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On Bishop Frames of Any Regular Curve in Euclidean 3-Space

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3-Boyutlu Öklid Uzayında Regüler Bir Eğrinin Bishop Çatıları Üzerine

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Abstract
Relationships between type-1 Bishop and Frenet, type-2 Bishop and Frenet, alternative and Frenet, N-Bishop and alternative frames of any regular curve in Euclidean 3-space are known. In this study, relationships between N-Bishop and Frenet frames and relationships between type-1 Bishop, type-2 Bishop and N-Bishop frames of any regular curve in Euclidean 3-space are given. In addition, pole vectors (unit vectors in the direction of Darboux vectors) belonging to these frames are computed. Last, pole and Darboux vectors belonging to these frames are compared with each other.

Anahtar Kelimeler *Type-1 Bishop Frame;Type-2 Bishop Frame; Alternative Frame; N-Bishop Frame; Frenet Frame; Darboux Vector*

Öz
3-boyutlu Öklid uzayında herhangi bir regüler eğrinin tip-1 Bishop ve Frenet, tip-2 Bishop ve Frenet, alternatif ve Frenet, N-Bishop ve alternatif çatıları arasındaki ilişkiler bilinmektedir. Bu çalışmada, 3-boyutlu Öklid uzayında herhangi bir regüler eğrinin N-Bishop ve Frenet çatıları arasındaki ilişkiler ve tip-1 Bishop, tip-2 Bishop ve N-Bishop çatıları arasındaki ilişkiler verilmiştir. Ayrıca bu çatılara ait pol vektörleri (Darboux vektörü yönündeki birim vektörler) hesaplanmıştır. Son olarak pol ve Darboux vektörleri birbirleriyle karşılaştırılmıştır.

Keywords *Tip-1 Bishop Çatısı; Tip-2 Bishop Çatısı; Alternatif Çatı; N-Bishop Çatısı; Frenet Çatısı;Darboux Vektörü*

1.Introduction

Frenet frame, a tool used to determine the characteristic features of a curve, was defined by J. F. Frenet in 1847. This frame is also known as Serret-Frenet frame, as J. A. Serret also defined the same frame in his thesis independently of Frenet in 1851. Frenet frame consists of tangent T, principal normal N and binormal B vectors of a curve, (Hacısalihoğlu 1983). Different frames can be defined on the curve, depending on the character, geometrical properties or location of any curve. One of them is Bishop frame (or parallel transport frame) defined by Bishop (1975). This frame is a relatively parallel frame obtained by rotating around T, of Frenet frame by a certain angle. The Bishop frame is more advantageous than the Frenet frame, which works even when the second derivative of the curve is zero. Subsequently, numerous studies related to curves and surfaces have been conducted using this frame, and new variants of this frame (type-2 Bishop and N-Bishop) have even been defined, (Bükcü and Karacan 2009, Kelleci et al. 2019, Kılıçoğlu and Hacısalihoğlu 2013, Masal and Azak 2019). Therefore, Bishop frame is also referred to as type-1 Bishop frame in some studies. The type-2 Bishop frame was advertised by Yılmaz and Turgut (2010) with the same logic as type-1 Bishop, that is, by rotating around B of Frenet frame by a certain angle. In addition, an alternative frame to Frenet frame was defined by Scofield (1995). This frame, which consists of N and unit Darboux vector W of a curve, and a third vector C attained by vector product of two vectors, is called alternative frame. Keskin and Yaylı (2017) obtained a new frame by rotating around N of alternative frame by a certain angle and called it N-Bishop frame. As can be understood from their definitions, tangential vector fields of type-1 Bishop and Frenet frames, binormal vector fields of type-2 Bishop and Frenet frames, and principal normal vector fields of N-Bishop and Frenet frames are common. There are many studies on this new types of Bishop and alternative frame (Alıç and Yılmaz 2021, Çakmak and Şahin 2022, Damar et al. 2017, Kızıltuğ et al. 2013, Masal and Azak 2015, Ourab et al. 2018, Samancı and Sevinç 2022, Şenyurt 2018, Şenyurt et al. 2023, Yılmaz

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and Has 2022, Şenyurt and Kaya 2018.). In these studies, the relationships between Frenet and various Bishop frames of a curve are given. In addition, just as the Darboux vector belonging to the Frenet frame a curve were defined, the Darboux vectors belonging to the Bishop frames were also defined in the studies (Bükcü and Karacan 2008, Yılmaz and Savcı 2017, Uzunoğlu et al. 2016, Samancı and İncesu 2020).

2. Preliminaries

2.1. Frenet Frame of Any Regular Curve in E^3

Frenet frame $\{T,N,B\}$ of any regular curve γ in E^3 is

$$T=\frac{\gamma'}{\|\gamma'\|},\qquad N=B\wedge T,\qquad B=\frac{\gamma'\wedge\gamma''}{\|\gamma'\wedge\gamma''\|},$$

curvature κ and torsion τ are

$$\kappa=\frac{\|\gamma'\wedge\gamma''\|}{\|\gamma'\|^3},\qquad \tau=\frac{\langle\gamma',\gamma''\wedge\gamma'''\rangle}{\|\gamma'\wedge\gamma''\|^2}.$$

The matrix representation of Frenet derivative formulas of γ is

$$\begin{bmatrix}T'\\N'\\B'\end{bmatrix}=\begin{bmatrix}0&\|\gamma'\|\kappa&0\\-\|\gamma'\|\kappa&0&\|\gamma'\|\tau\\0&-\|\gamma'\|\tau&0\end{bmatrix}\begin{bmatrix}T\\N\\B\end{bmatrix}.\tag{1}$$

Darboux and pole vector (unit vector in the direction of Darboux vector) belonging to Frenet frame are

$$\begin{cases}F=\|\gamma'\|(\tau T+\kappa B),\\ W=\dfrac{\tau}{\sqrt{\kappa^2+\tau^2}}T+\dfrac{\kappa}{\sqrt{\kappa^2+\tau^2}}B.\end{cases}\tag{2}$$

2.2 Type-1 Bishop Frame of Any Regular Curve in E^3

Type-1 Bishop Frame (or Bishop frame) is attained by rotating Frenet frame $\{T,N,B\}$ around tangent vector T, by an angle θ and so it is a relatively parallel adapted frame in with Frenet frame. Let $\{T,N_1,B_1\}$ be the Bishop frame of any regular curve γ. Here T, is tangent vector of Frenet frame of γ, N_1 is any unit vector perpendicular to T, by obtained rotating N by angle θ and $B_1=T\wedge N_1$ is a unit vector perpendicular to both T and N_1, Figure 1, (Bishop 1975).

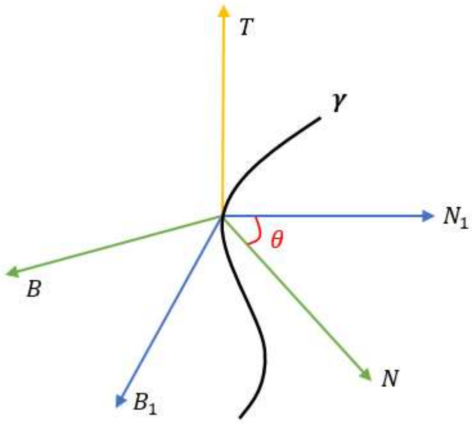

Figure 1. Frenet and Bishop frames

There are the following matrix relationships between Frenet and Bishop frames:

$$\begin{cases}\begin{bmatrix}T\\N\\B\end{bmatrix}=\begin{bmatrix}1&0&0\\0&\cos\theta&\sin\theta\\0&-\sin\theta&\cos\theta\end{bmatrix}\begin{bmatrix}T\\N_1\\B_1\end{bmatrix}\\ \text{or}\\ \begin{bmatrix}T\\N_1\\B_1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{bmatrix}\begin{bmatrix}T\\N\\B\end{bmatrix}\end{cases}\tag{3}$$

where

$$\theta=\int\|\gamma'\|\tau\,.\tag{4}$$

From (1), (3) and (4),

$$\begin{aligned}T'&=\|\gamma'\|\kappa(\cos\theta N_1+\sin\theta B_1)\\&=\|\gamma'\|\kappa_1N_1+\|\gamma'\|\tau_1B_1\end{aligned}$$

$$\begin{aligned}N_1'&=-\theta'\sin\theta N+\cos\theta N'-\theta'\cos\theta B-\sin\theta B'\\&=-\|\gamma'\|\kappa\cos\theta T\\&=-\|\gamma'\|\kappa_1T\end{aligned}$$

$$\begin{aligned}B_1'&=\theta'\cos\theta N+\sin\theta N'-\theta'\sin\theta B+\cos\theta B'\\&=-\|\gamma'\|\kappa\sin\theta T\\&=-\|\gamma'\|\tau_1T\end{aligned}$$

are obtained, where

$$\kappa_1 = \kappa \cos\theta, \qquad \tau_1 = \kappa \sin\theta \tag{5}$$

are curvatures of type-1 Bishop frame. Thus, the matrix representation of Bishop derivative formulas is

$$\begin{bmatrix} T' \\ N_1' \\ B_1' \end{bmatrix} = \begin{bmatrix} 0 & \|\gamma'\|\kappa_1 & \|\gamma'\|\tau_1 \\ -\|\gamma'\|\kappa_1 & 0 & 0 \\ -\|\gamma'\|\tau_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ B_1 \end{bmatrix} \tag{6}$$

Darboux vector belonging to Bishop frame is (Bükcü and Karacan, 2008)

$$F_1 = T \wedge T' = \|\gamma'\|\left(-\tau_1 N_1 + \kappa_1 B_1\right), \tag{7}$$

where

$$T' = F_1 \wedge T, \qquad N_1' = F_1 \wedge N_1, \qquad B_1' = F_1 \wedge B_1 .$$

2.3. Type-2 Bishop Frame of Any Regular Curve in E^3

Type-2 Bishop frame, similar to type-1 Bishop frame, is obtained by rotating Frenet frame $\{T, N, B\}$ around binormal vector B by an angle ϕ. Let $\{N_2, B_2, B\}$ be type-2 Bishop frame of any regular curve γ . Here B is binormal vector of Frenet frame of γ , N_2 is any unit vector perpendicular to B by obtained rotating N by angle ϕ and $B_2 = B \wedge N_2$ is a unit vector perpendıcular to both B and N_2 , Figure 2, (Yılmaz and Turgut 2010).

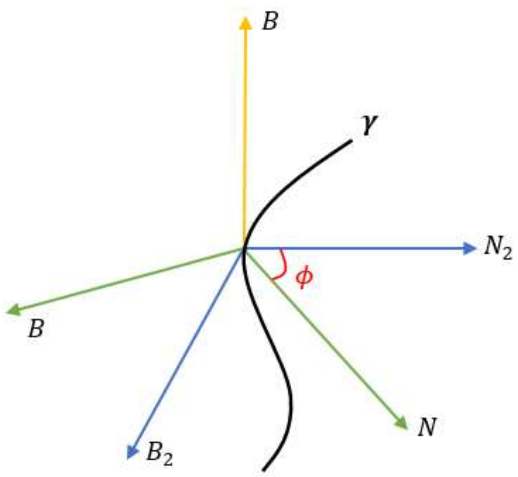

Figure 2. Frenet and type-2 Bishop frames

There are the following matrix relations between Frenet and type-2 Bishop frames:

$$\left\{ \begin{array}{l} \begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sin\phi & -\cos\phi & 0 \\ \cos\phi & \sin\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_2 \\ B_2 \\ B \end{bmatrix} \\ \text{or} \\ \begin{bmatrix} N_2 \\ B_2 \\ B \end{bmatrix} = \begin{bmatrix} \sin\phi & \cos\phi & 0 \\ -\cos\phi & \sin\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \end{array} \right. \tag{8}$$

where

$$\phi = \int \|\gamma'\|\kappa . \tag{9}$$

From (1), (8) and (9),

$$\begin{aligned} N_2' &= \phi' \cos\phi T + \sin\phi T' - \phi' \sin\phi N + \cos\phi N' \\ &= \|\gamma'\|\tau \cos\phi B \\ &= -\|\gamma'\|\kappa_2 B \end{aligned}$$

$$\begin{aligned} B_2' &= \phi' \sin\phi T - \cos\phi T' + \phi' \cos\phi N + \sin\phi N' \\ &= \|\gamma'\|\tau \sin\phi B \\ &= -\|\gamma'\|\tau_2 B \end{aligned}$$

$$\begin{aligned} B' &= -\|\gamma'\|\tau\left(\cos\phi N_2 + \sin\phi B_2\right) \\ &= \|\gamma'\|\kappa_2 N_2 + \|\gamma'\|\tau_2 B_2 \end{aligned}$$

are obtained, where

$$\kappa_2 = -\tau \cos\phi, \qquad \tau_2 = -\tau \sin\phi \tag{10}$$

are curvatures of type-2 Bishop. Thus, the matrix representation of type-2 Bishop derivative formulas is

$$\begin{bmatrix} N_2' \\ B_2' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\|\gamma'\|\kappa_2 \\ 0 & 0 & -\|\gamma'\|\tau_2 \\ \|\gamma'\|\kappa_2 & \|\gamma'\|\tau_2 & 0 \end{bmatrix} \begin{bmatrix} N_2 \\ B_2 \\ B \end{bmatrix} .$$

(11)

Darboux vector belonging to type-2 Bishop frame is (Yılmaz and Savcı 2017)

$$F_2 = B \wedge B' = \|\gamma'\|\left(-\tau_2 N_2 + \kappa_2 B_2\right), \tag{12}$$

where

$$N_2' = F_2 \wedge N_2, \qquad B_2' = F_2 \wedge B_2, \qquad B' = F_2 \wedge B .$$

2.4. Alternative Frame of Any Regular Curve in E^3

Alternative frame is a new frame obtained from Frenet vectors of any regular curve. Let $\{N, C, W\}$ be alternative frame of any regular curve γ. Here N is principal normal vector of Frenet frame of γ, $C = \dfrac{N'}{\|N'\|} = \dfrac{-\kappa T + \tau B}{\sqrt{\kappa^2 + \tau^2}}$ is a unit vector obtained from the first derivative of principal normal vector N and $W = N \wedge C = \dfrac{\tau T + \kappa B}{\sqrt{\kappa^2 + \tau^2}}$ is a unit vector (unit Darboux vector belonging to Frenet frame) perpendicular to both N and W, Figure 3, (Scofield, 1995).

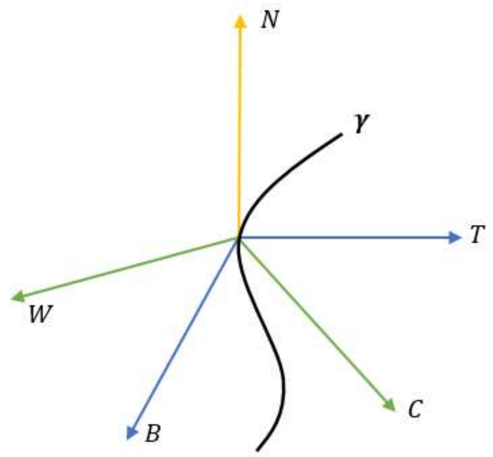

Figure 3. Frenet and alternative frames

From (1), there are the following matrix relationships between Frenet and alternative frames:

$$\begin{cases} \begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & -\overline{\kappa} & \overline{\tau} \\ 1 & 0 & 0 \\ 0 & \overline{\tau} & \overline{\kappa} \end{bmatrix} \begin{bmatrix} N \\ C \\ W \end{bmatrix} \\ \text{or} \\ \begin{bmatrix} N \\ C \\ W \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\overline{\kappa} & 0 & \overline{\tau} \\ \overline{\tau} & 0 & \overline{\kappa} \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \end{cases} \tag{13}$$

where

$$\overline{\kappa} = \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}, \qquad \overline{\tau} = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}} \tag{14}$$

From (1), (13) and (14),

$$N' = \|\gamma'\| fC$$

$$\begin{aligned} C' &= -\frac{\kappa' f - \kappa f'}{f^2} T - \frac{\kappa}{f} T' + \frac{\tau' f - \tau f'}{f^2} B + \frac{\tau}{f} B' \\ &= \frac{\tau(\kappa\tau' - \kappa'\tau)}{f^3} T - \|\gamma'\| fN + \frac{\kappa(\kappa\tau' - \kappa'\tau)}{f^3} B \\ &= gW - \|\gamma'\| fN \end{aligned}$$

$$\begin{aligned} W' &= \frac{\tau' f - \tau f'}{f^2} T + \frac{\tau}{f} T' + \frac{\kappa' f - \kappa f'}{f^2} B + \frac{\kappa}{f} B' \\ &= \frac{(\kappa'\tau - \kappa\tau')}{f^2} C \\ &= -gC \end{aligned}$$

are obtained, where

$$f = \sqrt{\kappa^2 + \tau^2}, \qquad g = \frac{\kappa\tau' - \kappa'\tau}{f^2} \tag{15}$$

are curvatures of alternative frame. Thus, the matrix representation of derivative formulas of alternative frame is

$$\begin{bmatrix} N' \\ C' \\ W' \end{bmatrix} = \begin{bmatrix} 0 & \|\gamma'\| f & 0 \\ -\|\gamma'\| f & 0 & g \\ 0 & -g & 0 \end{bmatrix} \begin{bmatrix} N \\ C \\ W \end{bmatrix}. \tag{16}$$

Darboux vector belonging to alternative frame is (Uzunoğlu et al. 2016)

$$\overline{F} = C \wedge C' = gN + \|\gamma'\| fW, \tag{17}$$

where

$$N' = \overline{F} \wedge N, \qquad C' = \overline{F} \wedge C, \qquad W' = \overline{F} \wedge W.$$

2.5. N-Bishop Frame of Any Regular Curve in E^3

N-Bishop frame, similar to type-1 Bishop frame, is obtained by rotating alternative frame $\{N, C, W\}$ around principal normal vector N by an angle φ. Let $\{N, N_3, B_3\}$ be N-Bishop frame of any regular curve γ. Here N is principal normal vector of Frenet frame of γ, N_3 is a unit vector perpendicular to N by obtained rotating C by angle φ and $B_3 = N \wedge N_3$ is a unit vector perpendicular to both N and N_3, Figure 4, (Keskin and Yaylı 2017).

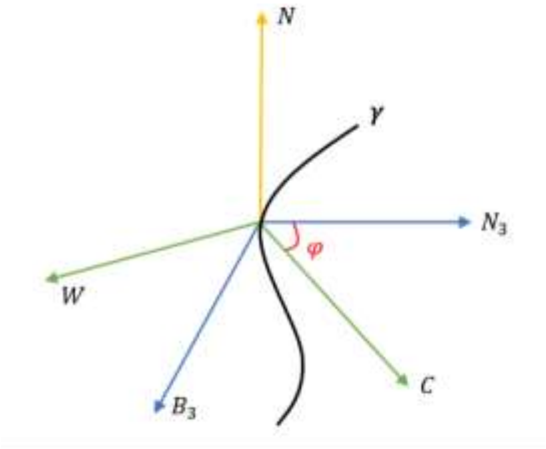

Figure 4. Frenet and N-Bishop frames

There are the following matrix relationships between the Frenet and N-Bishop frames:

$$\begin{cases} \begin{bmatrix} N \\ C \\ W \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} N \\ N_3 \\ B_3 \end{bmatrix} \\ \text{or} \\ \begin{bmatrix} N \\ N_3 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} N \\ C \\ W \end{bmatrix} \end{cases} \tag{18}$$

where

$$\varphi = \int g . \tag{19}$$

From (1), (18) and (19),

$$N' = \|\gamma'\| fC$$

$$\begin{aligned} N_3' &= -\varphi' \sin\varphi C + \cos\varphi C' - \varphi' \cos\varphi W - \sin\varphi W' \\ &= -\|\gamma'\| f \cos\varphi N \\ &= -\|\gamma'\| \kappa_3 N \end{aligned}$$

$$\begin{aligned} B_3' &= \varphi' \cos\varphi C + \sin\varphi C' - \varphi' \sin\varphi W + \cos\varphi W' \\ &= -\|\gamma'\| f \sin\varphi N \\ &= -\|\gamma'\| \tau_3 N \end{aligned}$$

are obtained, where

$$\kappa_3 = f\cos\varphi, \qquad \tau_3 = f\sin\varphi \tag{20}$$

are curvatures of N-Bishop. Thus, the matrix representation of N-Bishop derivative formulas is

$$\begin{bmatrix} N' \\ N_3' \\ B_3' \end{bmatrix} = \begin{bmatrix} 0 & \|\gamma'\|\kappa_3 & \|\gamma'\|\tau_3 \\ -\|\gamma'\|\kappa_3 & 0 & 0 \\ -\|\gamma'\|\tau_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} N \\ N_3 \\ B_3 \end{bmatrix}.$$

(21)

Darboux vector belonging to N-Bishop frame is (Samancı and İncesu 2020)

$$F_3 = N \wedge N' = \|\gamma'\| (-\tau_3 N_3 + \kappa_3 B_3), \tag{22}$$

where

$$N' = F_3 \wedge N, \qquad N_3' = F_3 \wedge N_3, \qquad B_3' = F_3 \wedge B_3 .$$

3. On Bishop Frames of Any Regular Curve in E^3

3.1. Relationships Between Bishop Frames of Any Regular Curve in E^3

First, let's get the matrix relation between Frenet and N-Bishop frames.

Theorem 3.1. There are the following matrix relationships between Frenet frame $\{T, N, B\}$ and N-Bishop frame $\{N, N_3, B_3\}$ of any curve γ in E^3:

$$\begin{cases} \begin{bmatrix} N \\ N_3 \\ B_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -X & 0 & Y \\ Y & 0 & X \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \\ \text{or} \\ \begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 0 & -X & Y \\ 1 & 0 & 0 \\ 0 & Y & X \end{bmatrix} \begin{bmatrix} N \\ N_3 \\ B_3 \end{bmatrix} \end{cases} \tag{23}$$

where

$$X = \frac{\kappa}{f}\cos\varphi + \frac{\tau}{f}\sin\varphi, \quad Y = -\frac{\kappa}{f}\sin\varphi + \frac{\tau}{f}\cos\varphi ,$$

$$\varphi = \int g .$$

Proof: N_3 is written as a linear combination of T, N, B as follows:

$$N_3 = a_0 T + b_0 N + c_0 B , \tag{24}$$

where a_0, b_0, c_0 are coefficients. If the inner product with T, N, B is applied to both sides of (24),

$$a_0 = \langle T, N_3 \rangle, \quad b_0 = \langle N, N_3 \rangle, \quad c_0 = \langle B, N_3 \rangle$$

are gotten. From (18),

$$\begin{cases} a_0 = \left\langle T, \cos\varphi\left(-\frac{\kappa}{f}T + \frac{\tau}{f}B\right) - \sin\varphi\left(\frac{\tau}{f}T + \frac{\kappa}{f}B\right)\right\rangle \\ \quad = -X, \\ b_0 = \left\langle N, \cos\varphi\left(-\frac{\kappa}{f}T + \frac{\tau}{f}B\right) - \sin\varphi\left(\frac{\tau}{f}T + \frac{\kappa}{f}B\right)\right\rangle \\ \quad = 0, \\ c_0 = \left\langle B, \cos\varphi\left(-\frac{\kappa}{f}T + \frac{\tau}{f}B\right) - \sin\varphi\left(\frac{\tau}{f}T + \frac{\kappa}{f}B\right)\right\rangle \\ \quad = Y. \end{cases}$$

(25)

If (25) is substituted in (24),

$$N_3 = -XT + YB$$

is obtained. And similarly,

$$B_3 = YT + XB$$

is gotten. On the other hand, T is written as a linear combination of N, N_3, B_3 as follows:

$$T = d_0 N + e_0 N_3 + f_0 B_3, \tag{26}$$

where d_0, e_0, f_0 are coefficients. If the inner product with N, N_3, B_3 is applied to both sides of (26),

$$d_0 = \langle N, T \rangle, \quad e_0 = \langle N_3, T \rangle, \quad f_0 = \langle B_3, T \rangle$$

are gotten. So,

$$\begin{cases} d_0 = 0, \\ e_0 = \langle -XT + YB, T \rangle = -X, \\ f_0 = \langle -XT + YB, B \rangle = Y. \end{cases} \tag{27}$$

If (27) is substituted in (26),

$$T = -XN_3 + YB_3$$

is obtained. And similarly,

$$B = YN_3 + XB_3$$

is gotten. Thus, the proof is completed.

Theorem 3.2. There are the following matrix relationships between type-1 Bishop frame $\{T, N_1, B_1\}$ and type-2 Bishop frame $\{N_2, B_2, B\}$ of any curve γ in E^3:

$$\begin{bmatrix} N_2 \\ B_2 \\ B \end{bmatrix} = \begin{bmatrix} \sin\phi & \cos\phi\cos\theta & \cos\phi\sin\theta \\ -\cos\phi & \sin\phi\cos\theta & \sin\phi\sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ B_1 \end{bmatrix}$$

or

$$\begin{bmatrix} T \\ N_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} \sin\phi & -\cos\phi & 0 \\ \cos\phi\cos\theta & \sin\phi\cos\theta & -\sin\theta \\ \cos\phi\sin\theta & \sin\phi\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} N_2 \\ B_2 \\ B \end{bmatrix},$$

where $\theta = \int \|\gamma'\| \tau$, $\phi = \int \|\gamma'\| \kappa$.

Proof: From (3),

$$B = -\sin\theta N_1 + \cos\theta B_1.$$

N_2 is written as a linear combination of T, N_1, B_1 as follows:

$$N_2 = a_1 T + b_1 N_1 + c_1 B_1, \tag{28}$$

where a_1, b_1, c_1 are coefficients. If the inner product with T, N_1, B_1 is applied to both sides of (28),

$$a_1 = \langle T, N_2 \rangle, \quad b_1 = \langle N_1, N_2 \rangle, \quad c_1 = \langle B_1, N_2 \rangle$$

are gotten. From (3) and (8),

$$\begin{cases} a_1 = \langle T, \sin\phi T + \cos\phi N \rangle \\ \quad = \sin\phi, \\ b_1 = \langle \cos\theta N - \sin\theta B, \sin\phi T + \cos\phi N \rangle \\ \quad = \cos\theta\cos\phi, \\ c_1 = \langle \sin\theta N + \cos\theta B, \sin\phi T + \cos\phi N \rangle \\ \quad = \sin\theta\cos\phi. \end{cases}$$

(29)

If (29) is substituted in (28),

$$N_2 = \sin\phi T + \cos\theta\cos\phi N_1 + \sin\theta\cos\phi B_1$$

is obtained. And similarly,

$$B_2 = -\cos\phi T + \cos\theta\sin\phi N_1 + \sin\theta\sin\phi B_1$$

is gotten. On the other hand, from (8)

$$T = \sin\phi N_2 - \cos\phi B_2.$$

N_1 is written as a linear combination of N_2, B_2, B as follows:

$$N_1 = d_1 N_2 + e_1 B_2 + f_1 B, \tag{30}$$

where d_1, e_1, f_1 are coefficients. If the inner product with N_2, B_2, B is applied to both sides of (30),

$$d_1 = \langle N_2, N_1 \rangle, \quad e_1 = \langle B_2, N_1 \rangle, \quad f_1 = \langle B, N_1 \rangle$$

are gotten. From (3) and (8),

$$\begin{cases} d_1 = \cos\theta\cos\phi, \\ e_1 = \langle -\cos\phi T + \sin\phi N, \cos\theta N - \sin\theta B \rangle \\ \quad = \cos\theta\sin\phi, \\ f_1 = \langle B, \cos\theta N - \sin\theta B \rangle = -\sin\theta. \end{cases}$$

(31)

If (31) is substituted in (30),

$$N_1 = \cos\theta\cos\phi N_2 + \cos\theta\sin\phi B_2 - \sin\theta B$$

is obtained. And similarly,

$$B_1 = \sin\theta\cos\phi N_2 + \sin\theta\sin\phi B_2 + \cos\theta B$$

is gotten. Thus, the proof is completed.

Theorem 3.3. There are the following matrix relationships between type-1 Bishop frame $\{T, N_1, B_1\}$ and N-Bishop frame $\{N, N_3, B_3\}$ of any curve γ in E^3:

$$\begin{bmatrix} N \\ N_3 \\ B_3 \end{bmatrix} = \begin{bmatrix} 0 & \cos\theta & \sin\theta \\ -X & -Y\sin\theta & Y\cos\theta \\ Y & -X\sin\theta & X\cos\theta \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ B_1 \end{bmatrix}$$

or

$$\begin{bmatrix} T \\ N_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 & -X & Y \\ \cos\theta & -Y\sin\theta & -X\sin\theta \\ \sin\theta & Y\cos\theta & X\cos\theta \end{bmatrix} \begin{bmatrix} N \\ N_3 \\ B_3 \end{bmatrix}$$

where

$$X = \frac{\kappa}{f}\cos\varphi + \frac{\tau}{f}\sin\varphi, \; Y = -\frac{\kappa}{f}\sin\varphi + \frac{\tau}{f}\cos\varphi,$$

$$\theta = \int \|\gamma'\|\tau, \;\; \varphi = \int g.$$

Proof: From (3),

$$N = \cos\theta N_1 + \sin\theta B_1.$$

N_3 is written as a linear combination of T, N_1, B_1 as follows:

$$N_3 = a_2 T + b_2 N_1 + c_2 B_1, \tag{32}$$

where a_2, b_2, c_2 are coefficients. If the inner product with T, N_1, B_1 is applied to both sides of (32), respectively

$$a_2 = \langle T, N_3 \rangle, \quad b_2 = \langle N_1, N_3 \rangle, \quad c_2 = \langle B_1, N_3 \rangle$$

are gotten. From (3) and (23),

$$\begin{cases} a_2 = \langle T, -XT + YB \rangle = -X, \\ b_2 = \langle \cos\theta N - \sin\theta B, -XT + YB \rangle = -Y\sin\theta, \\ c_2 = \langle \sin\theta N + \cos\theta B, -XT + YB \rangle = Y\cos\theta. \end{cases}$$

(33)

If (33) is substituted in (32),

$$N_3 = -XT - Y\sin\theta N_1 + Y\cos\theta B_1$$

is obtained. Similarly,

$$B_3 = YT - X\sin\theta N_1 + X\cos\theta B_1$$

is gotten. On the other hand, N_1 is written as a linear combination of N, N_3, B_3 as follows:

$$N_1 = d_2 N + e_2 N_3 + f_2 B_3, \tag{34}$$

where d_2, e_2, f_2 are coefficients. If the inner product with N, N_3, B_3 is applied to both sides of (34),

$$d_2 = \langle N, N_1 \rangle, \quad e_2 = \langle N_3, N_1 \rangle, \quad f_2 = \langle B_3, N_1 \rangle$$

are gotten. From (3) and (23),

$$\begin{cases} d_2 = \langle N, \cos\theta N - \sin\theta B \rangle = \cos\theta, \\ e_2 = \langle -XT - Y\sin\theta N_1 + Y\cos\theta B_1, N_1 \rangle \\ \quad = -Y\sin\theta, \\ f_2 = \langle YT - X\sin\theta N_1 + X\cos\theta B_1, N_1 \rangle \\ \quad = -X\sin\theta. \end{cases} \tag{35}$$

If (35) is substituted in (34),

$$N_1 = \cos\theta N - Y\sin\theta N_3 - X\sin\theta B_3$$

is obtained. And similarly,

$$B_1 = \sin\theta N + Y\cos\theta N_3 + X\cos\theta B_3,$$

$$T = -XN_3 + YB_3$$

are gotten. Thus, the proof is completed.

Theorem 3.4. There are the following matrix relationships between type-2 Bishop frame $\{N_2, B_2, B\}$ and N-Bishop frame $\{N, N_3, B_3\}$ of any curve γ in E^3:

$$\begin{bmatrix} N \\ N_3 \\ B_3 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -X\sin\phi & X\cos\phi & Y \\ Y\sin\phi & -Y\cos\phi & X \end{bmatrix} \begin{bmatrix} N_2 \\ B_2 \\ B \end{bmatrix}$$

or

$$\begin{bmatrix} N_2 \\ B_2 \\ B \end{bmatrix} = \begin{bmatrix} \cos\phi & -X\sin\phi & Y\sin\phi \\ \sin\phi & X\cos\phi & -Y\cos\phi \\ 0 & Y & X \end{bmatrix} \begin{bmatrix} N \\ N_3 \\ B_3 \end{bmatrix}$$

where

$$X = \frac{\kappa}{f}\cos\varphi + \frac{\tau}{f}\sin\varphi,\ Y = -\frac{\kappa}{f}\sin\varphi + \frac{\tau}{f}\cos\varphi,$$

$$\phi = \int \|\gamma'\|\kappa,\ \ \varphi = \int g.$$

Proof: From (8),

$$N = \cos\phi N_2 + \sin\phi B_2.$$

N_3 is written as a linear combination of N_2, B_2, B as follows:

$$N_3 = a_3 N_2 + b_3 B_2 + c_3 B, \tag{36}$$

where a_3, b_3, c_3 are coefficients. If the inner product with N_2, B_2, B is applied to both sides of (36), respectively

$$a_3 = \langle N_2, N_3\rangle, \quad b_3 = \langle B_2, N_3\rangle, \quad c_3 = \langle B, N_3\rangle$$

are gotten. From (8) and (23),

$$\begin{cases} a_3 = \langle \sin\phi T + \cos\phi N, -XT + YB\rangle \\ \quad = -X\sin\phi, \\ b_3 = \langle -\cos\phi T + \sin\phi N, -XT + YB\rangle \\ \quad = X\cos\phi, \\ c_3 = \langle B, -XT + YB\rangle = Y. \end{cases} \tag{37}$$

If (37) is substituted in (36),

$$N_3 = -X\sin\phi N_2 + X\cos\phi B_2 + YB$$

is obtained. Similarly,

$$B_3 = Y\sin\phi N_2 - Y\cos\phi B_2 + XB$$

is gotten. On the other hand,
N_2 is written as a linear combination of N, N_3, B_3 as follows:

$$N_2 = d_3 N + e_3 N_3 + f_3 B_3, \tag{38}$$

where d_3, e_3, f_3 are coefficients. If the inner product with N, N_3, B_3 is applied to both sides of (38),

$$d_3 = \langle N, N_2\rangle, \quad e_3 = \langle N_3, N_2\rangle, \quad f_3 = \langle B_3, N_2\rangle$$

are gotten. From (8),

$$\begin{cases} d_3 = \langle N, \sin\phi T + \cos\phi N\rangle = \cos\phi, \\ e_3 = \langle -X\sin\phi N_2 + X\cos\phi B_2 + YB, N_2\rangle \\ \quad = -X\sin\phi, \\ f_3 = \langle Y\sin\phi N_2 - Y\cos\phi B_2 + XB, N_2\rangle \\ \quad = Y\sin\phi. \end{cases}$$

(39)

If (39) is substituted in (38),

$$N_2 = \cos\phi N - X\sin\phi N_3 + Y\sin\phi B_3$$

is obtained. And similarly,

$$B_2 = \sin\phi N + X\cos\phi N_3 - Y\cos\phi B_3,$$

$$B = YN_3 + XB_3$$

are obtained. Thus, the proof is completed.

3.2. Relationships Between Darboux Vectors Belonging to Frenet and Bishop Frames of Any Regular Curve in E^3

Theorem 3.5. There is the following equation between Darboux vector F belonging to Frenet frame and Darboux vector F_1 belonging to type-1 Bishop frame of any curve γ in E^3:

$$F_1 = F - \|\gamma'\|\tau T. \tag{40}$$

Proof: If (3) and (5) is substituted in (7),

$$F_1 = \|\gamma'\|\kappa B$$

is obtained. From (2), the proof is completed.

Theorem 3.6. There is the following equation between Darboux vector F belonging to Frenet frame and Darboux vector F_2 belonging to type-2 Bishop frame of any curve γ in E^3:

$$F_2 = F - \|\gamma'\|\kappa B. \quad (41)$$

Proof: If (8) and (10) is substituted in (12),

$$F_2 = \|\gamma'\|\tau T$$

is obtained. From (2), the proof is completed.

Theorem 3.7. There is the following equation between Darboux vector F belonging to Frenet frame and Darboux vector $\overline{F}$ belonging to alternative frame of any curve γ in E^3:

$$\overline{F} = F + gC.$$

Proof: Since $W = \dfrac{\tau T + \kappa B}{\sqrt{\kappa^2 + \tau^2}}$, from (2) and (15),

$$F = \|\gamma'\| fW \quad (42)$$

is gotten. (42) is substituted in (17), the proof is completed.

Theorem 3.8. There is the following equation between Darboux vector F belonging to Frenet frame and Darboux vector F_3 belonging to N-Bishop frame of any curve γ in E^3:

$$F_3 = F. \quad (43)$$

Proof: If (18) and (20) is substituted in (22),

$$F_3 = \|\gamma'\| fW$$

is obtained. From (42), the proof is completed.

Corollary 3.1. There is the following equation between Darboux vectors F, F_1, F_2 and F_3 belonging to Frenet, type-1 Bishop, type-2 Bishop and N-Bishop frame of any curve γ in E^3, respectively:

$$F = F_3 = F_1 + F_2.$$

Proof: From (40), (41) and (43), the proof is completed.

3.3. Their Relationships and Pole Vectors Belonging to Bishop Frames of Any Regular Curve in E^3

Theorem 3.9. Pole vector W_1 belonging to type-1 Bishop frame of any curve γ in E^3:

$$W_1 = -\sin\theta N_1 + \cos\theta B_1. \quad (44)$$

Proof: If (5) is substituted in (7),

$$W_1 = \frac{F_1}{\|F_1\|} = -\frac{\tau_1}{\kappa} N_1 + \frac{\kappa_1}{\kappa} B_1$$

is obtained. So, the proof is completed.

Corollary 3.2. There is the following equation between binormal vector B and pole vector W_1 belonging to type-1 Bishop frame of any curve γ in E^3:

$$W_1 = B.$$

Proof: From (3) and (44), it is clear.

Theorem 3.10. Pole vector W_2 belonging to type-2 Bishop frame of any curve γ in E^3:

$$W_2 = \sin\phi N_2 - \cos\phi B_2. \quad (45)$$

Proof: If (10) is substituted in (12),

$$W_2 = \frac{F_2}{\|F_2\|} = -\frac{\tau_2}{\tau} N_2 + \frac{\kappa_2}{\tau} B_2$$

is obtained. So, the proof is completed.

Corollary 3.3. There is the following equation between tangent vector T and pole vector W_2 belonging to type-2 Bishop frame of any curve γ in E^3:

$$W_2 = T.$$

Proof: From (8) and (45), it is clear.

Theorem 3.11. Pole vector $\overline{W}$ belonging to alternative frame of any curve γ in E^3:

$$\overline{W} = \frac{g}{\sqrt{g^2 + \|\gamma'\|^2 f^2}} N + \frac{\|\gamma'\| f}{\sqrt{g^2 + \|\gamma'\|^2 f^2}} W.$$

Proof: From (17), the proof is completed.

Theorem 3.12. Pole vector W_3 belonging to N-Bishop frame of any curve γ in E^3:

$$W_3 = -\sin\varphi N_3 + \cos\varphi B_3. \quad (46)$$

Proof: If (20) is substituted in (22),

$$W_3 = \frac{F_3}{\|F_3\|} = -\frac{\tau_3}{f} N_3 + \frac{\kappa_3}{f} B_3$$

is obtained. So, the proof is completed.

Corollary 3.4. There is the following equation between pole vector W and pole vector W_3 belonging to type-1 Bishop frame of any curve γ in E^3:

$$W_3 = W .$$

Proof: From (2), (18) and (46), it is clear.

Corollary 3.5. There is the following equation between pole vectors W , W_1, W_2 and W_3 belonging to Frenet, type-1 Bishop, type-2 Bishop, N-Bishop frames of any curve γ in E^3, respectively:

$$W = W_3 = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}} W_1 + \frac{\tau}{\sqrt{\kappa^2+\tau^2}} W_2 .$$

Proof: From (2), (44), (45) and (46), it is clear.

4. Conclusion

In this study, the relationships between various Bishop frames and their Darboux vectors are discussed and new results are obtained. These results will be evaluated in the studies that have been done or will be done on Bishop frames. In addition, these results can be examined in various spaces such as Galilean space, Lorentz space, Dual Lorentz space.

Declaration of Ethical Standards
The authors declare that they comply with all ethical standards.

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Author 1: Conceptualization, Methodology/Study design, Software, Validation, Formal analysis, Investigation, Resources, Data curation, Writing–original draft, Writing–review and editing, Visualization, Supervision

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