}$; $i, j = 1, ..., 2n$ and $\bar{i}, \bar{j} = 2n+1, ..., 4n$. Consequently we have the following.

Proposition 2.3. *Consider (M_{2n}, J, g) as an anti-paraKähler manifold and $(TM, \tilde{g})$ as its tangent bundle equipped with the ciconia metric. The mapping $\pi : (TM, \tilde{g}) \to (M_{2n}, J_1, h)$ is harmonic if and only if the metric h is harmonic with respect to g.*

2.2. Schouten-Van Kampen and Vranceanu connections

The Schouten-Van Kampen connection was introduced in the third decade of the previous century to investigate non-holomorphic manifolds, driven by the necessity for a geometric interpretation of non-holomorphic mechanical systems [28, 30]. Bejancu [10] conducted a study of the Schouten-Van Kampen connection on foliated manifolds. More recently, Olszak [22] explored the Schouten-Van Kampen connection in the context of almost (para) contact metric structures. Through this connection, certain classes of almost (para) contact metric manifolds were characterized, and specific curvature properties of this connection were identified. Let $\overline{\nabla}$ (resp. $\overline{\nabla}^m$) represent the Schouten-Van Kampen connection (and the mean connection of the Schouten-Van Kampen connection) associated with the Levi-Civita connection of the ciconia metric $\widetilde{g}$. The Schouten-Van Kampen connection $\overline{\nabla}$ linked with $\widetilde{\nabla}$ is defined as follows:

$$\overline{\nabla}_{\widetilde{X}}\widetilde{Y} = V\widetilde{\nabla}_{\widetilde{X}}V\widetilde{Y} + H\widetilde{\nabla}_{\widetilde{X}}H\widetilde{Y},$$

where $\widetilde{X}, \widetilde{Y}$ are vector fields on TM, and V and H represent the vertical and horizontal projections.

The results derived in this section will be employed later in this paper.

Proposition 2.4. *Consider an anti-paraKähler manifold (M_{2n}, J, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the ciconia metric $\widetilde{g}$. Let $\overline{\nabla}$ be the Schouten-Van Kampen connection associated with the Levi-Civita connection of the ciconia metric $\widetilde{g}$. The components of $\overline{\nabla}$ in local coordinates are given by:*

$$\begin{aligned}
\overline{\nabla}_{E_i}E_j &= \left(\Gamma^k_{ij} + \frac{b}{2\alpha}y^s J^t_i[R_{stj}{}^k + R_{sjt}{}^k]\right)E_k,\\
\overline{\nabla}_{E_i}E_{\overline{j}} &= \left(\Gamma^k_{ij} + \frac{b}{2\alpha}y^s J^k_l R_{jsi}{}^l\right)E_{\overline{k}},\\
\overline{\nabla}_{E_{\overline{i}}}E_j &= \frac{b^2}{2\alpha}y^s R_{sij}{}^k E_k,\\
\overline{\nabla}_{E_{\overline{i}}}E_{\overline{j}} &= 0.
\end{aligned}$$

Proof. If the Proposition 1.2 is used in equation $\overline{\nabla}_{\widetilde{X}}\widetilde{Y} = V\widetilde{\nabla}_{\widetilde{X}}V\widetilde{Y} + H\widetilde{\nabla}_{\widetilde{X}}H\widetilde{Y}$ we have

$$\begin{aligned}
i.\quad \overline{\nabla}_{E_i}E_j &= V\widetilde{\nabla}_{E_i}VE_j + H\widetilde{\nabla}_{E_i}HE_j = H\widetilde{\nabla}_{E_i}E_j\\
&= \left(\Gamma^k_{ij} + \frac{b}{2\alpha}y^s J^t_i[R_{stj}{}^k + R_{sjt}{}^k]\right)E_k.\\
ii.\quad \overline{\nabla}_{E_i}E_{\overline{j}} &= V\widetilde{\nabla}_{E_i}VE_{\overline{j}} + H\widetilde{\nabla}_{E_i}HE_{\overline{j}} = V\widetilde{\nabla}_{E_i}E_{\overline{j}}\\
&= \left(\Gamma^k_{ij} + \frac{b}{2\alpha}y^s J^k_l R_{jsi}{}^l\right)E_{\overline{k}}.\\
iii.\quad \overline{\nabla}_{E_{\overline{i}}}E_j &= V\widetilde{\nabla}_{E_{\overline{i}}}VE_j + H\widetilde{\nabla}_{E_{\overline{i}}}HE_j\\
&= H\widetilde{\nabla}_{E_{\overline{i}}}E_j = \left(\frac{b^2}{2\alpha}y^s R_{sij}{}^k\right)E_k.\\
iv.\quad \overline{\nabla}_{E_{\overline{i}}}E_{\overline{j}} &= V\widetilde{\nabla}_{E_{\overline{i}}}VE_{\overline{j}} + H\widetilde{\nabla}_{E_{\overline{i}}}HE_{\overline{j}}\\
&= V\widetilde{\nabla}_{E_{\overline{i}}}E_{\overline{j}} = 0.
\end{aligned}$$

□

Proposition 2.5. *Consider the torsion tensor $\overline{T}$ of the Schouten-Van Kampen connection $\overline{\nabla}$ associated with the Levi-Civita connection $\widetilde{\nabla}$ of the ciconia metric $\widetilde{g}$. The components* *of $\overline{T}$ in local coordinates are given by:*

$$\begin{aligned}
\overline{T}\left(E_i, E_j\right) &= -y^s R_{jis}{}^k E_{\overline{k}},\\
\overline{T}\left(E_{\overline{i}}, E_j\right) &= \frac{b^2}{2\alpha} y^s R_{sij}{}^k E_k - \frac{b}{2\alpha} y^s J_l^k R_{isj}{}^l E_{\overline{k}},\\
\overline{T}\left(E_i, E_{\overline{j}}\right) &= -\frac{b^2}{2\alpha} y^s R_{sji}{}^k E_k + \frac{b}{2\alpha} y^s J_l^k R_{jsi}{}^l E_{\overline{k}},\\
\overline{T}\left(E_{\overline{i}}, E_{\overline{j}}\right) &= 0.
\end{aligned}$$

Proof. By utilizing the expression for the torsion tensor, namely $\overline{T}(X, Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y]$ and performing the required tensor computations, the proof of the proposition immediately follows. □

Proposition 2.6. *Let $\overline{T}$ denote the torsion tensor of the Schouten-Van Kampen connection $\overline{\nabla}$. The mean connection of the Schouten-Van Kampen connection $\overline{\nabla}$, denoted as $\overline{\nabla}^m$, is defined as $\overline{\nabla}^m = \overline{\nabla} - \frac{1}{2}\overline{T}$. The local components of $\overline{\nabla}^m$ are given as follows:*

$$\begin{aligned}
\overline{\nabla}^m_{E_i} E_j &= \left(\Gamma_{ij}^k + \frac{b}{2\alpha} y^s J_i^t [R_{stj}{}^k + R_{sjt}{}^k]\right) E_k - \frac{1}{2} y^s R_{ijs}{}^k E_{\overline{k}},\\
\overline{\nabla}^m_{E_{\overline{i}}} E_j &= \frac{b^2}{4\alpha} y^s R_{sij}{}^k E_k + \frac{b}{4\alpha} y^s J_l^k R_{isj}{}^l E_{\overline{k}},\\
\overline{\nabla}^m_{E_i} E_{\overline{j}} &= \frac{b^2}{4\alpha} y^s R_{sji}{}^k E_k + \left(\Gamma_{ij}^k + \frac{b}{4\alpha} y^s J_l^k R_{jsi}{}^l\right) E_{\overline{k}},\\
\overline{\nabla}^m_{E_{\overline{i}}} E_{\overline{j}} &= 0.
\end{aligned}$$

In [30], Vrănceanu introduced a linear connection tailored for the investigation of differential geometry on non-holonomic manifolds. The connection is called Vrănceanu connection. Then, Bejancu and Farran [11] demonstrated that the Vrănceanu connection, originally designed for non-holonomic manifolds, is applicable to the study of foliated manifolds. Their proof establishes that a foliation achieves total geodesy with a bundle-like metric if and only if the Vrănceanu connection functions as a metric connection. Introducing the concept of a foliated Riemannian manifold with constant transversal Vrănceanu curvature, as well as the concept of a transversal Einstein foliated Riemannian manifold, they conducted an in-depth exploration of the geometry associated with these two classes of manifolds. Furthermore, they determined the relationship between them. Now, let $\widetilde{\nabla}^*$ be the Vrănceanu connection associated with the Levi-Civita connection of the ciconia metric $\widetilde{g}$. The Vrănceanu connection $\widetilde{\nabla}^*$ associated with $\widetilde{\nabla}$ is defined by

$$\widetilde{\nabla}^*_{\widetilde{X}} \widetilde{Y} = H \widetilde{\nabla}_{H\widetilde{X}} H\widetilde{Y} + V \widetilde{\nabla}_{V\widetilde{X}} V\widetilde{Y} + H\left[V\widetilde{X}, H\widetilde{Y}\right] + V\left[H\widetilde{X}, V\widetilde{Y}\right], \tag{2.4}$$

where $\widetilde{X}, \widetilde{Y}$ are vector fields on TM, and V and H are the vertical and horizontal projections. Taking into the Levi-Civita connection of the ciconia metric $\widetilde{g}$ in (2.4), with the help of Proposition 1.2, we have the coefficients of the Vranceanu connection $\widetilde{\nabla}^*$ associated with $\widetilde{\nabla}$ as follows:

$$\begin{aligned}
\widetilde{\nabla}^*_{E_i} E_j &= \left[\Gamma_{ij}^k - \frac{b}{2\alpha} y^s J_i^t \left(R_{tsj}{}^k + R_{jst}{}^k\right)\right] E_k,\\
\widetilde{\nabla}^*_{E_i} E_{\overline{j}} &= \Gamma_{ji}^k E_{\overline{k}},\\
\widetilde{\nabla}^*_{E_{\overline{i}}} E_j &= 0, \quad \widetilde{\nabla}^*_{E_{\overline{i}}} E_{\overline{j}} = 0.
\end{aligned}$$

Proposition 2.7. *Consider the torsion tensor $\widetilde{T}^*$ of the Vrănceanu connection $\widetilde{\nabla}^*$ associated with the Levi-Civita connection $\widetilde{\nabla}$ of the ciconia metric $\widetilde{g}$. The components of $\widetilde{T}^*$*

in local coordinates are given by:

$$\begin{aligned}\widetilde{T}^{*}\left(E_{i},E_{j}\right) &= y^{s}R_{ijs}{}^{k}E_{\overline{k}},\\ \widetilde{T}^{*}\left(E_{\overline{i}},E_{j}\right) &= \widetilde{T}^{*}\left(E_{i},E_{\overline{j}}\right)=\widetilde{T}^{*}\left(E_{\overline{i}},E_{\overline{j}}\right)=0.\end{aligned}$$

Proposition 2.8. *Let $\widetilde{T}^{*}$ denote the torsion tensor of the Vrănceanu connection $\widetilde{\nabla}^{*}$. The mean connection of the Vranceanu connection $\widetilde{\nabla}^{*}$, denoted as $\overline{\nabla}^{m}$, is defined as $\widetilde{\nabla}^{*m}=\widetilde{\nabla}^{*}-\frac{1}{2}\widetilde{T}^{*}$. The local components of $\widetilde{\nabla}^{*m}$ are given as follows:*

$$\begin{aligned}\widetilde{\nabla}^{*m}_{E_{i}}E_{j} &= \left[\Gamma^{k}_{ij}-\frac{b}{2\alpha}y^{s}J^{t}_{i}\left(R_{tsj}{}^{k}+R_{jst}{}^{k}\right)\right]E_{k}+\frac{1}{2}y^{s}R_{jis}{}^{k}E_{\overline{k}},\\ \widetilde{\nabla}^{*m}_{E_{i}}E_{\overline{j}} &= \Gamma^{k}_{ij}E_{\overline{k}},\\ \widetilde{\nabla}^{*m}_{E_{\overline{i}}}E_{j} &= \widetilde{\nabla}^{*m}_{E_{\overline{i}}}E_{\overline{j}}=0.\end{aligned}$$

2.3. Harmonicity of the map $\xi : M\to TM$

Proposition 2.9. *In the context of an anti-paraKähler manifold (M_{2n},J,g) and its tangent bundle $(TM,\tilde{g})$ equipped with the ciconia metric, $\xi:(M_{2n},J,g)\to(TM,\tilde{g})$ is an isometric immersion if and only if $(\mathcal{L}_{J\xi}g)(X,Y)=(1-a)g(X,Y)-bg(\nabla_{X}\xi,\nabla_{Y}\xi)$, where $(\mathcal{L}_{J\xi}g)$ is the Lie derivative of $J\xi$ with respect to g.*

Proof. For $\forall p\in M$, the following equation holds

$$\xi_{*,p}X=\left({}^{H}X+\ {}^{V}(\nabla_{X}\xi)\right)_{\xi(p)}.$$

If we suppose

$$\begin{aligned}\overset{1}{g}(X,Y) &= \tilde{g}_{\xi(p)}\left(\xi_{*,p}X,\xi_{*,p}Y\right)\\ &= \tilde{g}_{\xi(p)}\left(\left({}^{H}X+\ {}^{V}(\nabla_{X}\xi)\right)_{\xi(p)},\left({}^{H}Y+\ {}^{V}(\nabla_{Y}\xi)\right)_{\xi(p)}\right)\\ &= \tilde{g}_{\xi(p)}\left({}^{H}X,{}^{V}(\nabla_{Y}\xi)_{\xi(p)}\right)+\tilde{g}_{\xi(p)}\left({}^{V}(\nabla_{X}\xi)_{\xi(p)},{}^{H}Y\right)\\ &\quad+\tilde{g}_{\xi(p)}\left({}^{V}(\nabla_{X}\xi)_{\xi(p)},{}^{V}(\nabla_{Y}\xi)_{\xi(p)}\right)+\tilde{g}_{\xi(p)}\left({}^{H}X,{}^{H}Y\right)\\ &= ag(X,Y)+g(JX,\nabla_{Y}\xi)+g(\nabla_{X}\xi,JY)+bg(\nabla_{X}\xi,\nabla_{Y}\xi)\\ &= ag(X,Y)+(\mathcal{L}_{J\xi}g)(X,Y)+bg(\nabla_{X}\xi,\nabla_{Y}\xi).\end{aligned}$$

ξ is isometric immersion if and only if $\overset{1}{g}=g$. Then the result follows. □

Remark 2.10. Unlike the typical Sasaki metric and certain other natural metrics, there are cases where non-parallel vector fields $\xi:(M_{2n},J,g)\to(TM,\tilde{g})$ can serve as isometric immersions.

Example 2.11. Let $(\mathbb{R}^{2},J,<>)$ be the real space equipped with the standard metric, J is the canonical paracomplex structure defined by

$$J(\partial x)=\partial y \qquad J(\partial y)=\partial x$$

For $\xi=-x\partial y-y\partial x$ we have $J(\xi)=-x\partial x-y\partial y$ and

$$\nabla_{X}J\xi=-X \quad \forall X\in\chi\left(\mathbb{R}^{2}\right)$$

A direct calculation shows that for all $a+b=3$ we have

$$\tilde{g}_{\xi(p)}\left(\xi_{*,p}X,\xi_{*,p}Y\right)=g(X,Y) \quad \forall X,Y\in\chi\left(\mathbb{R}^{2}\right)$$

which means that ξ is an isometric immersion.

Proposition 2.12. *In the context of an anti-paraKähler manifold (M_{2n}, J, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the ciconia metric, the second fundamental form $\beta(\xi)$ and the tension field $\tau(\xi)$ of the map $\xi : (M_{2n}, J, g) \to (TM, \tilde{g})$ can be explicitly stated as follows:*

$$\begin{aligned}
\beta(\xi)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) &= \left[\frac{b}{2\alpha}\xi^s J_l^\gamma (R_{sij}^{\ \ \ l} + R_{sji}^{\ \ \ l}) + \frac{b^2}{2\alpha}\xi^s [R_{smj}^{\ \ \ \ \gamma}(\nabla_i \xi^m) + R_{sni}^{\ \ \ \ \gamma}(\nabla_j \xi^n)]\right] E_\gamma \\
&+ \left[\nabla_i \nabla_j \xi^\gamma - \frac{1}{2}\xi^s R_{ijs}^{\ \ \ \gamma} + \frac{1}{2\alpha}\xi^s \left(R_{isj}^{\ \ \ \gamma} + R_{jsi}^{\ \ \ \gamma}\right)\right. \\
&\left. + \frac{b}{2\alpha} J_l^\gamma \xi^s \left(R_{msj}^{\ \ \ \ l} \nabla_i \xi^m + R_{nsi}^{\ \ \ \ l} \nabla_j \xi^n\right)\right] E_{\overline{\gamma}},
\end{aligned}$$

$$\begin{aligned}
\tau(\xi) &= \xi^s g^{ij}\left[\frac{b}{\alpha} J_l^\gamma R_{sij}^{\ \ \ l} + \frac{b^2}{\alpha} R_{smj}^{\ \ \ \ \gamma}(\nabla_i \xi^m)\right] E_\gamma \\
&+ g^{ij}\left[\frac{b}{\alpha}\xi^s J_l^\gamma \left(R_{msj}^{\ \ \ \ l}\nabla_i \xi^m\right) + \frac{1}{\alpha}\xi^s R_{isj}^{\ \ \ \gamma} + \nabla_i \nabla_j \xi^\gamma\right] E_{\overline{\gamma}}.
\end{aligned}$$

Proof. By using equation (2.1), we get

$$\begin{aligned}
\beta(\xi)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)^\gamma &= \left[\frac{\partial^2 \xi^\gamma}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial \xi^\gamma}{\partial x_k} + \widetilde{\Gamma}_{mn}^\gamma \frac{\partial \xi^m}{\partial x_i}\frac{\partial \xi^n}{\partial x_j} + \widetilde{\Gamma}_{\overline{m}n}^\gamma \frac{\partial \xi^{\overline{m}}}{\partial x_i}\frac{\partial \xi^n}{\partial x_j}\right. \\
&\left. + \widetilde{\Gamma}_{m\overline{n}}^\gamma \frac{\partial \xi^m}{\partial x_i}\frac{\partial \xi^{\overline{n}}}{\partial x_j} + \widetilde{\Gamma}_{\overline{mn}}^\gamma \frac{\partial \xi^{\overline{m}}}{\partial x_i}\frac{\partial \xi^{\overline{n}}}{\partial x_j}\right] E_\gamma \\
&= \left[\frac{b}{2\alpha}\xi^s J_l^\gamma (R_{sij}^{\ \ \ l} + R_{sji}^{\ \ \ l}) + \frac{b^2}{2\alpha}\xi^s [R_{smj}^{\ \ \ \ \gamma}(\nabla_i \xi^m) + R_{sni}^{\ \ \ \ \gamma}(\nabla_j \xi^n)]\right] E_\gamma,
\end{aligned}$$

$$\begin{aligned}
\beta(\xi)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)^{\overline{\gamma}} &= \left[\frac{\partial^2 \xi^{\overline{\gamma}}}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial \xi^{\overline{\gamma}}}{\partial x_k} + \widetilde{\Gamma}_{mn}^{\overline{\gamma}} \frac{\partial \xi^m}{\partial x_i}\frac{\partial \xi^n}{\partial x_j} + \widetilde{\Gamma}_{\overline{m}n}^{\overline{\gamma}} \frac{\partial \xi^{\overline{m}}}{\partial x_i}\frac{\partial \xi^n}{\partial x_j}\right. \\
&\left. + \widetilde{\Gamma}_{m\overline{n}}^{\overline{\gamma}} \frac{\partial \xi^m}{\partial x_i}\frac{\partial \xi^{\overline{n}}}{\partial x_j} + \widetilde{\Gamma}_{\overline{mn}}^{\overline{\gamma}} \frac{\partial \xi^{\overline{m}}}{\partial x_i}\frac{\partial \xi^{\overline{n}}}{\partial x_j}\right] E_{\overline{\gamma}} \\
&= \left[\nabla_i \nabla_j \xi^\gamma - \frac{1}{2}\xi^s R_{ijs}^{\ \ \ \gamma} + \frac{1}{2\alpha}\xi^s \left(R_{isj}^{\ \ \ \gamma} + R_{jsi}^{\ \ \ \gamma}\right)\right. \\
&\left. + \frac{b}{2\alpha} J_l^\gamma \xi^s \left(R_{msj}^{\ \ \ \ l} \nabla_i \xi^m + R_{nsi}^{\ \ \ \ l} \nabla_j \xi^n\right)\right] E_{\overline{\gamma}}.
\end{aligned}$$

For the tension field we have

$$\begin{aligned}
\tau(\xi) &= g^{ij}\beta(\xi)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \\
&= \xi^s g^{ij}\left[\frac{b}{\alpha} J_l^\gamma R_{sij}^{\ \ \ l} + \frac{b^2}{\alpha} R_{smj}^{\ \ \ \ \gamma}(\nabla_i \xi^m)\right] E_\gamma \\
&+ g^{ij}\left[\frac{b}{\alpha}\xi^s J_l^\gamma \left(R_{msj}^{\ \ \ \ l}\nabla_i \xi^m\right) + \frac{1}{\alpha}\xi^s R_{isj}^{\ \ \ \gamma} + \nabla_i \nabla_j \xi^\gamma\right] E_{\overline{\gamma}}.
\end{aligned}$$

□

Theorem 2.13. *In the context of an anti-paraKähler manifold (M_{2n}, J, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the ciconia metric, the map $\xi : (M_{2n}, J, g) \to (TM, \tilde{g})$ is harmonic if and only if*

$$\xi^s g^{ij}\left[\frac{b}{\alpha} J_l^\gamma R_{sij}^{\ \ \ l} + \frac{b^2}{\alpha} R_{smj}^{\ \ \ \ \gamma}(\nabla_i \xi^m)\right] = 0$$

and

$$g^{ij}\left[\frac{b}{\alpha}\xi^s J_l^\gamma\left(R_{msj}{}^l\nabla_i\xi^m\right)+\frac{1}{\alpha}\xi^s R_{isj}{}^\gamma+\nabla_i\nabla_j\xi^\gamma\right]=0.$$

As a direct consequence of Proposition 2.12, we obtain the theorem below.

Theorem 2.14. *Consider (M_{2n},J,g) as an anti-paraKähler manifold and $(TM,\tilde{g})$ as its tangent bundle with the ciconia metric. If the map $\xi:(M_{2n},J,g)\to(TM,\tilde{g})$ is parallel, then ξ is totally geodesic. Furthermore, ξ also exhibits the property of being harmonic.*

Next, we investigate the harmonicity of the vector field ξ as maps from the anti-paraKähler manifold (M_{2n},J,g) to the tangent bundle $\left(TM,\widetilde{\nabla}^m\right)$ (resp. $\left(TM,\widetilde{\nabla}^{*m}\right)$). Here $\left(TM,\widetilde{\nabla}^m\right)$ (resp. $\widetilde{\nabla}^{*m}$) correspond respectively to the mean connection of Schouten-Van Kampen and the Vranceanu connection, both of which are associated with the ciconia metric as presented in Proposition 2.6 and Proposition 2.8.

Proposition 2.15. *In the context of an anti-paraKähler manifold (M_{2n},J,g) and its tangent bundle $(TM,\tilde{g})$ equipped with the ciconia metric, the second fundamental form $\overline{\beta}(\xi)$ and the tension field $\overline{\tau}(\xi)$ of the map $\xi:(M_{2n},J,g)\to\left(TM,\widetilde{\nabla}^m\right)$ are given by*

$$\begin{aligned}\overline{\beta}(\xi)\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right) &= \left[\frac{b^2}{4\alpha}\xi^s[R_{smi}{}^l\nabla_j\xi^m+R_{snj}{}^l\nabla_i\xi^n]-\frac{b}{2\alpha}\xi^s J_l^\gamma(R_{isj}{}^l+R_{jsi}{}^l)\right]E_\gamma\\ &\quad+\left[\nabla_i\nabla_j\xi^\gamma-\frac{1}{2}\xi^s R_{ijs}{}^\gamma-\frac{b}{4\alpha}\xi^s[R_{smi}{}^l\nabla_j\xi^m+R_{snj}{}^l\nabla_i\xi^n]\right]E_{\overline{\gamma}}\end{aligned}$$

and

$$\overline{\tau}(\xi)=\frac{b}{\alpha}\xi^s g^{ij}\left[\frac{b}{2}(R_{smi}{}^l\nabla_j\xi^m)-J_l^\gamma R_{isj}{}^l\right]E_\gamma+g^{ij}\left[\nabla_i\nabla_j\xi^\gamma-\frac{b}{2\alpha}(R_{smi}{}^l\nabla_j\xi^m)\right]E_{\overline{\gamma}}.$$

Theorem 2.16. *Consider an anti-paraKähler manifold (M_{2n},J,g) and its tangent bundle $(TM,\tilde{g})$ equipped with the ciconia metric, the map $\xi:(M_{2n},J,g)\to\left(TM,\widetilde{\nabla}^m\right)$ is harmonic if and only if*

$$\xi^s g^{ij}\left[\frac{b}{2}(R_{smi}{}^l\nabla_j\xi^m)-J_l^\gamma R_{isj}{}^l\right]=0$$

and

$$g^{ij}\left[\nabla_i\nabla_j\xi^\gamma-\frac{b}{2\alpha}(R_{smi}{}^l\nabla_j\xi^m)\right]=0.$$

Proposition 2.17. *In the context of an anti-paraKähler manifold (M_{2n},J,g) and its tangent bundle $(TM,\tilde{g})$ equipped with the ciconia metric, the second fundamental form $\tilde{\beta}(\xi)$ and the tension field $\tilde{\tau}(\xi)$ of the map $\xi:(M_{2n},J,g)\to\left(TM,\widetilde{\nabla}^{*m}\right)$ are given by*

$$\tilde{\beta}(\xi)\left(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right)=\left[\frac{b}{2\alpha}\xi^s J_l^\gamma(R_{sij}{}^l+R_{sji}{}^l)\right]E_\gamma+\left[\nabla_i\nabla_j\xi^\gamma-\frac{1}{2}\xi^s R_{ijs}{}^\gamma\right]E_{\overline{\gamma}}$$

and

$$\tilde{\tau}(\xi)=\xi^s g^{ij}\left[\frac{b}{\alpha}J_l^\gamma R_{sij}{}^l\right]E_\gamma+g^{ij}\left[\nabla_i\nabla_j\xi^\gamma\right]E_{\overline{\gamma}}.$$

Theorem 2.18. *Consider an anti-paraKähler manifold (M_{2n},J,g) and its tangent bundle $(TM,\tilde{g})$ equipped with the ciconia metric, the map $\xi:(M_{2n},J,g)\to\left(TM,\widetilde{\nabla}^{*m}\right)$ is harmonic if and only if*

$$\nabla_i\nabla_j\xi=0\qquad\text{and}\qquad Q(\xi)=0,$$

where Q is the Ricci operator.

2.4. Harmonicity of the identity map $I : TM \to TM$

We now delve into the examination of the harmonicity relationship between the ciconia metric $\tilde{g}$ and the Sasaki metric ${}^{S}g$. Through an analysis of their respective Christoffel symbols, we establish the subsequent two propositions (for the Christoffel symbols of the Sasaki metric see [27]). Here $I : (TM, \tilde{g}) \to \left(TM, {}^{S}g\right)$ represents the identity map, ${}^{S}g$ denotes the Sasaki metric, and $\tilde{g}$ represents the ciconia metric. Upon performing the required operations, we arrive at the following expressions:

$$\beta(I)(E_i, E_j) = \left[\frac{b}{2\alpha} y^s J_i^t \left(R_{tsj}{}^{\gamma} + R_{jst}{}^{\gamma}\right)\right] E_\gamma - \left[\frac{1}{2\alpha} y^s \left(R_{isj}{}^{\gamma} + R_{jsi}{}^{\gamma}\right)\right] E_{\overline{\gamma}}$$

and

$$\beta(I)(E_{\overline{i}}, E_j) = \left[\frac{\alpha - b^2}{2\alpha} y^s R_{sij}{}^{\gamma}\right] E_\gamma - \left[\frac{b}{2\alpha} y^s J_i^t R_{tsj}{}^{\gamma}\right] E_{\overline{\gamma}}$$

$$\beta(I)\left(E_{\overline{i}}, E_{\overline{j}}\right) = 0.$$

Proposition 2.19. *Consider an anti-paraKähler manifold (M_{2n}, J, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the ciconia metric. Suppose that $I : (TM, \tilde{g}) \to \left(TM, {}^{S}g\right)$ is the identity map. Then, the following holds:*

- **i):** *If M_{2n} is locally flat, then I is totally geodesic;*
- **ii):** *The tension field $\tau_{\tilde{g}}(I)$ of I is given by*

$$\tau_{\tilde{g}}(I_{TM}) = \left[\frac{1}{\alpha} y^s G^{ij} R_{isj}{}^{\gamma}\right] E_\gamma.$$

Proof.

$$\begin{aligned}
\tau_{\tilde{g}}(I_{TM}) &= \tilde{g}^{\alpha\beta} \beta(I)(E_\alpha, E_\beta) \\
&= \tilde{g}^{ij} \beta(I)(E_i, E_j) + \tilde{g}^{i\overline{j}} \beta(I)\left(E_i, E_{\overline{j}}\right) \\
&\quad + \tilde{g}^{\overline{i}j} \beta(I)(E_{\overline{i}}, E_j) + \tilde{g}^{\overline{i}\overline{j}} \beta(I)\left(E_{\overline{i}}, E_{\overline{j}}\right) \\
&= \frac{b}{\alpha} g^{ij} \beta(I)(E_i, E_j) - \frac{2}{\alpha} G^{ij} \beta(I)\left(E_i, E_{\overline{j}}\right) \\
&= \left[\frac{b^2}{\alpha^2} g^{ij} y^s J_l^{\gamma} R_{isj}{}^{l} + \frac{b^2 - \alpha}{\alpha^2} y^s G^{ij} R_{sij}{}^{\gamma}\right] E_\gamma \\
&\quad + \left[\frac{b}{\alpha^2} y^s g^{ij} R_{sij}{}^{\gamma} + \frac{b}{\alpha^2} G^{ij} y^s J_l^{\gamma} R_{isj}{}^{l}\right] E_{\overline{\gamma}} \\
&= \left[\frac{1}{\alpha} y^s G^{ij} R_{isj}{}^{\gamma}\right] E_\gamma.
\end{aligned}$$

□

Corollary 2.20. *Let (M_{2n}, J, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent bundle with ciconia metric. then the identity map $I : (TM, \tilde{g}) \to \left(TM, {}^{S}g\right)$ is harmonic if and only if M_{2n} is locally Ricci-flat.*

Proposition 2.21. *Consider an anti-paraKähler manifold (M_{2n}, J, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the ciconia metric. Suppose that $I : \left(TM, {}^{S}g\right) \to (TM, \tilde{g})$ is the identity map. Then the tension field $\tau_{S_g}(I)$ of I is given by*

$$\tau_{S_g}(I_{TM}) = \left(-\frac{b}{\alpha} y^s g^{ij} J_l^{\gamma} R_{isj}{}^{l}\right) E_\gamma + \left(\frac{1}{\alpha} y^s g^{ij} R_{isj}{}^{\gamma}\right) E_{\overline{\gamma}}.$$

Corollary 2.22. *Consider an anti-paraKähler manifold (M_{2n}, J, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the ciconia metric. Then the identity map $I : \left(TM,^{S} g\right) \to (TM, \tilde{g})$ is harmonic if and only if M_{2n} is Ricci-flat.*

Next, we examine the criterion under which the second fundamental form of the identity map between $(TM, \tilde{g})$ and $\left(TM, \overline{\nabla}^{m}\right)$, as well as between $(TM, \tilde{g})$ and $\left(TM, \widetilde{\nabla}^{*m}\right)$, vanishes. Here $\overline{\nabla}^{m}$ (resp. $\widetilde{\nabla}^{*m}$) denotes the mean connection associated with the Schouten-Van Kampen connection (resp. Vrănceanu connection) linked to the Levi-Civita connection of the ciconia metric $\widetilde{g}$. Utilizing Propositions 2.6, 2.8, we arrive at the following result:

Proposition 2.23. *Consider an anti-paraKähler manifold (M_{2n}, J, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the ciconia metric. The second fundamental form of the map $I : (TM, \tilde{g}) \to \left(TM, \overline{\nabla}^{m}\right)$ and the tension field $\tau\left(I_{TM}\right)$ of I are given by*

$$\begin{aligned}
\beta(I)\left(E_i, E_j\right) &= \left[\frac{1}{2\alpha} y^s \left(R_{sij}{}^{\gamma} + R_{sji}{}^{\gamma}\right)\right] E_{\overline{\gamma}}, \\
\beta(I)\left(E_i, E_{\overline{j}}\right) &= \frac{b^2}{4\alpha} y^s R_{jsi}{}^{\gamma} E_{\gamma} + \left[\frac{b}{4\alpha} y^s J_l^{\gamma} R_{sji}{}^{l}\right] E_{\overline{\gamma}}, \\
\beta(I)\left(E_{\overline{i}}, E_{\overline{j}}\right) &= 0
\end{aligned}$$

and

$$\tau\left(I_{TM}\right) = \left[\frac{b^2}{2\alpha^2} y^s G^{ij} R_{isj}{}^{\gamma}\right] E_{\gamma} + \left[\frac{b}{2\alpha^2} y^s g^{ij} R_{sij}{}^{\gamma}\right] E_{\overline{\gamma}}.$$

Consequently we have the following.

Corollary 2.24. *Let (M_{2n}, J, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ its tangent bundle with ciconia metric. The map $I : (TM, \tilde{g}) \to \left(TM, \overline{\nabla}^{m}\right)$ is:*

1: *totally geodesic if and only if M_{2n} is locally flat;*
2: *harmonic if and only if M_{2n} is locally Ricci-flat.*

Proposition 2.25. *Consider an anti-paraKähler manifold (M_{2n}, J, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the ciconia metric. The second fundamental form of the map $I : (TM, \tilde{g}) \to \left(TM, \widetilde{\nabla}^{*m}\right)$ and the tension field $\tau\left(I_{TM}\right)$ of I are given by*

$$\begin{aligned}
i.\ \ \beta(I)\left(E_i, E_j\right) &= -\frac{1}{2\alpha} y^s \left[R_{isj}{}^{\gamma} + R_{jsi}{}^{\gamma}\right] E_{\overline{\gamma}}, \\
ii.\ \beta(I)\left(E_{\overline{i}}, E_j\right) &= \left[\frac{b^2}{2\alpha} y^s R_{isj}{}^{\gamma}\right] E_{\gamma} + \frac{b}{2\alpha} y^s J_i^t R_{tsj}{}^{\gamma} E_{\overline{\gamma}}, \\
iv.\beta(I)\left(E_{\overline{i}}, E_{\overline{j}}\right) &= 0
\end{aligned}$$

and

$$\tau\left(I_{TM}\right) = \left[-\frac{b^2}{\alpha^2} y^s G^{ij} R_{isj}{}^{\gamma}\right] E_{\gamma}.$$

Corollary 2.26. *Consider an anti-paraKähler manifold (M_{2n}, J, g) and its tangent bundle $(TM, \tilde{g})$ equipped with the ciconia metric. The map $I : (TM, \tilde{g}) \to \left(TM, \overline{\nabla}^{*m}\right)$ is:*

1: *totally geodesic if and only if M_{2n} is locally flat;*
2: *harmonic if and only if M_{2n} is locally Ricci-flat.*

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