INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 16 NO. 2 PAGE 680–688 (2023)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1343784

Fuzzy Counterpart of Klein Quadric

Ziya Akça * and Abdilkadir Altıntaş

(Communicated by Cihan Özgür)

ABSTRACT

Many techniques have been proposed to project the high-dimensional space into a low-dimensional space, one of the most famous methods being principal component analysis. The Klein quadric is a geometric shape defined by a second-degree homogeneous equation. The lines of projective three-space are, via the Klein mapping, in one-to-one correspondence with points of a hyperbolic quadric of the projective $5-$space. This paper presents some results on the images under the Klein mapping of the projectice $3-$space order of 4 and the fuzzification of the Klein quadric in $5-$dimensional projective space.

Keywords: *Fuzzy point, fuzzy vector space, fuzzy projective space, Klein quadric.*

AMS Subject Classification (2020): *Primary: 20N25; 51E20; 51E30*

1. Introduction

Mathematicians also use Galois theory to examine geometry, physics, chemistry, and many other fields of study. We introduce the $n-$dimensional projective space $PG(n, K)$ for $n > 0$ and K, any Galois field $GF(q^r)$ where q is prime number and r is a positive integer. In a field of order q^r, adding q copies of any element always results in zero; that is, the characteristic of the field is q.

Let V be any vector space of dimension $(n + 1)$ over K. Then $PG(n, K)$, the $n-$dimensional projective space over K, is the set of all subspaces of V distinct from the trivial subspaces $\{0\}$ and V. The $1-$dimensional subspaces are called the points of $PG(n, K)$, the $2-$dimensional subspaces are called the (projective) lines and the $3-$dimensional ones are called (projective) planes. We can see that by going from a vector space to the associated projective space, the dimension drops by one unit. Hence an $(n + 1)-$dimensional vector space V gives rise to an $n-$dimensional projective space $PG(n, K)$ [7].

It is shown in [3] that the lax generalized Veronesean embeddings of $PG(2, 3)$. They relate to the quadric Veronesean of $PG(n, K)$ in $PG(d, K)$ and its projections from subspaces of $PG(d, K)$ generated by sub-Veroneseans (the point sets corresponding to subspaces of $PG(n, K)$, if K is commutative, and to a degenerate analogue of this, if K is noncommutative.

Since the Klein correspondence preserves incidence relation, two concurrent lines of $PG(n, K)$ are mapped to two collinear points of the Klein quadric. If we take all the lines incident with a single point p of $PG(n, K)$, then the image is a set of mutually collinear points of Klein quadric, that is, a plane of the quadric. Likewise, if we take all the lines contained in a single projective plane of $PG(n, K)$, then the image is again a plane (for the same reason). So points and planes of $PG(n, K)$ are the planes of Klein quadric. Two points of $PG(n, K)$ span a line, and so the corresponding planes must meet in a point. It is not difficult to see that an incident point and plane must be mapped to the set of points on a line of the quadric, and a non-incident point and plane yield two planes of the Klein quadric that are disjoint. In other words, we have an equivalence relation on planes of the Klein quadric that divides them into two equivalence classes:

The two equivalence classes are known as the latins and the greeks. The latins are in one-to-one correspondence with the points of $PG(3, K)$, whereas the greeks are in one-to-one correspondence with the planes of $PG(3, K)$. Projective space is a mathematical concept with nonlinear geometry, and the Klein quadric is defined as a surface within projective space and expressed by a specific equation.

Received : *15–08–2023,* ***Accepted :*** *28–09–2023*
***** Corresponding author***

A lot of theories in mathematics have the so-called *fuzzy counterparts*. Basically this means that certain elements of an object get a membership degree as an alternative for the classical black and white situation of belonging to or not belonging to. In the theory of fuzzy sets, Lubczonok [10] gives the notion of *fuzzy vector space*, he defines fuzzy dimension for all fuzzy vector spaces as a non-negative real number or infinity and he investigates the properties of the introduced concepts. In [9], a general definition of a fuzzy $n-$dimensional projective space $\lambda^{'}$ which is obtained from fuzzy $(n+1)-$dimensional vector space λ on V over some field K and a method to find a fuzzy projective line and a fuzzy projective plane are given. The notion of a fuzzy spread of a fuzzy projective space and the fuzzy line spreads of the smallest finite projective space are inroduced in [1].

And then the classification of fuzzy vector planes and 3-dimensional vector spaces of fuzzy $4-$dimensional vector space are given in [2],[6]. In [5], the role of triangular norm, fibered harmonic conjugates and a fibered version of Reidemeister's condition are considered. In [4], the classifications of fuzzy vector planes of fuzzy $(n+1)-$dimensional vector space and fuzzy projective lines of fuzzy $n-$dimensional projective space from fuzzy $(n+1)-$dimensional vector space λ on V for $n>2$ were given.

2. Preliminaries

First recall that, projective space, Klein quadric and fuzzy set will be introduced.

Definition 2.1. We define the n-dimensional projective space $PG(n,K)$ for $n>0$ and K any (skew) field. Let V be any vector space of dimension $n+1$ over K. Then $PG(n,K)$, the n-dimensional projective space over K, is the set of all subspaces of V distinct from the trivial subspaces $\{\overline{0}\}$ and V. The one-dimensional subspaces of V are called the points of $PG(n,K)$, the two-dimensional subspaces are called the (projective) lines and the three-dimensional ones are called (projective) planes. A non-trivial $(k+1)$-dimensional subspaces of V is also called a k-subspace of $PG(n,K)$, or simply, a subspace. Since every subspace of V is itself a vector space, we may view any subspace of $PG(n,K)$ as a projective space. For two subspaces U, U' of $PG(n,K)$, we write $U\leq U'$ if U is contained in U'.

Let $PG(3,q)$ be a $3-$dimensional projective space over a field $GF(q)$ where q is prime, such that the points and planes are represented by homogeneous coordinates. The method is due to Julius Plucker (1801-1868). In [11], the homogeneity of Plucker coordinates suggest to view the the coordinates of a line as homogenous coordinates of points in ve dimensional space $PG(5,q)$. This is a particular case of construction of the Grassmannian of lines. Homogenous coordinates in $PG(5,q)$ are written as in the form $(l_{01},l_{02},l_{03},l_{23},l_{31},l_{12})$ where

$$l_{ij}=x_iy_j-x_jy_i.$$

We denote $(X_0,X_1,X_2,X_3,X_4,X_5)=(l_{01},l_{02},l_{03},l_{23},l_{31},l_{12})$.

Definition 2.2. (see [8]) The Klein mapping,

$$\gamma:\ \mathcal{L}\rightarrow PG(5,q)$$

assigns to a line of $\mathcal{L}$ in $P(3,q)$ the point $(l_{01},l_{02},l_{03},l_{23},l_{31},l_{12})$ of $PG(5,q)$ where $(l_{01},l_{02},l_{03},l_{23},l_{31},l_{12})$ are the line's Plucker coordinates (F.Klein, 1868). The lines of projective three-space are, via the Klein mapping, in one-to-one correspondence with points of a hyperbolic quadric of the projective $5-$space. The quadric of $PG(5,q)$ denoted by the equation,

$$X_0X_3+X_1X_4+X_2X_5=O$$

is called the Klein Quadric and is denoted by the symbol $\mathcal{H}^5$.

The Klein quadric, being a hyperbolic quadric in five dimensions, contains points, lines and planes. The two equivalence classes are known as the Latin planes and the Greek planes. The points of $PG(3,2)$ are mapped to the Latin Planes, whereas the planes are mapped to the Greek planes, [5,9]. Since $PG(3,2)$ have 15 points, 35 lines and 15 planes, the Klein Quadric $\mathcal{H}^5$ have 35 point, 15 the Latin Plane and 15 the Greek plane.

Fuzzy sets were introduced by Zadeh in the fundamental paper [12]. A *fuzzy set* λ of a set X is a function $\lambda:X\rightarrow[0,1]$. We assume that the reader is familiar with the basic notions of fuzzy mathematics and get down the following definitions and theorems.

Definition 2.3. (see [10]) Let $\lambda:V\rightarrow[0,1]$ be a fuzzy set on V. Then we call λ a fuzzy vector space on V if and only if $\lambda(a.\overline{u}+b.\overline{v})\geq\lambda(\overline{u})\wedge\lambda(\overline{v})$, $\forall\overline{u},\overline{v}\in V$ and $\forall a,b\in K$.

Definition 2.4. (see [9]) Let λ is a fuzzy vector space on V. The subspace L, (linearly) generated by $Supp(\lambda)$ $(\sup p(\lambda) = \{x \in V : \lambda(x) = 0\}$, is called the base vector space of λ. The dimension $d(\lambda)$ of a fuzzy vector space of V is the dimension of its base subspace.

Definition 2.5. (see [9]) If U is an $i-$dimensional subspace of V, and (λ, U) is a fuzzy vector space, then it is called a fuzzy $i-$dimensional vector space on U. If $i = 1$, i.e. U is a vector line, then (λ, U) is a fuzzy vector line on U, if $i = 2$, i.e. U is a plane, (λ, U) will be called a fuzzy vector plane on U. If $i = n-1$, then (λ, U) is called a fuzzy vector hyperplane on U.

Let V be an $n-$dimensional vector space over some field K, with $n \geq 2$. Let L be a vector line of V, so L is uniquely defined by some nonzero vector $\overline{u}$. Let α be a vector plane of the $n-$dimensional vector space V $(n \geq 3)$, then we know that α is uniquely defined by two linearly independent vectors $\overline{u}$ and $\overline{v}$.

Theorem 2.1. *(see[9]) If $\lambda : L \to [0,1]$ is a fuzzy vector line on L, then $\lambda(\overline{u}) = \lambda(\overline{v})$, $\forall \overline{u}, \overline{v} \in L \backslash \{\overline{o}\}$, and $\lambda(\overline{o}) \geq \lambda(\overline{u})$, $\forall \overline{u} \in L$.*

Theorem 2.2. *(see [9]) If $\lambda : \alpha \to [0,1]$ is a fuzzy vector plane on α, then there exists a vector line L of α and real numbers $a_0 \geq a_1 \geq a_2 \in [0,1]$ such that λ is of the following form:*

$$\begin{aligned} \lambda : \quad & \alpha \to [0,1] \\ & \overline{o} \to a_0 \\ & \overline{u} \to a_1 \text{ for } \overline{u} \in L \backslash \{\overline{o}\} \\ & \overline{u} \to a_2 \text{ for } \overline{u} \in \alpha \backslash L, \end{aligned}$$

Definition 2.6. (see [9]) Suppose V is an $n-$dimensional vector space. A flag in V is a sequence of distinct, non-trivial subspaces $(U_0, U_1, ..., U_m)$ such that $U_j \subset U_i$ for all $j < i < n-1$. The rank of a flag is the number of subspaces it contains. A maximal flag in V is a flag of length n.

Theorem 2.3. *(see [2]) Let V be a 4-dimensional vector space over some field K and $\lambda : V \to [0,1]$ be a fuzzy vector space on V. Then the fuzzy 4-dimensional vector space λ has exactly six kinds of fuzzy vector planes.*

Theorem 2.4. *(see [2]) Fuzzy $3-$dimensional projective space λ' from fuzzy $4-$dimensional vector space λ over some field K has exactly six kinds of fuzzy projective lines.*

Theorem 2.5. *(see [6]) Fuzzy $3-$dimensional projective space λ' from fuzzy $4-$dimensional vector space λ over some field K has exactly four kinds of fuzzy projective planes.*

3. The images under the Klein mapping of the projectice 3-space order of 4

Let $PG(3,4)$ be a 3-dimensional projective space over a field $GF(2^2)$ such that the points and planes are represented by homogeneous coordinates using irreducible polynomial t^2+t+1 over $GF(2)$. $PG(3,4)$ has 85 points, 357 lines and 85 planes. We will give the full list of points and lines of $\alpha-$plane and $\beta-$plane with the following propositions.

Proposition 3.1. *The points of $PG(3,4)$ are mapped to $\alpha-$planes in projective space $PG(5,4)$ via Klein mapping.*

Proof. We prove for a point. The other proofs are similar. Let $P = (0,0,0,1)$ be a point in $PG(3,4)$.There are 21 lines through P. These lines with their Plücker coordinates listed below as l_i, $i = 1$ to 21.

$$\begin{aligned} & l_1(0,0,0,0,1,0), l_2(0,0,0,1,0,0), l_3(0,0,0,1,1,0), l_4(0,0,0,t^2,1,0), \\ & l_5(0,0,0,t,1,0), l_6(0,0,1,1,1,0\}, l_7(0,0,1,1,0,0), l_8(0,0,1,1,t,0), \\ & l_9(0,0,1,0,t^2,0), l_{10}(0,0,1,0,1,0), l_{11}(0,0,1,t^2,1,0), l_{12}(0,0,1,t,1,0), \\ & l_{13}(0,0,1,t^2,t,0), l_{14}(0,0,1,t,t^2,0), l_{15}(0,0,1,0,0,0), l_{16}(0,0,1,0,t,0), \\ & l_{17}(0,0,1,1,t^2,0), l_{18}(0,0,1,t^2,0,0), l_{19}(0,0,1,t,0,0), l_{20}(0,0,1,t^2,t^2,0), \\ & l_{21}(0,0,1,t,t,0) \end{aligned}$$

under the Klein mapping these lines are points of $PG(5,4)$.We show that these 21 points of $PG(5,4)$ form a projective plane in $PG(5,4)$. Let L_1 be line through point $l_1 l_2$. Equation of this line is;

$$(0,0,0,0,1,0) + \lambda(0,0,0,1,0,0)$$

where $\lambda = 1, t, t^2$. For $\lambda = 1$ we obtain the point $l_3(0,0,0,1,1,0)$, for $\lambda = t, t^2$ we obtain the points $l_5(0,0,0,t,1,0)$, $l_4(0,0,0,t^2,1,0)$ respectively. The points l_1, l_2, l_3, l_4, l_5 lie on the line L_1. The complete list of lines and points;

Lines	Points on the line
L_1	l_1, l_2, l_3, l_4, l_5
L_2	$l_1, l_6, l_7, l_8, l_{17}$
L_3	$l_1, l_9, l_{10}, l_{15}, l_{16}$
$\mathcal{L}_4$	$l_1, l_{11}, l_{13}, l_{18}, l_{20}$
L_5	$l_1, l_{12}, l_{14}, l_{19}, l_{21}$
L_6	$l_2, l_6, l_{10}, l_{11}, l_{12}$
L_7	$l_2, l_7, l_{15}, l_{18}, l_{19}$
L_8	$l_2, l_8, l_{13}, l_{16}, l_{21}$
L_9	$l_2, l_9, l_{14}, l_{17}, l_{20}$
L_{10}	$l_3, l_6, l_{15}, l_{20}, l_{21}$
L_{11}	$l_3, l_7, l_{10}, l_{13}, l_{14}$
L_{12}	$l_3, l_8, l_9, l_{12}, l_{18}$
L_{13}	$l_3, l_{11}, l_{16}, l_{17}, l_{19}$
L_{14}	$l_4, l_6, l_9, l_{13}, l_{19}$
L_{15}	$l_4, l_7, l_{12}, l_{16}, l_{20}$
L_{16}	$l_4, l_8, l_{11}, l_{14}, l_{15}$
L_{17}	$l_4, l_{10}, l_{17}, l_{18}, l_{21}$
L_{18}	$l_5, l_6, l_{14}, l_{16}, l_{18}$
L_{19}	$l_5, l_7, l_9, l_{11}, l_{21}$
L_{20}	$l_5, l_8, l_{10}, l_{19}, l_{20}$
L_{21}	$l_5, l_{12}, l_{13}, l_{15}, l_{17}$

These 21 lines $\mathcal{L}_i$ and 21 points l_i ($i = 1$ to 21)satisfy the projective plane axioms. So its a plane of $PG(5,4)$ that is called α-plane of point $P = (0,0,0,1)$ in $PG(3,4)$. □

Proposition 3.2. *The planes of $PG(3,4)$ are mapped to $\beta-$planes in projective space $PG(5,4)$ via Klein mapping.*

Proof. We prove for a plane. The other proofs are similar. Let $P = [0,0,0,1]$ be a projective plane in $PG(3,4)$.There are 21 points and 21 lines of P in $PG(3,4)$. The points and lines with their Plücker coordinates listed below as:

$N_1(0,1,0,0)$, $N_2(0,1,0,0)$, $N_3(0,1,0,0)$, $N_4(0,1,0,0)$, $N_5(0,1,0,0)$,
$N_6(0,1,0,0)$, $N_7(0,1,0,0)$, $N_8(0,1,0,0)$, $N_9(0,1,0,0)$, $N_{10}(0,1,0,0)$,
$N_{11}(0,1,0,0)$, $N_{12}(0,1,0,0)$, $N_{13}(0,1,0,0)$, $N_{14}(0,1,0,0)$, $N_{15}(0,1,0,0)$,
$N_{16}(0,1,0,0)$, $N_{17}(0,1,0,0)$, $N_{18}(0,1,0,0)$, $N_{19}(0,1,0,0)$, $N_{20}(0,1,0,0)$,
$N_{21}(0,1,0,0)$
and
$L_1(0,0,0,0,0,1)$, $L_2(1,0,0,0,0,1)$, $L_3(1,0,0,0,0,0)$, $L_4(1,0,0,0,0,t^2)$,
$L_5(1,0,0,0,0,t)$, $L_6(0,1,0,0,0,1)$, $L_7(0,1,0,0,0,1)$, $L_8(0,1,0,0,0,t)$,
$L_9(0,1,0,0,0,t^2)$, $L_{10}(1,1,0,0,0,0)$, $L_{11}(1,1,0,0,0,1)$, $L_{12}(1,1,0,0,0,t^2)$,
$L_{13}(1,1,0,0,0,t)$, $L_{14}(1,t^2,0,0,0,t)$, $L_{15}(1,t^2,0,0,0,1)$, $L_{16}(1,t^2,0,0,0,0)$,
$L_{17}(1,t^2,0,0,0,t^2)$, $L_{18}(1,t,0,0,0,t^2)$, $L_{19}(1,t,0,0,0,1)$, $L_{20}(1,t,0,0,0,t)$,
$L_{21}(1,t,0,0,0,0)$.

Incidence relation for these points and lines are as follows:

Points	Lines through the point
N_1	L_1, L_2, L_3, L_4, L_5
N_2	L_1, L_6, L_7, L_8, L_9
N_3	$L_1, L_{10}, L_{11}, L_{12}, L_{13}$
N_4	$L_1, L_{14}, L_{15}, L_{16}, L_{17}$
N_5	$L_1, L_{18}, L_{19}, L_{20}, L_{21}$
N_6	$L_2, L_6, L_{10}, L_{14}, L_{18}$
N_7	$L_2, L_6, L_{11}, L_{15}, L_{19}$
N_8	$L_2, L_8, L_{12}, L_{16}, L_{20}$
N_9	$L_3, L_9, L_{12}, L_{14}, L_{19}$
N_{10}	$L_3, L_6, L_{11}, L_{17}, L_{20}$
N_{11}	$L_4, L_6, L_{13}, L_{16}, L_{19}$
N_{12}	$L_5, L_6, L_{12}, L_{15}, L_{21}$
N_{13}	$L_4, L_8, L_{12}, L_{14}, L_{21}$
N_{14}	$L_5, L_9, L_{11}, L_{16}, L_{21}$
N_{15}	$L_3, L_7, L_{10}, L_{16}, L_{21}$
N_{16}	$L_3, L_8, L_{13}, L_{15}, L_{18}$
N_{17}	$L_2, L_9, L_{13}, L_{17}, L_{21}$
N_{18}	$L_4, L_7, L_{12}, L_{17}, L_{18}$
N_{19}	$L_5, L_7, L_{13}, L_{14}, L_{20}$
N_{20}	$L_4, L_9, L_{10}, L_{15}, L_{20}$
N_{21}	$L_5, L_8, L_{10}, L_{17}, L_{19}$

Under the Klein mapping γ these lines are points of $PG(5,4)$.We show that these 21 points of $PG(5,4)$ form a projective plane in $PG(5,4)$. Let $L_1^{\shortmid}$ be line through points L_{1L2}. Equation of this line is;

$$(0,0,0,0,0,1) + \lambda(1,0,0,0,0,1)$$

where $\lambda = 1, t, t^2$. For $\lambda = 1$ we obtain the point $L_3(1,0,0,0,0,0)$, for $\lambda = t, t^2$ we obtain the points $L_5(1,0,0,0,0,t)$, $L_4(1,0,0,0,0,t^2)$ respectively. The points L_1, L_2, L_3, L_4, L_5 lie on the line $L_1^{\shortmid}$. Similarly we get $L_2^{\shortmid}, ...L_{21}^{\shortmid}$.These 21 lines $L_i^{\shortmid}$ and 21 points L_i $(i = 1$ to $21)$ satisfy the projective plane axioms. So it is a plane of $PG(5,4)$ that is called β-plane $P = [0,0,0,1]$ in $PG(3,4)$. □

Theorem 3.1. *The quadric Veronesean $\mathcal{V}_2^4$ can be embedded to Klein quadric with a linear transformation.*

Proof. Let (u, v, w) be a point of $PG(2,4)$, the Veronese surface in $PG(5,4)$ has parametric equation

$$\mathcal{V}_2^4 = (u^2, uv, uw, w^2, vw, v^2).$$

Veronese surface is embedded in $\mathcal{H}^5$ by the linear map;

$$\Phi(X_0, X_1, X_2, X_3, X_4, X_5) = (X_0, X_1, X_2, X_3, -X_4, X_5 - X_2).$$

Since $\mathcal{H}^5$ has equation

$$X_0X_3 + X_1X_4 + X_2X_5 = 0,$$

we obtain

$$\mathcal{H}^5(X_0, X_1, X_2, X_3, -X_4, X_5 - X_2) = u^2w^2 - uv^2w + uw(v^2 - uw) = 0.$$

□

4. Fuzzy $(n+1)-$Dimensional Vector Space and Fuzzy $n-$Dimensional Projective Space

A general definition of fuzzy $(n+1)-$dimensional vector space (λ, V) and fuzzy $n-$dimensional projective space λ' on V' are well-known [9]. Here, we restrict ourselves to the case a subspace should necessarily have the same values in its points as the whole space.

$$\begin{array}{rl}\lambda: & V \to [0,1] \\ & \overline{o} \to a_0 \\ & \overline{u} \to a_1 \text{ for } \overline{u} \in U_1 \backslash \{\overline{o}\} \\ & \overline{u} \to a_2 \text{ for } \overline{u} \in U_2 \backslash U_1 \\ & \overline{u} \to a_3 \text{ for } \overline{u} \in U_3 \backslash U_2 \\ & ... \\ & ... \\ & \overline{u} \to a_n \text{ for } \overline{u} \in U_n \backslash U_{n-1} \\ & \overline{u} \to a_{n+1} \text{ for } \overline{u} \in V \backslash U_n \end{array} \tag{1}$$

with U_i an $i-$dimensional subspace of V, containing all U_j for $j < i$, and $a_0 \geq a_1 \geq a_2 \geq ... \geq a_n \geq a_{n+1}$ are reals in $[0, 1]$.

Suppose V' is the $n-$ dimensional projective space corresponding to the $(n + 1)-$dimensional vector space V. Now, we give fuzzy $n-$dimensional projective space $.\lambda'$ on V'

$$\begin{array}{rl}\lambda': & V' \to [0,1] \\ & q \to a_1 \\ & p \to a_2 \text{ for } p \in U_1' \backslash \{q\} \\ & p \to a_3 \text{ for } p \in U_2' \backslash U_1' \\ & ... \\ & ... \\ & p \to a_n \text{ for } p \in U_{n-1}' \backslash U_{n-2}' \\ & p \to a_{n+1} \text{ for } p \in V' \backslash U_{n-1}'. \end{array}$$

with q the projective point corresponding to the fuzzy vector line U_1 and U_i' the $i-$dimensional projective space, corresponding to the vector space U_{i+1}. Then, the sequence $(q, U_1', U_2', ..., V')$ is a maximal flag and $a_1 \geq a_2 \geq ... \geq a_{n+1}$ are reals in $[0, 1]$.

5. Fuzzy Klein Quadric

Many techniques have been proposed to project the high-dimensional space into a low-dimensional space, one of the most famous methods being principal component analysis. Conclusion, this paper presents a research study on the fusion of the Klein quadric in $5-$dimensional projective space. The fusion of the Klein quadric is an important topic in the field of geometry and requires the utilization of different mathematical tools. This study contributes to a better understanding of the geometric properties of the Klein quadric and the discovery of new methods that can serve as foundations for various operations.

In this study, we first provide the fuzzy $5-$dimensional projective space.

$$\begin{array}{rl}\lambda': & PG(5,2) \to [0,1] \\ & q \to a_1 \\ & p \to a_2 \text{ for } p \in U_1' \backslash \{q\} \\ & p \to a_3 \text{ for } p \in U_2' \backslash U_1' \\ & p \to a_4 \text{ for } p \in U_3' \backslash U_2' \\ & p \to a_5 \text{ for } p \in U_4' \backslash U_3' \\ & p \to a_6 \text{ for } p \in PG(5,2) \backslash U_4' \end{array}$$

the sequence $(q, U_1', U_2', ..., PG(5, 2))$ is a maximal flag and $a_1 \geq a_2 \geq ... \geq a_6$ are reals in $[0, 1]$.

Theorem 5.1. *Let P be a $5-$dimensional projective space over a finite field. The fuzzy projective space $[P, \lambda']$, λ' : $PG(5, 2) \to [0, 1]$, with $(q, U_1', U_2', ..., PG(5, 2))$ a maximal flag in $PG(5, 2)$ and $a_1 \geq a_2 \geq ... \geq a_6$ are reals in $[0, 1]$,* *can be constructed from the following the points of fuzzyy Klein quadric* $\lambda^{''}$*:*

$$\begin{aligned} \lambda^{''}: \quad & \mathcal{H}^5 \to [0,1] \\ & q \to a_1 \\ & p \to a_2 \textit{ for } p \in U_1^{''} \backslash \{q\} \\ & p \to a_3 \textit{ for } p \in U_2^{''} \backslash U_1^{''} \\ & p \to a_4 \textit{ for } p \in U_3^{''} \backslash U_2^{''} \\ & p \to a_5 \textit{ for } p \in U_4^{''} \backslash U_3^{''} \\ & p \to a_6 \textit{ for } p \in \mathcal{H}^5 \backslash U_4^{''} \end{aligned}$$

where $q \subseteq U_1^{''} \subseteq U_2^{''} \subseteq U_3^{''} \subseteq U_4^{''} \subseteq \mathcal{H}^5$ *is a chain of subspaces of* $\mathcal{H}^5$ *such that* $U_i^{''}$ *stabilizes the* $[i,5]$*-flag, for all* $i \in \{0,1,2,...,5\}$.

Proof. The quadric of $PG(5,q)$ denoted by the equation,

$$X_0X_3 + X_1X_4 + X_2X_5 = O$$

is called the Klein Quadric and is denoted by the symbol $\mathcal{H}^5$. We want to classifie the points of the Klein quadric with the following membership degrees. The base point q has a a_0 membership degree, the remaining two points of base line have a_1 membership degrees, the remaining four points of base plane have a_2 membership degrees, four points have a_3 membership degrees, 9 points have a_4 membership degrees, and the remaining 15 points of $\mathcal{H}^5$ have a_5 membership degrees.

We will consider an algorithm and fuzzy theory to improve the maximal flag of the Klein model. Now, we provide an algorithm that contains some steps to complete the proof.

STEP1. Now, we determine the maximal flag covering the points of the Klein quadric and give an algoritym.

For this, in the first step, we select a base point on the Klein quadric $\mathcal{H}^5$.

In the second step, we take a base line passing through this base point.

In the third step, we consider a plane of $\mathcal{H}^5$ containing the base line selected in the second step. This plane is called the Greek plane.

In the fourth step, outside the Greek plane, we choose a point on the Klein quadric. Using this point, we determine the four points in the $3-$dimensional subspace of the maximal flag, which consists of the 28 points outside the Greek plane. Additionally, we establish the equation of the 3-dimensional space.

In the fifth step, we select a point on the Klein quadric that is not in the $3-$dimensional space determined previously. Using this point, we determine the equation of the 4-dimensional subspace of maximal flag and identify its 9 points.

In the final step, we select a point on the Klein quadric that is not in the 4-dimensional space previously determined. Using this point, we determine the equation of the 5-dimensional space in the maximal flag and identify its remaining 15 points.

Exactly! Through the process of selecting points and determining the equations of the lower-dimensional spaces within the Klein quadric, we establish the distribution of its 35 points into the subspaces of the maximal flag. This provides valuable information about the geometric structure and arrangement of points in the Klein quadric space.

STEP2. Thus, we obtain;

0-dimensional space: A point represented by "q" (base point)

1-dimensional space: A line represented by "$U_1^{''}$" (Base line passing through the base point)

2-dimensional space: A plane represented by "$U_2^{''}$" (The Greek plane containing the base line passing through the base point).

3-dimensional space: A subspace represented by "$U_3^{''}$" (3-dimensional subspace consisting of the 28 points outside the Greek plane).

4-dimensional space: A subspace represented by "$U_4^{''}$" (4-dimensional subspace consisting of a point outside the 3-dimensional space and the remaining 9 points).

5-dimensional space: A subspace represented by "$\mathcal{H}^5$" ($5-$dimensional subspace consisting of a point outside the 4-dimensional space and the remaining 15 points).

STEP3. These subspaces represent the arrangement of the 35 points in the Klein quadric space according to the maximal flag.

The distribution of points forming the maximal flag is obtained as follows:

The base point $q = (0,0,0,1,0,0)$, the base line $U_1^{''} = \{(0,0,0,1,0,0), (0,0,0,0,1,0), (0,0,0,1,1,0)\}$,

the base plane $U_2^{''} = \{(0,0,0,1,0,0), (0,0,0,0,1,0),$
$(0,0,1,1,1,0), (0,0,1,1,0,0), (0,0,0,1,1,0), (0,0,1,0,1,0), (0,0,1,0,0,0)\}$,
$U_3^{''} = U_2^{''} \cup \{(0,1,0,1,0,0), (0,1,1,0,0,0), (0,1,0,0,0,0), (0,1,1,1,0,0)\}$.

Here, we take a $3-$space with the equation $S_3 = [x_1, 0, 0, 0, 0, x_6]$ of $PG(5,2)$, which encompasses the Greek plane $U_2^{''}$, is obtained as $U_3^{''} = S_3 \cap \mathcal{H}^5$ and a $4-$space with the equation $S_4 = [x_1, 0, 0, 0, 0, 0]$ of $PG(5,2)$, which encompasses $U_3^{''}$, is obtained as $U_4^{''} = S_4 \cap \mathcal{H}^5$.

One can easily calculate that the points of the Klein quadric are as follows: $U_3^{''} \backslash U_2^{''}$ has 4 points, $U_4^{''} \backslash U_3^{''}$ has 9 points, and $\mathcal{H}^5 \backslash U_4^{''}$ has 15 points. This chain allows us to write $\lambda^{''}$ as the following fuzzy Klein quadric on $PG(5,2)$.

If we suppose that $a_0, a_1, ..., a_5$ are mutually distinct, we can give the membership degrees the points of $\mathcal{H}^5$.

1. Put $\lambda^{''}(q) = a_0 \geq \lambda^{''}(p), \forall p \in \mathcal{H}^5$.

2. We chose two points in the base line $U_1^{''}$, so that every point $p \in U_1^{''}$ can be written as a linear combination. We see that $\lambda^{''}(p) = a_0 \wedge a_1$ for $p \in U_1^{''}$.

3. We chose three points in the base plane $U_2^{''}$, so that every point $p \in U_2^{''}$ can be written as a linear combination. We see that $\lambda^{''}(p) = a_1 \wedge a_2$ for $p \in U_2^{''}$.

4. We chose a point $(0,1,0,0,0,0)$ in the $U_3^{''} = S_3 \cap \mathcal{H}^5$, so that $\lambda^{''}(p) = a_2 \wedge a_3$ for $p \in U_3^{''}$.

5.We chose a point $(0,0,0,0,0,1)$ in the $U_4^{''} = S_4 \cap \mathcal{H}^5$, so that $\lambda^{''}(p) = a_3 \wedge a_4$ for $p \in U_4^{''}$.

6. the remaining 15 points of $\mathcal{H}^5 \backslash U_4^{''}$ will have the same a_5-membership degree.

Therefore, the membership degrees of the points of the Klein quadric and the fuzzy version of the Klein quadric are obtained as follows:

$$\begin{aligned} \lambda^{''}: \quad & \mathcal{H}^5 \to [0,1] \\ & q \to a_0 \\ & p \to a_1 \quad \text{for } p \in U_1^{''} \backslash \{q\} \\ & p \to a_2 \quad \text{for } p \in U_2^{''} \backslash U_1^{''} \\ & p \to a_3 \quad \text{for } p \in U_3^{''} \backslash U_2^{''} \\ & p \to a_4 \quad \text{for } p \in U_4^{''} \backslash U_3^{''} \\ & p \to a_5 \quad \text{for } p \in \mathcal{H}^5 \backslash U_4^{''} \end{aligned}$$

the sequence $(q, U_1^{''}, U_2^{''}, U_3^{''}, U_4^{''}, \mathcal{H}^5)$ is a maximal flag and $a_0 \geq a_2 \geq ... \geq a_5$ are reals in $[0,1]$. □

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Akça, Z., Bayar, A., Ekmekçi, S., Van Maldeghem, H.: *Fuzzy Projective Spreads of Fuzzy Projective Spaces*, Fuzzy Sets and Systems, vol. 157, no. 24, pp. 3237-3247 (2006).

[2] Akça, Z., Bayar, A., Ekmekçi, S.: *On the classification of Fuzzy projective lines of Fuzzy 3-dimensional projective spaces*, Communications Mathematics and Statistics, Vol. 55(2), 17-23 (2007).

[3] Akça, Z., Bayar, A., Ekmekçi, S., Kaya, R., Thas, J. A., Van Maldeghem H.: *Generalized lax Veronesean embeddings of projective spaces*, Ars Combin. 103, 65-80 (2012).

[4] Akça, Z.: *On fuzzify the notion of Grassmannian in fuzzy n dimensional projective space*, International Journal of the Physical Sciences, vol. 6, no. 25, pp. 5877-5882 (2011).

[5] Bayar, A., Akça, Z., Ekmekçi, S.: *A Note on Fibered Projective Plane Geometry*, Information Science 178, 1257-1262 (2008).

[6] Ekmekçi, S., Bayar, A., Akça, Z.: *On the classification of Fuzzy projective planes of Fuzzy 3-dimensional projective spaces*, Chaos, Solitons and Fractals 40, 2146-2151 (2009).

[7] Hirschfeld J.W.P.: *Projective Geometries over Finite Fields*, Oxford Mathematical Monographs, 576 (1998).
[8] Klein, F.: *Uber die Transformation der allgemeinen Gleichung des zweiten Grades zwischen Linien-Koordinaten auf eine kanonische*, Form Math., p. 539-578 (1868).
[9] Kuijken L., Van Maldeghem H., Kerre E.E.: *Fuzzy projective geometries from fuzzy vector spaces*, in: A. Billot et al. (Eds.), Information Processing and Management of Uncertainty in Knowledge-based Systems, Editions Medicales et Scientifiques, Paris, La Sorbonne, 1331-1338 (1998).
[10] Lubczonok P.: *Fuzzy Vector Spaces*, Fuzzy Sets and Systems 38, 329-343 (1990).
[11] Plucker, J.: *On a New Geometry of Space*, Philosophical Transactions of the Royal Society of London, Vol. 155, pp. 725-791 (1865).
[12] Zadeh, L.: *Fuzzy sets*, Information control 8, 338-353 (1965).

Affiliations

Ziya Akça
Address: Eskişehir Osmangazi University, Science Foculty, Department of Mathematics and Computer Science, 26480 Eskişehir, Turkey.
e-mail: zakca@ogu.edu.tr
ORCID ID:0000-0001-6379-0546

Abdilkadir Altintaş
Address: Eskişehir Osmangazi University, Science Foculty, Department of Mathematics and Computer Science, 26480 Eskişehir, Turkey.
e-mail: kadiraltintas1977@gmail.com
ORCID ID:0000-0002-7012-352X